Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 10: 10-04-21: https://youtu.be/0mFJTq4ASRE

Lecture 10: 10/01/21: Independence, Derangements, Inclusion-Exclusion, Induction:
https://youtu.be/lJKSKEUZ69k

Plan for the day: Lecture 10: October 4, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Independence
- Inclusion / Exclusion: Pentium Bug
- Derangements (permute and nothing returns to where started)

General items.

- Proof techniques: Induction
- Need to be careful: what did we actually prove (derivative of a sum… infinite vs finite)
- Definitions versus theorems: What is $e^x\ e^y$, and why? Generalizations…..
- Study the right object!

A big caveat for independence of three or more events is that any combination of two of those events may be independent of each other, but three or more might be dependent. For example, roll a die twice. Let

- A denote the event that the first time the die shows an even number,
- B the second time the die shows an even number, and
- C the sum of the first two numbers is even.

We can see that

$$\begin{aligned}
\Pr(A \cap B) &= \Pr(A) \cdot \Pr(B) \\
\Pr(A \cap C) &= \Pr(A) \cdot \Pr(C) \\
\Pr(B \cap C) &= \Pr(B) \cdot \Pr(C).
\end{aligned}$$

However, in this case

$$\Pr(A \cap B \cap C) \neq \Pr(A) \cdot \Pr(B) \cdot \Pr(C),$$

as $\Pr(A \cap B \cap C)$ is the probability of getting an even number the first time and an even number again the second time (if the first two rolls are even, then the sum *must* be even – this is what causes the problems). We thus have $\Pr(A \cap B \cap C) = \frac{1}{4}$, but if the three events were independent then, according to the formula,

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

Bayes' Theorem: The General Multiplication Rule implies

$$\Pr(B|A) \cdot \Pr(A) \;=\; \Pr(A|B) \cdot \Pr(B)$$

for events A and B. Therefore, so long as $\Pr(B) \neq 0$, we have

$$\Pr(A|B) \;=\; \Pr(B|A) \cdot \frac{\Pr(A)}{\Pr(B)}.$$

$P(A|B)$ and $P(A \cap B)$: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

OR : $P(A \cap B) = P(A|B)\,P(B) = P(B|A)\,P(A)$

symmetry $= P(B \cap A)$

Example: Nationwide, Tuberculosis (TB) affects about 1 in every 15,000 people. Suppose that there's a TB scare in your town, and for simplicity assume that the rate of incidence of TB in your town is the same as the national average. Just to be safe, you go to the doctor to get tested for the disease. The doctor tells you that the test has a 1% false positive rate – which is to say that for every 100 healthy people, one will test positive. The doctor also reveals that the test has a 0.1% false negative rate – similarly, for every 1000 sick people, only one will test negative. Suppose that you test positive. What's the probability that you have TB?

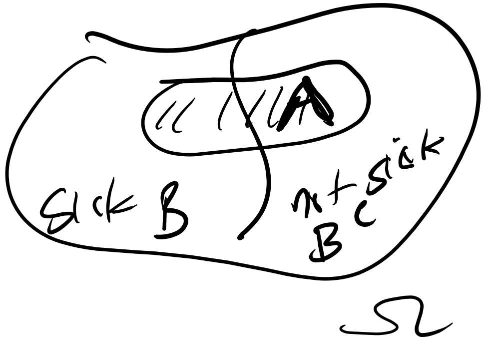
$\frac{1}{15000}$ easy to find

Bayes' Theorem tell us that

$$\Pr(\text{sick}|\text{positive}) = \frac{\Pr(\text{sick})}{\Pr(\text{positive})} \cdot \Pr(\text{positive}|\text{sick}).$$

ok ✓

Bayes' Theorem tell us that



$$\Pr(\text{sick}|\text{positive}) = \frac{\Pr(\text{sick})}{\Pr(\text{positive})} \cdot \Pr(\text{positive}|\text{sick}).$$

We use the partition "sick" and "not sick." So B is the event sick, B^c the event healthy (i.e., "not sick"), and A is the event of testing positive. We find

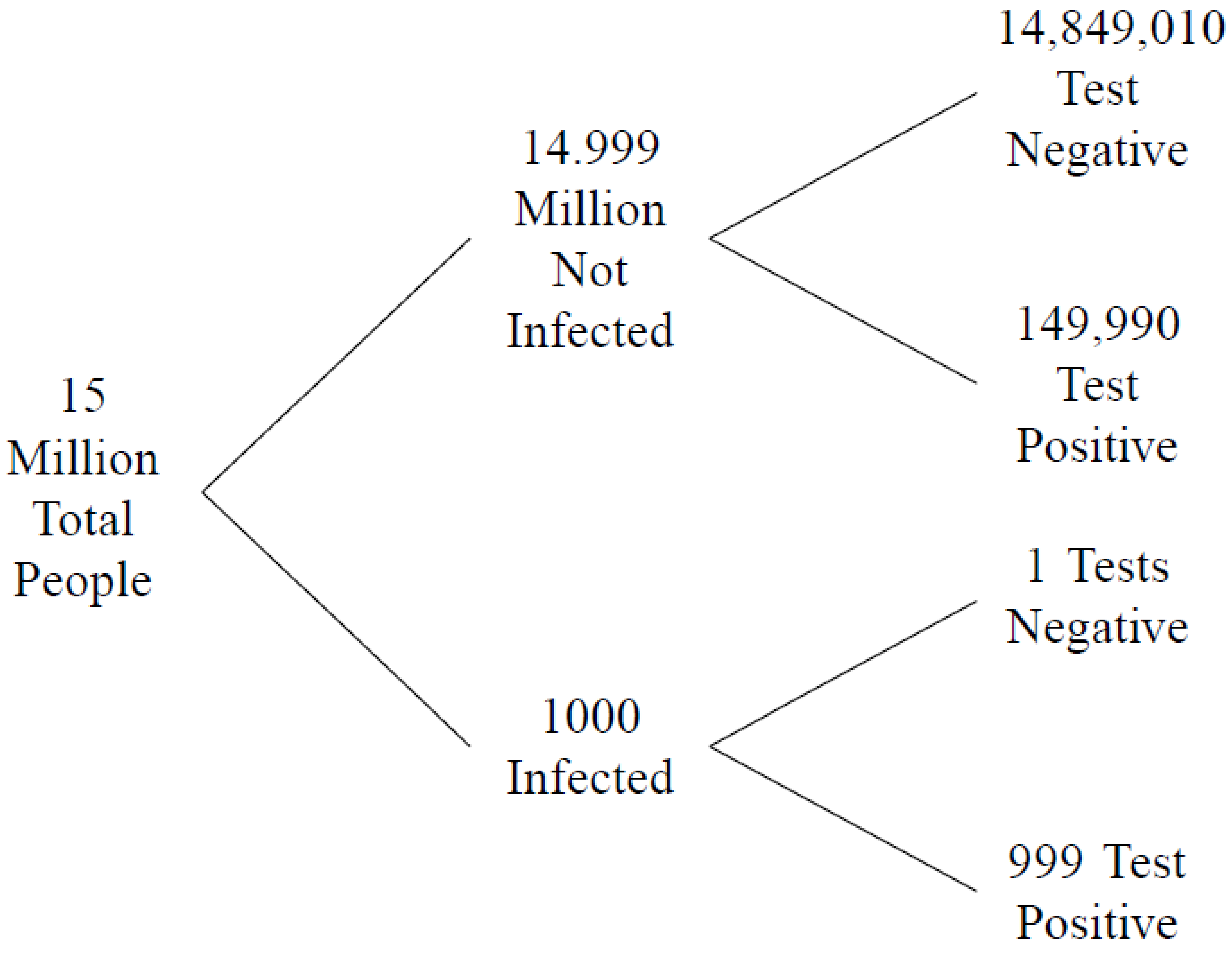
$$\begin{aligned} \Pr(\text{positive}) &= \Pr(\text{positive}|\text{sick})\Pr(\text{sick}) + \Pr(\text{positive}|\text{healthy})\Pr(\text{healthy}) \\ &= 0.999 \cdot \frac{1}{15000} + 0.01 \cdot \frac{14999}{15000} \approx 0.01. \end{aligned}$$

This gives

$$\Pr(\text{sick}|\text{positive}) = \frac{1/15000}{0.01} \cdot 0.999 \approx 0.0066.$$

Were you expecting the probability to be that low?

The expected counts approach can be seen graphically in the tree below.



Law of Total Probability: If $\{B_1, B_2, \ldots\}$ form a partition for the sample space S (into at most countably many pieces), then for any $A \subset S$ we have

$$\Pr(A) = \sum_n \Pr(A|B_n) \cdot \Pr(B_n).$$

We should have $0 < \Pr(B_n) < 1$ for all n as the conditional probabilities aren't defined otherwise (note if a B_n has probability zero then it isn't needed, as that piece is hit by the factor $\Pr(B_n) = 0$, while if it is 1 then all the other factors are unnecessary).

Bayes' Theorem: Let $\{A_i\}_{i=1}^n$ denote a partition of the sample space. Then

$$\Pr(A|B) = \frac{\Pr(B|A) \cdot \Pr(A)}{\sum_{i=1}^n \Pr(B|A_i) \cdot \Pr(A_i)}.$$

Frequently one takes A to be one of the sets A_i.

denom is P(B)

The At Least to Exactly Method: Let $N(k)$ be the number of ways for *at least* k things to happen, and let $E(k)$ be the number of ways for *exactly* k things to happen. Then $E(k) = N(k) - N(k+1)$. Equivalently,

$$\begin{aligned} & \text{Prob(exactly } k \text{ happen)} \\ = \; & \text{Prob(at least } k \text{ happen)} - \text{Prob(at least } k+1 \text{ happen)}. \end{aligned}$$

5.3.1 Counting Derangements

So, how many of the $n!$ orderings have no element returned to where it starts? This means the 1st element cannot be in the first spot, nor the 2nd element in the second spot, and so on. For example, $\{2, 3, 4, 1\}$ is a derangement as each number is moved, while $\{3, 2, 4, 1\}$ is not a derangement as 2 is in the second position.

It turns out to be much easier to look at the related problem, where we count how many ways there are for at least one element to return to its starting point. Why is this easier? Remember the statement of the inclusion-exclusion principle (see §5.2.2). We show how to write an *at least* event in terms of intersections of events, and intersections are often easy to compute. To get the number of derangements, we just subtract the number of non-derangements from $n!$.

Derangements: order n objects, nothing returns to start

1 2 3 4 5
5 3 2 1 4
Derangement

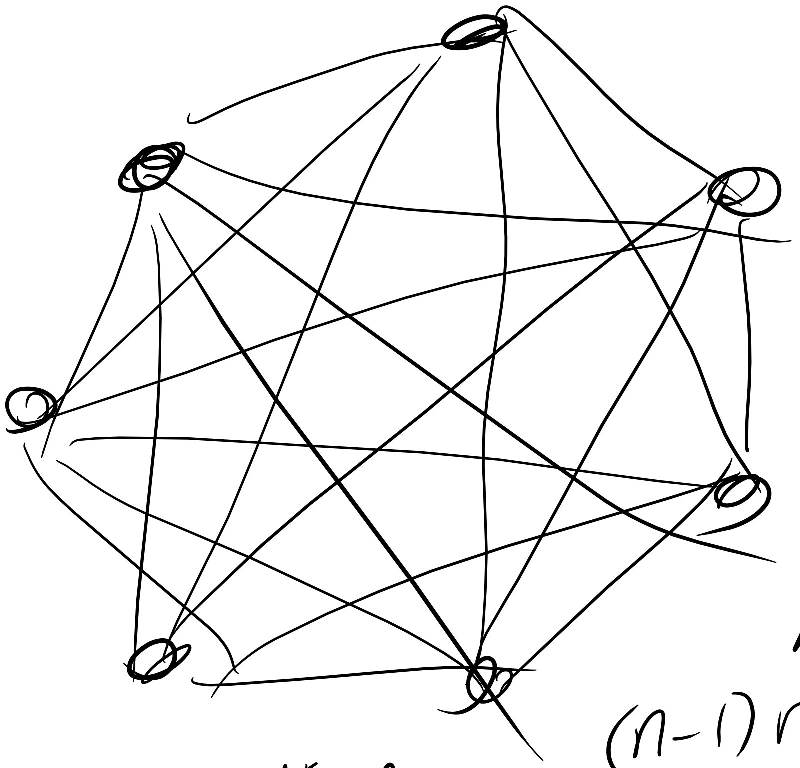
1 2 3 4 5
2 3 1 4 5
not

Network Applications

d-regular graphs

each vertex has exactly d neighbors

#edges $\frac{nd}{2} \sim$ linear in n



Complete Graph

Expensive

n vertices have

#edges $\frac{(n-1)n}{2} = \binom{n}{2} \sim \frac{n^2}{2}$

Random Graphs Work

1	2	3	– – – –	n
4	n−6	n−8	– – – –	7

need to avoid vertex connected to itself

Prob of a derangement with n objects

$= 1 -$ Prob(something connected to itself)

Number	Prob
n	$\frac{(n-1)!}{n!}$
$\binom{n}{2}$	$\frac{(n-2)!}{n!}$
$\binom{n}{3}$	$\frac{(n-3)!}{n!}$

A_i is event i is connected to i

A_{ij} is event $i \to i$ and $j \to j$ and $(i \neq j)$

A_{ijk} is event $i \to i$, $j \to j$, $k \to k$, all distinct

$A_{i_1 \dots i_l}$ have $\binom{n}{l}$ with prob $\frac{(n-l)!}{n!}$, product is $\binom{n}{l}\frac{(n-l)!}{n!}$

which is $\frac{n!}{l!(n-l)!} \frac{(n-l)!}{n!} = 1/l!$

ΣA_{ij}

$$P(A_1 \cup \dots \cup A_n) = \sum_{l=1}^{n} (-1)^{l+1} \frac{1}{l!} = \sum_{i=1}^{n} Pr(A_i) - \sum_{1 \le i < j \le n} Pr(A_i \cap A_j) + \dots$$

something returns

$$\text{Prob}(A_1 \cup \cdots \cup A_n) = \sum_{l=1}^{n} (-1)^{l+1} \frac{1}{l!} = 1 - \text{Prob}(\text{derangement})$$

$$\text{Prob}(\text{derangement}) = 1 - \sum_{l=1}^{n} (-1)^{l+1} \frac{1}{l!}$$

$$= 1 + \sum_{l=1}^{n} (-1)^{l} \frac{1}{l!} \quad \text{but } 1 = (-1)^0 \frac{1}{0!}$$

$$= \sum_{l=0}^{n} \frac{(-1)^l}{l!}$$

converges to $1/e$

$$\text{But } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{so } e^{-1} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}$$

Induction: Statement $P(n)$

Show $P(0)$ true

Show **if** $P(n)$ true **then** $P(n+1)$ true

If do this, $P(n)$ holds for all n

Proof:

$$\frac{P(0) \text{ true} \qquad P(0) \text{ true} \rightarrow P(1) \text{ true}}{P(1) \text{ true}} \quad n=0 \text{ gives}$$

$$\frac{P(1) \text{ true} \rightarrow P(2) \text{ true}}{P(2) \text{ true}} \quad n=1 \text{ gives}$$

Ex: $P(n) \Rightarrow 0+1+2+\cdots+n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: is $0 \stackrel{?}{=} \frac{0\cdot(0+1)}{2}$? Yes

Inductive Step: Assume $P(n)$ true

So $0+\cdots+n = \frac{n(n+1)}{2}$

Must show $0+\cdots+n+(n+1) = \frac{(n+1)(n+1+1)}{2}$

But $\underbrace{[0+1+\cdots+n]}_{\text{inductive assump}} + (n+1)$

$= \frac{n(n+1)}{2} + \frac{(n+1)2}{2} = \frac{(n+1)(n+2)}{2}$ ✓