Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 04: 9-17-21:
https://youtu.be/h6UrtHrwdbw

Plan for the day: Lecture 4: September 17, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Lecture from 2019: Lecture 04: 9/13/19: Axioms of Probability, Consequences, Sniffing out Formulas: https://youtu.be/qiek1ZM_KLE (2018 version)
- Also bonus lecture: Lecture 04': 9/13/19: Supplemental: Infinities, Generating Functions, Differentiating Identities: https://youtu.be/TQ20q4yZsho

General items.

- Sniffing out equations

Sniffing out Eqs

If could, $(fg)' = f'g'$ But not true!

$$A(x) = f(x)\,g(x)$$

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x+h) + f(x)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)\,g(x) + f(x)\,g'(x) \quad \blacksquare$$

Need fns f, g st know f', g' and A' with $A = fg$

Know $(x^r)' = r x^{r-1}$ if $r \in \mathbb{R}$

$f(x) = x^n$ $\quad g(x) = x^m$ $\quad A(x) = f(x) g(x) = x^{n+m}$

$f'(x) = n x^{n-1}$ $\quad g'(x) = m x^{m-1}$ $\quad A'(x) = (n+m) x^{n+m-1}$

(know need $f'(x)$, $g'(x)$ from taking $g(x) = 1$ and $f(x) = 1$)

(Suggesting $f'(x) g(x) + f(x) g'(x)$ from this analysis)

Check: $f'g + fg'$ is $n x^{n-1} x^m + x^n m x^{m-1} = (n+m) x^{n+m-1}$ ✓

Know f, g and $A = fg$ and know f', g', A'

$f(x) = \sin x$

$f'(x) = \cos x$

$g(x) = \cos x$

$g'(x) = -\sin x$

(minus sign)

$A(x) = \sin x \cos x$

$= \frac{1}{2}\sin(2x)$

$A'(x) = \frac{1}{2}\cos(2x)\cdot 2 = \cos(2x)$

$f'g + fg' = \cos x \cos x + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x$

$A'(x) = \cos 2x$

$= \cos x \cos x - \sin x \sin x$

✓

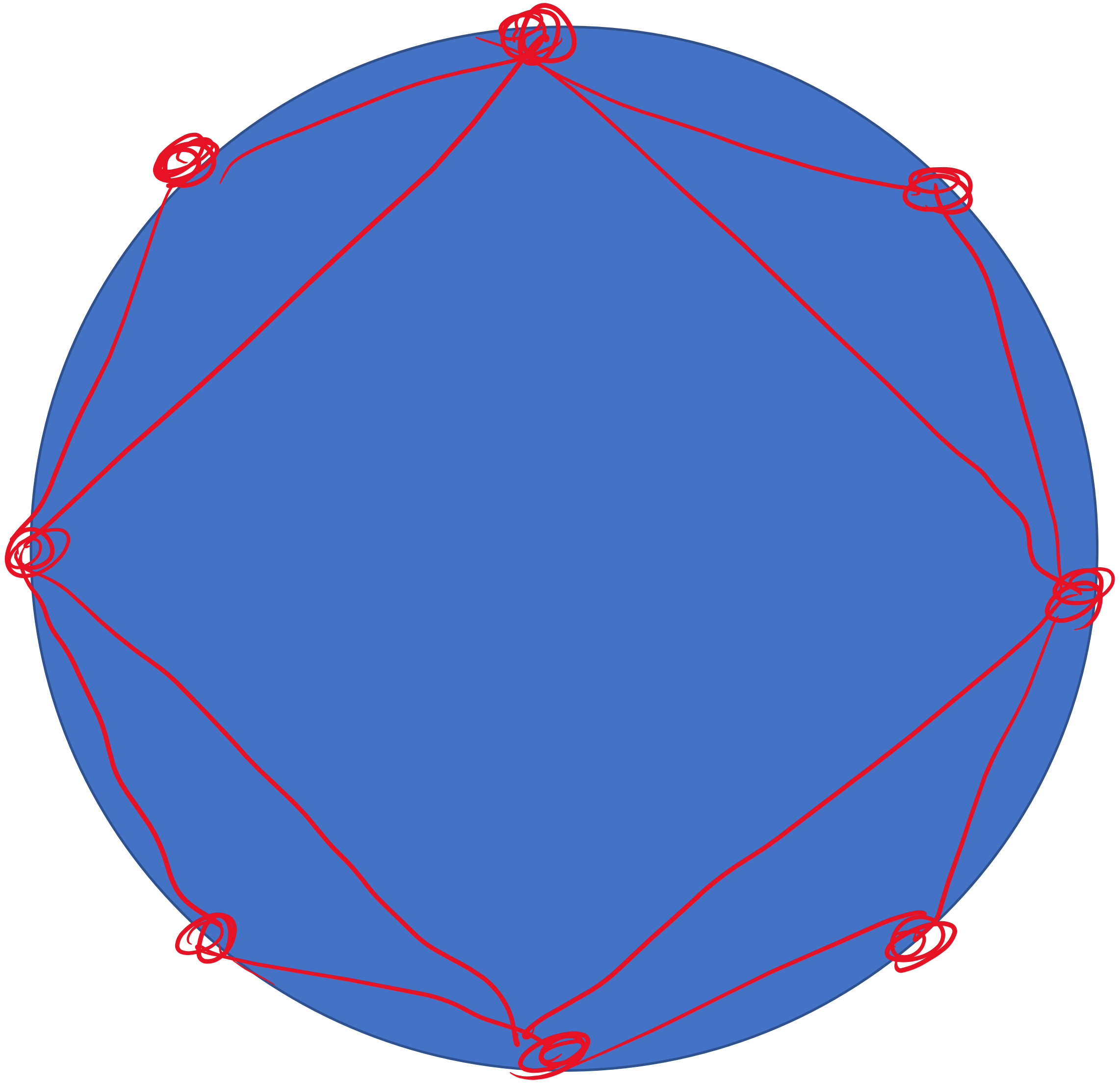
$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

$$\lim_{x \to \infty} \frac{x}{\cos x} = \text{not L'Hopitalable / L'Hopible}$$

$$\lim_{x \to 0} \frac{\sin x}{x} \underset{L'H}{=} \lim_{x \to 0} \frac{\cos x}{1} = 1$$ **NO!**

$$\sin'(0) = \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h}$$

Calculating Limit



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Generating Functions

Fibonacci $F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$

$0, 1, 1, 2, 3, 5, 8, \ldots$

$$g(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} \quad (\text{t Pmt})$$

$$= x \frac{1}{(1-\alpha x)(1-\beta x)} \xrightarrow{\text{Partial Fractions}} = x\left(\frac{A}{1-\alpha x} + \frac{B}{1-\beta x}\right)$$

$$= x \sum_{n=0}^{\infty} A\alpha^n x^n + \sum_{n=0}^{\infty} B\beta^n x^n \quad \text{Geometric series}$$

$$F_n = A\alpha^{n-1} + B\beta^{n-1}$$

Do we have convergence?

$$g(x) = \sum_{n=0}^{\infty} F_n x^n$$

$F_{n+1} = F_n + F_{n-1} \leq 2F_n$ as seq is non-decreasing

so basically $F_n \leq 2^n$

Converges if $|x| < \frac{1}{2}$ by comparison with the geometric series $|2x| < 1$ ratio

Binet's Formula: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

$$g(X) = \sum_{n=0}^{\infty} F_n X^n$$

$$\sum_{n=2}^{\infty} F_n X^n = \sum_{m=0}^{\infty} F_{m+2} X^{m+2} \qquad n = m+2$$

$$= \sum_{m=0}^{\infty} \left(F_{m+1} + F_m\right) X^{m+2}$$

$$= \sum_{m=0}^{\infty} F_{m+1} X^{m+2} + \sum_{m=0}^{\infty} F_m X^{m+2}$$

$$= X \sum_{m=0}^{\infty} F_{m+1} X^{m+1} + X^2 \sum_{m=0}^{\infty} F_m X^m$$

$$= X \sum_{k=1}^{\infty} F_k X^k + X^2 g(X) = X g(X) + X^2 g(X)$$

$$g(X) - X$$

$$g(x) - x = x\,g(x) + x^2 g(x)$$ Bring it over

$$(1 - x - x^2)\,g(x) = x$$

$$g(x) = \frac{x}{1 - x - x^2}$$