Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 05: 9-20-21: https://youtu.be/AV02q6VnSa0

Plan for the day: Lecture 5: September 20, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Lecture 05: 2018: Factorial Function, Binomial Coefficients, Poker Hands, Pascal's Triangle Mod 2: https://youtu.be/pEoW_hX4-oo
- Pascal's triangle modulo 1: Long version: https://www.youtube.com/watch?v=_vkGakVt1RA&t=243s and short version: https://www.youtube.com/watch?v=tt4_4YajqRM&t=94s

General items.

- Efficient coding

Pascal's triangle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$\binom{n}{1} = k \binom{n}{0}$$

Binomial theorem

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (y+x)$$

Proof of the Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Calculus: $f(x) = x^n$, $f(x+h) = x^n + nhx^{n-1} + (h^2$ and higher$)$

Proof: Induction: $(x+y)^{n+1} = (x+y)^n(x+y)$

$$= (x+y)^n x + (x+y)^n y$$

look for coeff of $x^k y^{n+1-k}$: need $x^{k-1}y^{n+1-k}$ x $x^k y^{n-k}$

by induction: $\binom{n}{k-1} x^k y^{n+1-k} + \binom{n}{k} x^k y^{n+1-k}$

Binomial Thm

$$(x+y)^n = (x+y)(x+y) \cdots (x+y)$$

Terms are of form $x^k y^{n-k}$ $0 \le k \le n$ integer

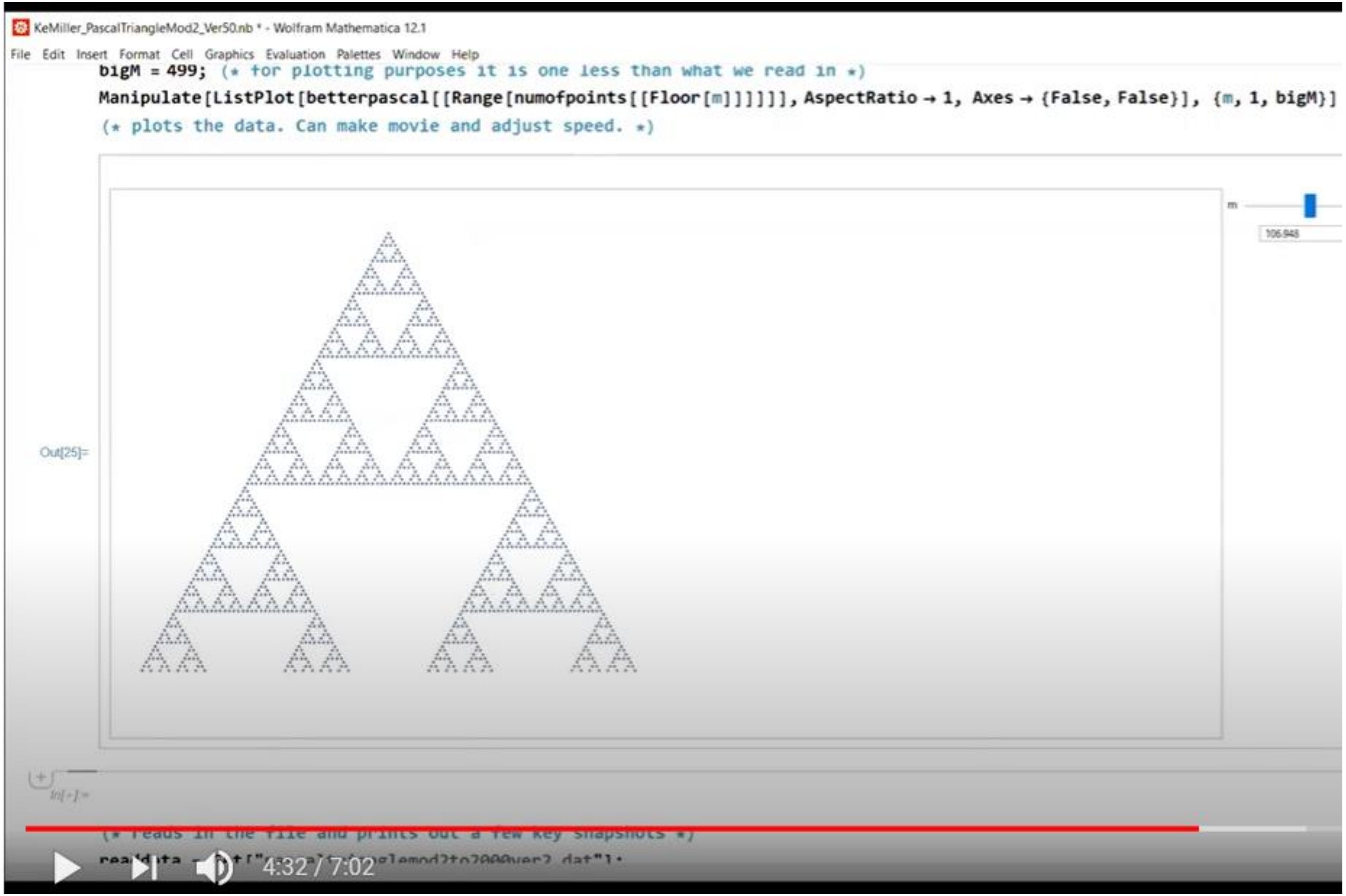
$$\binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$$

Given an x just 1

Multinomials

More useful

Application: $x = p$ and $y = 1-p$, $0 \le p \le 1$

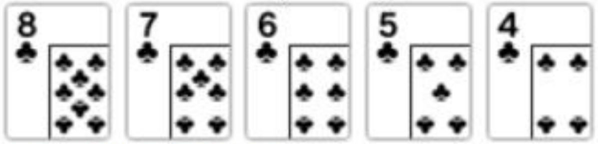


Pascal's triangle modulo 1: Long version: https://www.youtube.com/watch?v=_vkGakVt1RA&t=243s, and short version: https://www.youtube.com/watch?v=tt4_4YajqRM&t=94s

Poker hands from highest to lowest

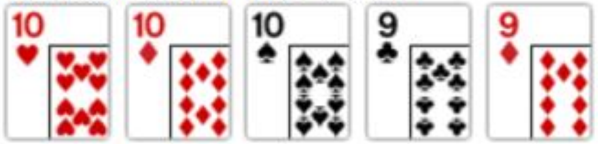
1. Royal flush
A, K, Q, J, 10, all the same suit.

2. Straight flush
Five cards in a sequence, all in the same suit.



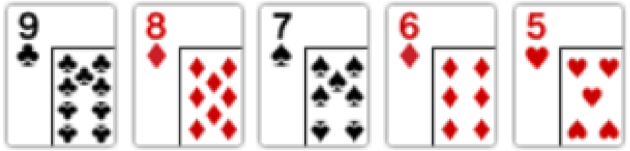
3. Four of a kind
All four cards of the same rank.

4. Full house
Three of a kind with a pair.

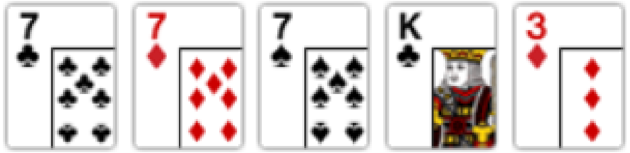


5. Flush
Any five cards of the same suit, but not in a sequenc

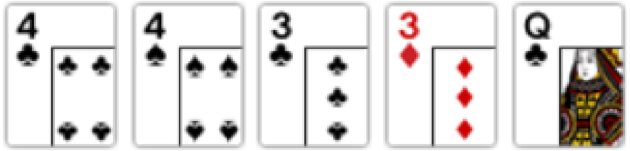
6. Straight
Five cards in a sequence, but not of the same suit.



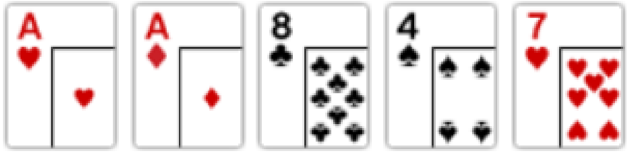
7. Three of a kind
Three cards of the same rank.



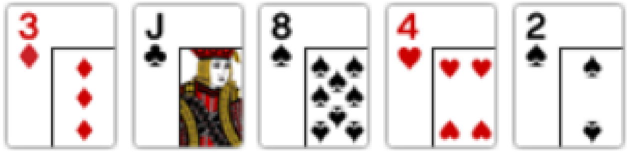
8. Two pair
Two different pairs.



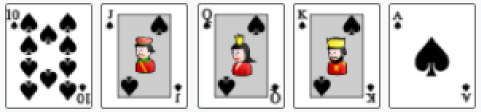
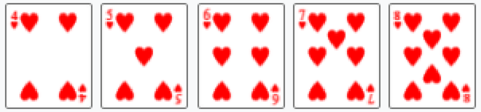
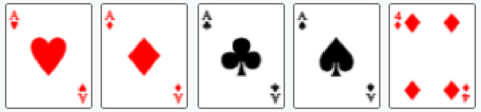
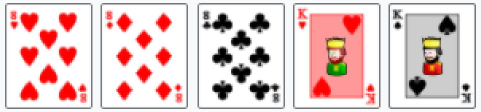
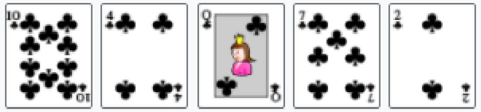
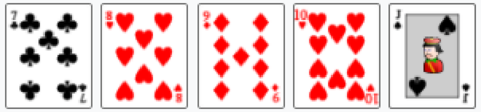
9. Pair
Two cards of the same rank.

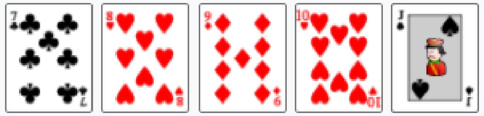
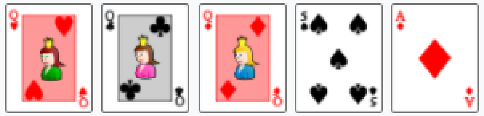
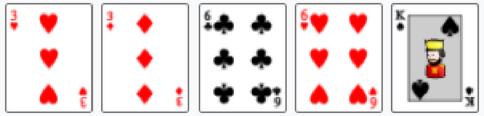
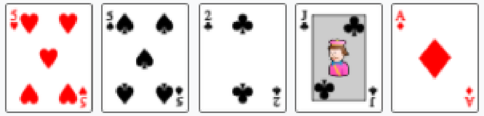
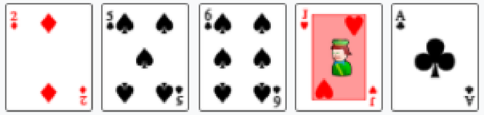


10. High Card
When you haven't made any of the hands above, the highest card plays.
In the example below, the jack plays as the highest card.



https://www.cardplayer.com/rules-of-poker/hand-rankings https://en.wikipedia.org/wiki/Poker_probability

Hand	Distinct hands	Frequency	Probability	Cumulative probability	Odds against	Mathematical expression of absolute frequency
Royal flush 	1	4	0.000154%	0.000154%	649,739 : 1	$\binom{4}{1}$
Straight flush (excluding royal flush) 	9	36	0.00139%	0.0015%	72,192.33 : 1	$\binom{10}{1}\binom{4}{1} - \binom{4}{1}$
Four of a kind 	156	624	0.02401%	0.0256%	4,164 : 1	$\binom{13}{1}\binom{4}{4}\binom{48}{1}$
Full house 	156	3,744	0.1441%	0.17%	693.1667 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$
Flush (excluding royal flush and straight flush) 	1,277	5,108	0.1965%	0.367%	507.8019 : 1	$\binom{13}{5}\binom{4}{1} - \binom{10}{1}\binom{4}{1}$
Straight (excluding royal flush and straight flush) 	10	10,200	0.3925%	0.76%	253.8 : 1	$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}$

Straight (excluding royal flush and straight flush) 	10	10,200	0.3925%	0.76%	253.8 : 1	$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}$
Three of a kind 	858	54,912	2.1128%	2.87%	46.32955 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2$
Two pair 	858	123,552	4.7539%	7.62%	20.03535 : 1	$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$
One pair 	2,860	1,098,240	42.2569%	49.9%	1.366477 : 1	$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$
No pair / High card 	1,277	1,302,540	50.1177%	100%	0.9953015 : 1	$\left[\binom{13}{5} - 10\right]\left[\binom{4}{1}^5 - 4\right]$

5 cards, hand has 2 Queens

Q Q ? ? ?

could have queens

Exactly vs At least

Be explict

Exactly 2 Queens: $\frac{\text{\# hands that work}}{\text{\# hands}} = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$

note: $4+48=52$

$2+3=5$

good check

At least 2 Queens

$$\frac{\binom{4}{2}\binom{50}{3}}{\binom{52}{5}} = \frac{10}{221} \approx 4.5\%$$

```
Sum[Binomial[4, k] Binomial[48, 5 - k], {k, 2, 4}] /
 Binomial[52, 5]
```

$$\frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{2257}{54145} \approx 4.2\%$$

$$1 - \frac{\binom{4}{0}\binom{48}{5} + \binom{4}{1}\binom{48}{4}}{\binom{52}{5}} =$$

same

```
1 - Sum[Binomial[4, k] Binomial[48, 5 - k], {k, 0, 1}] /
  Binomial[52, 5]
```

```
twoqueensinfive[num_, print_] := Module[{},
  (* num is the number of hands we play *)
  (* hands have 5 cards *)
  success = 0; (* counts how often we have 2 Q in 5 cards *)
  deck = {1, 1, 1, 1};
  For[m = 1, m ≤ 48, m++, deck = AppendTo[deck, 0]];
  (* deck of 4 queens and 48 non-queens *)
  For[n = 1, n ≤ num, n++,
   {
    If[print == 1,
     If[Mod[n, num / 10] == 0,
       Print["We have done ", 100.0 n / num, "%."]];
    ]; (* print out where we are if print = 1 *)
    hand = RandomSample[deck, 5];
    (* if hand sum is 2 or more, have at least 2 Q *)
    handsum = Sum[hand[[i]], {i, 1, 5}];
    If[handsum ≥ 2, success = success + 1];
   }]; (* end of n loop *)
  Print["Success percentage is ", 100.0 success / num, "%."];
  Print["Theory is ",
   100.0 Sum[Binomial[4, k] Binomial[48, 5 - k] / Binomial[52, 5],
     {k, 2, 4}], "%."];
 ] (* end of module *)
```

At least 2 Q and at least 2 K

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possibilities: QQKK0

QQQKK

KKKQQ

deck = {10, 10, 10, 10, 1, 1, 1, 1, 0 ⟶ 0}

sum of 22, 23, 32

1, 10, 100, 1000, ...   Sudoku add

111, 111, 111