Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 03: 9-15-21:

Set Theory, Probability Wish List, Coding: https://youtu.be/2agUkFQJtnU (2019 lecture)
https://youtu.be/ZhPypMaQWhc (2021 lecture)

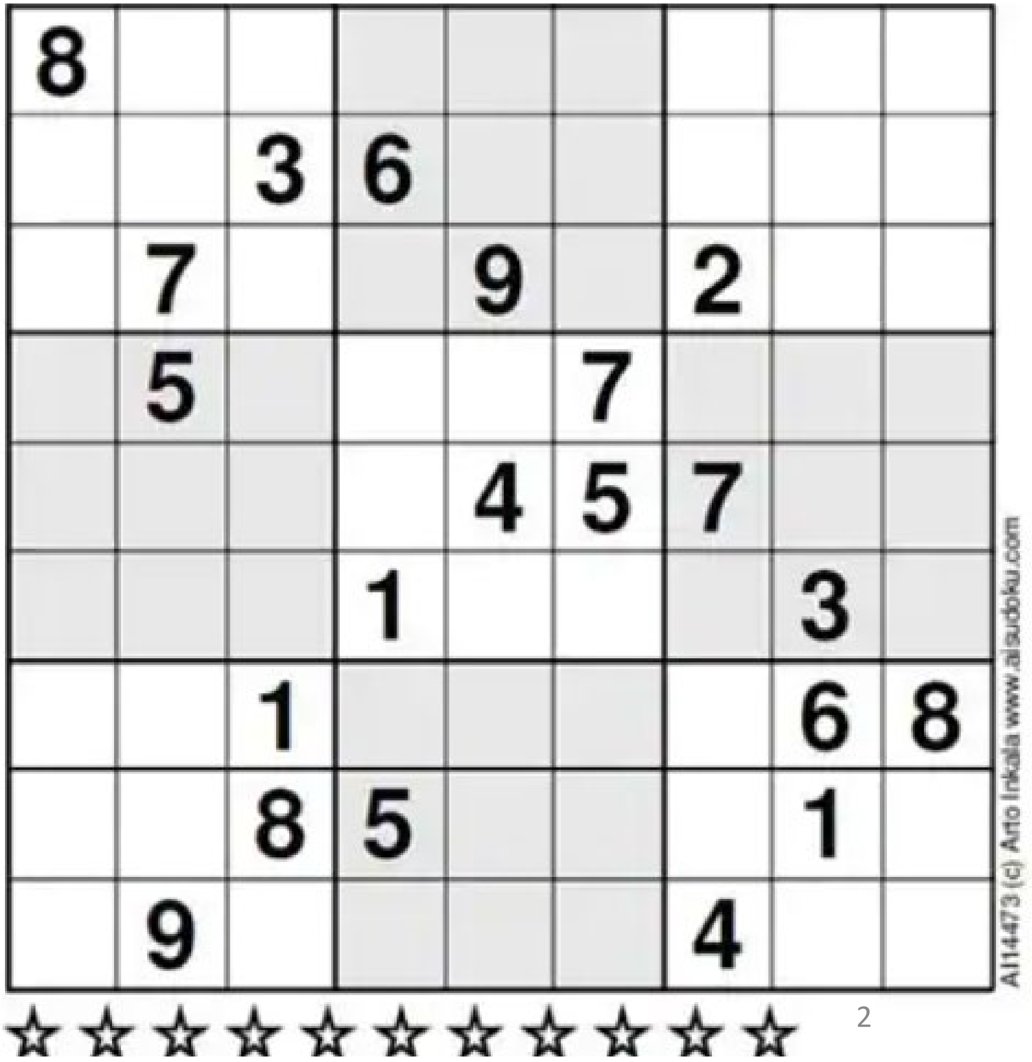
Plan for the day: Lecture 3: September 15, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- General review: combinatorics, integration, …?
- Power of coding

General items.

- Sniffing out formulas?



Die Problem: Do you take the bet?

I have a fair 6 sided die with[1 - 6] on the sides and you are paid the same amount of dollars as the number you roll.If you roll a 5 you get $5, if you roll a 1 you get $1, etc.The question was to find out "how much are you willing to pay to play the game?"

The game then evolved to how much are you willing to pay to play the game with two dice?In this case you get to keep your first roll or you get to roll again and are forced to take whatever the second roll is.Keep in mind, this is for a role that pays~$110, 000 a year out of college and I blurted the answers out within 15 seconds.Even if you don' t do well in probability at Williams, if you know basic concepts like expected value and can keep cool in an interview, then you can move your interview along quite nicely and put yourself in a good position to go from getting a D/D - in probability to not being broke😂.

Simple bounds for two die game

Recall 1 die game ok paying \$3.50

Clearly will play if \$1, not at \$6

If do not use second roll, first game, so play at \$3.50 for sure.

Expect better than \$3.50: can take first roll if 4, 5 or 6

(happens half the time)

$$\frac{1}{2}5 + \frac{1}{2}(3.5) = 4.25$$

$$\frac{1}{2}\left(\frac{1}{3}4 + \frac{1}{3}5 + \frac{1}{3}6\right) \qquad \left(\frac{1}{6}1 + \cdots + \frac{1}{6}6\right)$$

```
twodiesim[iterations_] := Module[{},
   get = 0; (* keeps a running sum of what you get *)
   For[n = 1, n ≤ iterations, n++,
    {
     x = RandomInteger[{1, 6}]; (* roll a fair die *)
     (* if roll > 3 keep and add, else re-roll and keep that *)
     If[x > 3, get = get + x, get = get + RandomInteger[{1, 6}]];
    }];
   get = get / iterations; (* average earning per game *)
   Print["Earn on average ", get 1.0, "."]
  ];
```

```
Timing[twodiesim[100000]]
Timing[twodiesim[1000000]]
Timing[twodiesim[10000000]]
Timing[twodiesim[100000000]]
```

```
Earn on average 4.25225.

{0.1875, Null}

Earn on average 4.25142.

{1.875, Null}

Earn on average 4.25019.

{18.6406, Null}

Earn on average 4.25026.

{182.156, Null}
```

My logic : second die on average 3.5, take the first if it is a 4, 5, 6 (average is 5).
So half the time get a 5, half the time a 3.5, average is 4.25

Integration

Change of variable (u-substitution)

↳ Trig substitution

∫ by parts (product rule)

partial fractions

Converting to double integral

Differentiating under the integral sign

Bring it over Method

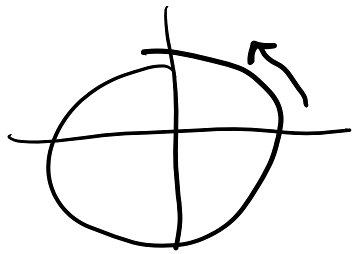
Integration by parts

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

or
$$\int_a^b u\,dv = u(x)v(x)\Big|_a^b - \int_a^b v\,du$$

$u\,dv =$
$= u(x)v'(x)dx$
$\frac{dv}{dx}dx$



Ex: $\int_0^\pi x\cos x\,dx$

$u(x) = x$

$du = dx$

$dv = \cos x\,dx$

$v = \sin x$

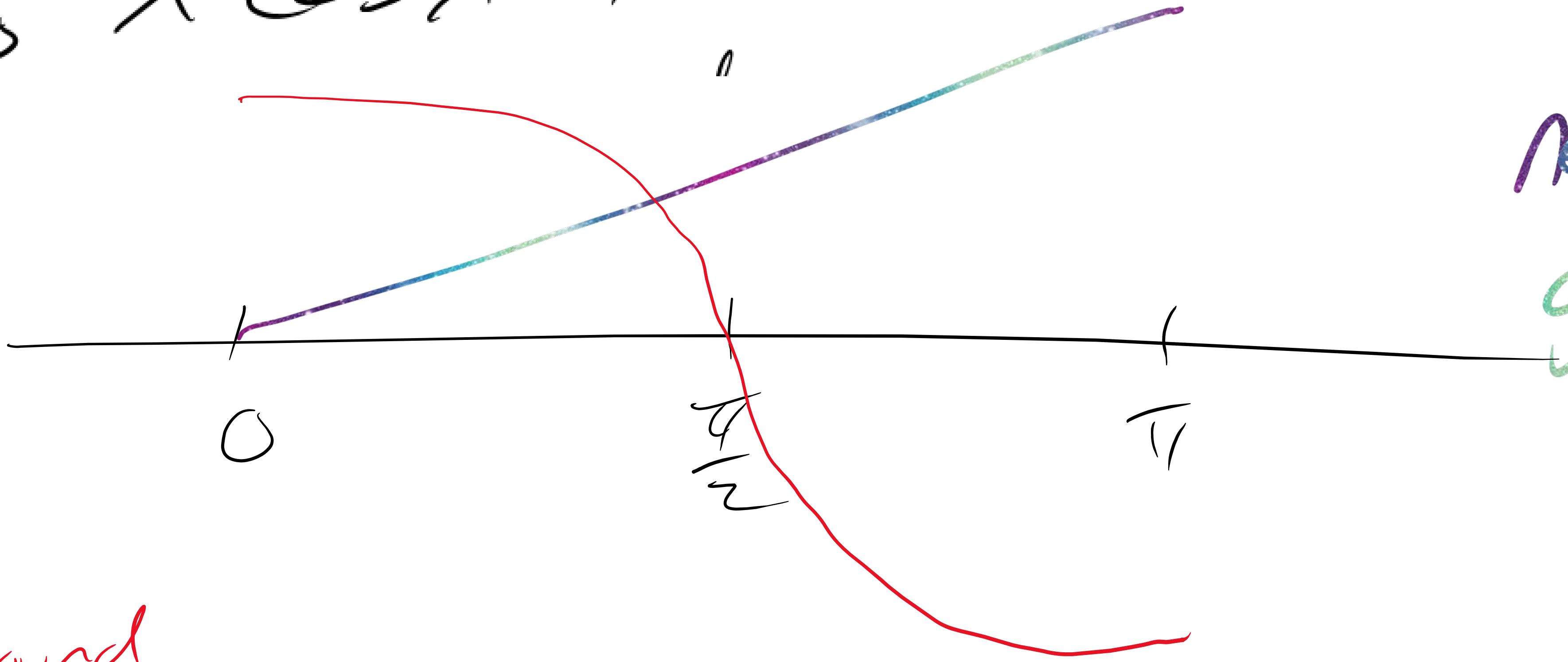
$\cos'(x) = -\sin x$

minus sign

$$= uv\Big|_0^\pi - \int_0^\pi v\,du = x\sin x\Big|_0^\pi - \int_0^\pi \sin x\,dx$$

$$= 0 + \cos x\Big|_0^\pi = -2$$

$\int_0^\pi x\cos x\,dx = -2$: is this reasonable?



neg region gets larger weight

Expect negative answer

Bound

$0 \geq \int_0^\pi x\cos x\,dx \geq -\pi^2$

improve to

$\geq -\frac{1}{2}\pi^2$

$$\binom{5}{3} = \frac{5!}{3!\,2!}$$ means # ways to choose 3 from 5 when order does not matter

$$= \frac{5 \cdot 4 \cdot 3!}{3!\, 2 \cdot 1} = 10$$

$$\binom{3}{5} = \frac{3!}{5!\,(-2)!} = 0 = \frac{6}{120 \cdot (-2)!}$$

Thus $(-2)!$ is infinite!

Gamma Fn: $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$

$\Gamma(n+1) = n!$ if $n \geq 0$ is an integer

Change of variable

$$\int_0^\infty x\, e^{-x^2/2}\, dx = \int_0^\infty e^{-x^2/2}\, \underline{\underline{x\,dx}}$$

$$u = x^2/2 \qquad du = x\,dx$$

$$\int_{u=0}^\infty e^{-u}\, du = -e^{-u}\Big|_0^\infty = e^{-u}\Big|_\infty^0 = 1$$

$$\int_{-\infty}^{\infty} x e^{-4x^2}\,dx = \int_{-\infty}^{\infty} e^{-4x^2}\,x\,dx$$

$u = 4x^2$ $\qquad\qquad$ $du = 8x\,dx$

$$\frac{1}{8}\int_{x=-\infty}^{\infty} e^{-4x^2}\,8x\,dx = \frac{1}{8}\int_{u=-\infty}^{\infty} e^{-u}\,du$$

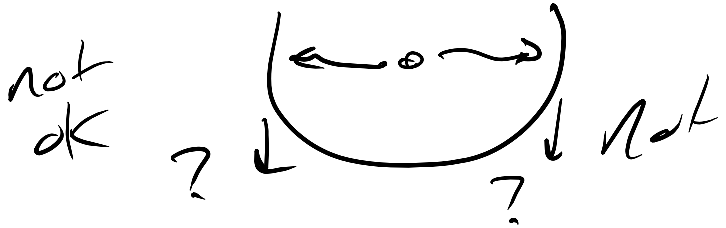
NO!

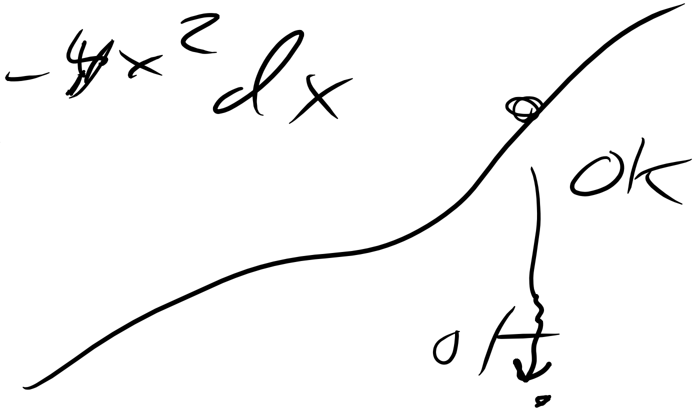
Be explicit: $x: -\infty$ to ∞, $u = 4x^2$

$u: \infty$ to ∞

not 1-1 : need to do $2\int_{0}^{\infty} x e^{-4x^2}\,dx$

onto





$$\frac{16}{64} = \frac{1}{4} \qquad \frac{19}{95} = \frac{1}{5} \qquad \frac{49}{98} = \frac{1}{2}$$

$$\frac{12}{24} = \frac{1}{4}$$

Cases must be exhaustive