Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 11: 10-06-21: https://youtu.be/a6bwRbs5sow

Lecture 11: 9/30/19: Basics of pdfs and Random Variables: https://youtu.be/sn-hypa9tRY (coding: https://youtu.be/sSgjBysixdQ; code file here, pdf here)

Plan for the day: Lecture 11: October 06, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Basics of PDFs
- Random Variables: Continuous (FTC) vs Discrete
- Moments and Expected Values

General items.

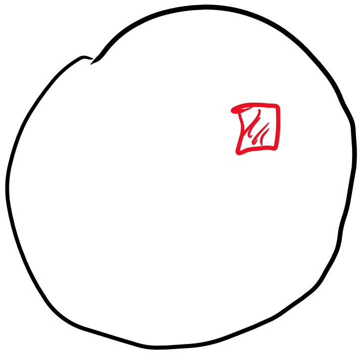
- Rescale
- Taylor Trick (several variables)
- More coding: https://youtu.be/sSgjBysixdQ; code file here, pdf here

Probability Density Functions and Cumulative Distributions (PDF, CDF): Continuous vs Discrete

X random variable, density p or f or p_X or f_X

- $f(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$ or $\sum_{n=0}^{\infty} f(x_n) = 1$

$$\text{Prob}(a \leq X \leq b) = \int_a^b f(x)\,dx \text{ or } \sum_{a \leq x_n \leq b} f(x_n)$$



Uniform distribution

What events get probabilities?

$$\frac{\text{Area of red}}{\text{Area of circle}}$$

Probability density of Y = g(X) in terms of X and its density

Ex: $X \sim \text{Unif}(0,1)$ $f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Try $Y = X^2$: what is $f_Y(y)$?

CDF: cumulative distribution fn: $CDF_X(x) = \text{Prob}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$

↳ prob X is at most x

↳ non-decreasing and $\lim_{x \to \infty} CDF_X(x) = 1$

often write $F_X(x) = CDF_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

Good notation: $F_X'(x) = f_X(x)$

F_X is ~~the~~ an anti-derivative of f_X, but $\lim_{x \to -\infty} F_X(x) = 0$ so the is ok!

Explicitly, if $F' = f_X$ then

$$CDF_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = F(x) - \underbrace{F(-\infty)}_{\text{constant}}$$

So $CDF_X'(x) = F'(x)$

$= f_X(x)$

CDF METHOD

$Y = X^2, \quad X \sim \text{Unif}(0,1)$

$\left(\frac{1}{2}\right)^2 \leq \frac{1}{4}$

but $\frac{1}{2} \not\leq \frac{1}{4}$

$$F_Y(y) = \text{Prob}(Y \leq y) = \text{Prob}(X^2 \leq y)$$

$$= \text{Prob}(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= \text{Prob}(0 \leq X \leq \sqrt{y}) \quad \text{as } X \sim \text{Unif}(0,1)$$

$$= \begin{cases} 1 & \text{if } y \geq 1 \\ \sqrt{y} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \Bigg\} \text{ know } CDF_Y = F_Y$$

$$f_Y(y) = F_Y'(y) = CDF_Y'(y) = \begin{cases} \frac{1}{2} y^{-1/2} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Check: pdf is non-neg, $\int$ to 1: both true

In the CDF method do not need F_X as only its derivative emerges....

Ex: $\text{Prob}(0 \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(0) = F_Y(y)$

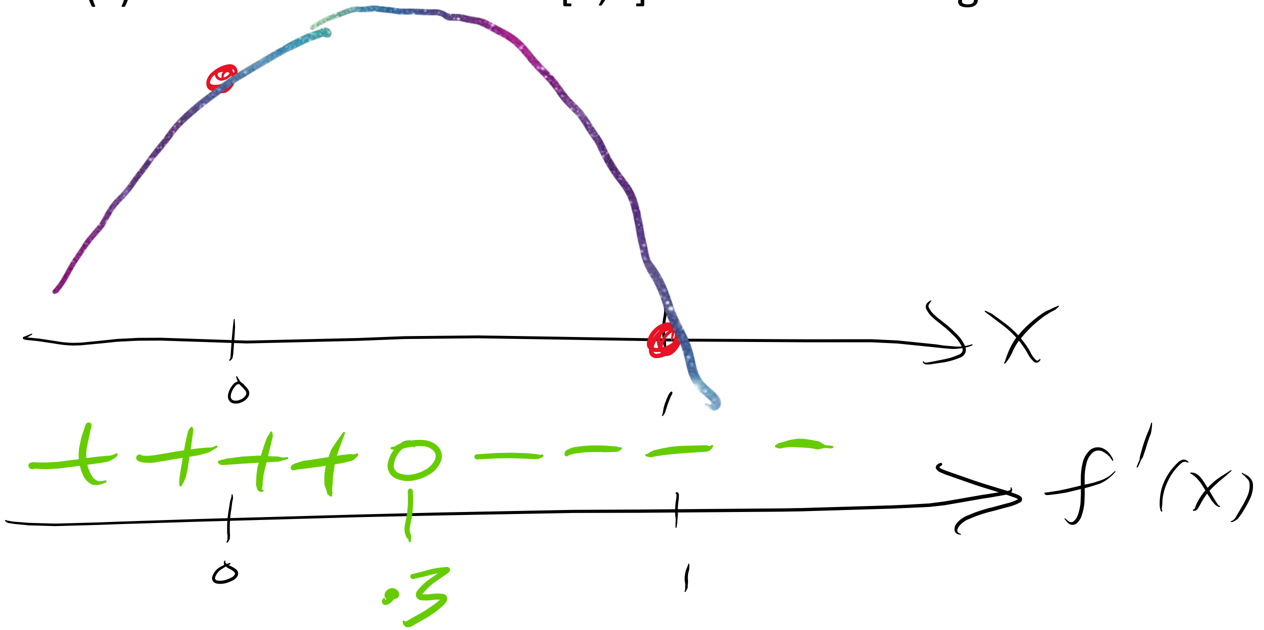
$$F_Y'(y) = f_Y(y) = F_X'(\sqrt{y}) * (\sqrt{y})'$$

$$= f_X(\sqrt{y}) * \frac{1}{2} y^{-1/2}$$

do not need F_X explicitly

↳ Good as $\int$ is hard!

Example: $f(x) = 2 + 3x - 5x^2$ for x in [0,1]: Curve sketching



$$f(x) = 2 + 3x - 5x^2$$

$$f'(x) = 3 - 10x$$

$$f(0) = 2 \qquad f(1) = 0$$

Critical point: $f'(x) = 0$

so $3 - 10x = 0 \to x = \frac{3}{10}$

$$\int_0^1 (2 + 3x - 5x^2)\,dx = \left[2x + \frac{3x^2}{2} - \frac{5x^3}{3}\right]_0^1$$

$$= 2 + \frac{3}{2} - \frac{5}{3} = \frac{12 + 9 - 10}{6} = \frac{11}{6}$$

Not a density but... $\frac{6}{11} f(x)$ is a density

If $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx$ is finite

Then $g(x) = \dfrac{f(x)}{\int_{-\infty}^{\infty} f(t)\,dt}$

is a probability density

→ Renormalizing

Sum of two independent, fair die…. Uniform + Uniform = Triangle

sum	
2	(1,1)
3	(1,2), (2,1)
4	(1,3), (2,2), (3,1)
5	
6	
7	
8	
9	
10	(4,6), (5,5), (6,4)
11	(5,6), (6,5)
12	(6,6)

Ex: write down a formula for sum of two indep fair die

$f_X(7) = 1/6$

$f_X(2) = f_X(12) = \frac{1}{36}$

Moments and expected value of functions of random variables

X, density f_X, $\mu_k = \int_{-\infty}^{\infty} x^k f_X(x)\,dx$

Hope: know the moments, know f_X

$$g(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

↳ ∞ differentiable

use L'Hopital (extra credit) $g^{(n)}(0) = 0$

$$g(x) \quad \text{vs} \quad g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \dots$$

$E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$, moments are $h(X) = X^n$

Fun facts: Taylor series of several variables trick, area-volume and perimeter-area connections

$f(x,y) = \cos(x^2 + xy^2)$

$\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots$

$\cos(x^2 + xy^2) = 1 - \frac{(x^2+xy^2)^2}{2!} + \frac{(x^2+xy^2)^4}{4!} - \cdots$

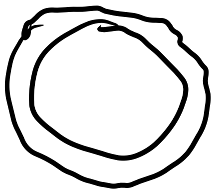
Vol Sphere of radius R: $\frac{4}{3}\pi R^3$

Surface Area ______ : $4\pi R^2$

Area circle radius R: πR^2

Perimeter ______ $2\pi R$

Note derivatives



I love now Steven can turn a game observation into a research question! No wonder he's so productive! Steven, let's put that on our list.

Steve: Tampa has lined up their rotation, but is it better to have a weaker person go against their #1 and maybe have the edge in the other match-ups?

Golf/Tennis?

1 v 1
2 v 2
3 v 3
⋮
6 v 6

1 v 5
2 v 6
3 v 1
4 v 2
5 v 3
6 v 4

https://www.latimes.com/archives/la-xpm-2003-nov-04-sp-briefing4-story.html

Bill James log-5

Drysdale gave up seven runs in three innings in an 8-2 loss.
Jane Leavy, author of “Sandy Koufax: A Lefty’s Legacy,” who spoke at the Pasadena Jewish Temple and Center on Sunday night, said that after the game Drysdale told Manager Walter Alston: “Hey, skip, bet you wish I was Jewish today too.”