Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 20: 11-1-21: https://youtu.be/qW-3bHAwdPU

Lecture 21: 10/28/19: Sums of Uniform Random Variables, Sums of Gaussian Random Variables, Cauchy Distribution, Gregory-Leibnitz Formula: https://youtu.be/N0ye_W-2MD4

Plan for the day: Lecture 20: November 1, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Sums of Uniform Random Variables
- Sums of Gaussian Random Variables
- Cauchy Distribution
- Gregory-Leibnitz Formula:

General items.

- Integrating without integrating! Find the functional form….

Sums of unif random variables (indep)

$X_1, X_2 \sim \text{Unif}(0,1)$

densities $p(x) = 1$ if $0 \le x \le 1$ and 0 otherwise

$X = X_1 + X_2$ convolution!

$$p_X(x) = (p_{X_1} * p_{X_2})(x) = (p * p)(x)$$

$$= \int_{-\infty}^{\infty} \underbrace{p(t)}_{\text{forces } 0 \le t \le 1} \, p(x-t)\, dt$$

$$= \int_0^1 1 \, \underbrace{p(x-t)}_{\substack{0 \le x-t \le 1 \\ t \le x \le 1+t}} \, dt$$

Symmetry: Claim $X = X_1 + X_2$ is symm about 1

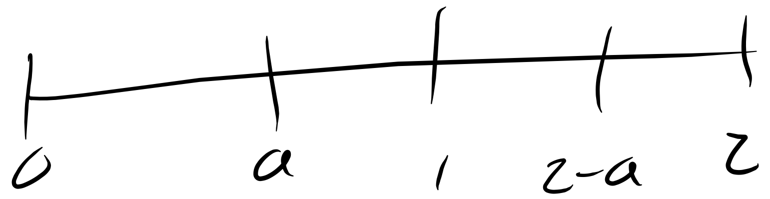
Proof: $\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 1$

each is Unif(0,1) mean 1/2

Let $Y_i = 1 - X_i \sim \text{Unif}(0,1)$

Study $Y_1 + Y_2$ same distr as $X_1 + X_2$

$\text{Prob}(X_1 + X_2 = a < 1) = \text{Prob}(X_1 + X_2 = 2 - a)$



$Y_1 + Y_2 = 2 - (X_1 + X_2)$

$$p_Y(x) = \int_0^1 1 \; \underline{p(x-t)} \, dt$$

wlog, $0 \le x \le 1$

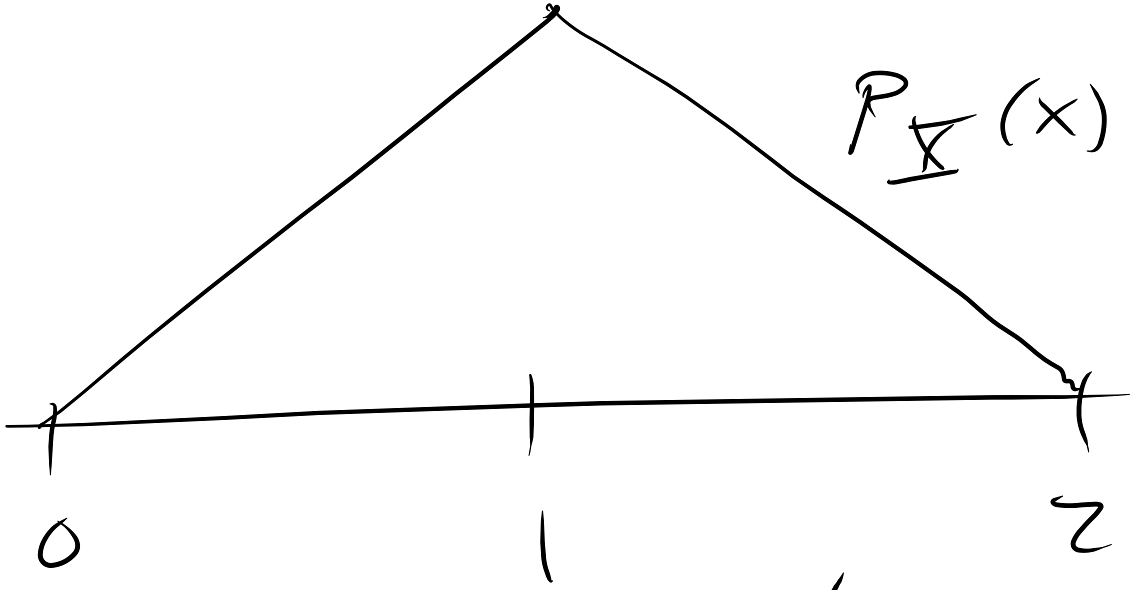
$0 \le x - t \le 1$

$t \le x \le 1 + t$

Get trivially $x \le 1+t$

integral restricted to $t \le x$

$$= \int_{t=0}^{x} 1 \cdot 1 \, dt = x$$



Check

(1) non-neg ✓

(2) area is 1 ✓

Caveat: region of integration!

<u>Gaussian</u>: wlog $\mu=0, \sigma=1$

X_1, X_2 indep $N(0,1)$

$X = X_1 + X_2$? $E[X] = 0 = E[X_1] + E[X_2]$

$Var(X) = Var(X_1) + Var(X_2) = 1+1 = 2$

so $StDev(X) = \sqrt{2}$

Stable distribution: same shape when sum indep elements

Generalize

$X_1 \sim N(0, \sigma_1^2) \qquad X_2 \sim N(0, \sigma_2^2)$

$$\underbrace{\underbrace{(Y_1 + Y_2 + Y_3 + \dots + Y_{n-1})}_{\text{Gaussian}} + \underbrace{Y_n}_{\text{Gaussian}}}_{\text{Gaussian}}$$

$$(((Y_1 + Y_2) + Y_3) + Y_4) + Y_5 \qquad \text{Grouping } (\ )$$

$$X_i \sim N(0, \sigma_i^2) \qquad p_i(x) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-x^2/2\sigma_i^2}$$

$$X = X_1 + X_2$$

$$p_X(x) = (p_1 * p_2)(x) = \int_{-\infty}^{\infty} p_1(t)\, p_2(x-t)\, dt$$

$$= \int_{t=-\infty}^{\infty} \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} e^{-t^2/2\sigma_1^2} e^{-(x-t)^2/2\sigma_2^2} dt$$

hope is $C(\sigma_1, \sigma_2)\, e^{-x^2/g(\sigma_1, \sigma_2)}$

$$\boxed{a_i = \frac{1}{2\sigma_i^2}}$$

$$p_X(x) = C_{12} \int_{t=-\infty}^{\infty} e^{-a_1 t^2} e^{-a_2 (x-t)^2} dt$$

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$$= C_{12} \int_{t=-\infty}^{\infty} e^{-a_1 t^2} e^{-a_2 (x^2 - 2xt + t^2)} dt$$

$$= C_{12} e^{-a_2 x^2} \int_{t=-\infty}^{\infty} e^{-a_1 t^2 - a_2 t^2 + 2 a_2 x t} dt$$

$$= C_{12} e^{-a_2 x^2} \int_{t=-\infty}^{\infty} e^{-(a_1 + a_2) \underbrace{\left[t^2 - \frac{2 a_2}{a_1 + a_2} x t \right]}_{\text{looking like } t^2 - 2\mu t + \mu^2 - \mu^2}} dt$$

$$(t - \mu)^2 - \mu^2$$

$$\mu = \mu(a_1, a_2, x)$$

$$= C_{12} e^{-a_2 x^2} \int_{t=-\infty}^{\infty} e^{-(a_1 + a_2)\left[t^2 - \frac{2 a_2 x}{a_1 + a_2} t + \left(\frac{a_2 x}{a_1 + a_2}\right)^2 - \left(\frac{a_2 x}{a_1 + a_2}\right)^2 \right]} dt$$

$$e^{-(a_1 + a_2)(t - \mu(a_1, a_2, x))^2} e^{+\left(\frac{a_2 x}{a_1 + a_2}\right)^2}$$

$\mu = \mu(a_1, a_2, x)$

$$= C_{12}\, e^{-a_2 x^2} \int_{t=-\infty}^{\infty} e^{-(a_1+a_2)\left[t^2 - \frac{2a_2 x}{a_1+a_2} t + \left(\frac{a_2 x}{a_1+a_2}\right)^2 - \left(\frac{a_2 x}{a_1+a_2}\right)^2\right]} dt$$

$$e^{-(a_1+a_2)(t-\mu(a_1,a_2,x))^2} \quad e^{+\left(\frac{a_2 x}{a_1+a_2}\right)^2}$$

$$= C_{12}\, e^{-a_2 x^2}\, e^{+\frac{a_2^2 x^2}{(a_1+a_2)^2}} \underbrace{\int_{t=-\infty}^{\infty} e^{-(a_1+a_2)(t-\mu(a_1,a_2,x))^2} dt}$$

Change var: $u = t - \mu(a_1, a_2, x)$

$$\int_{u=-\infty}^{\infty} e^{-(a_1+a_2)u^2} du = A_{12}$$

$$P_X(x) = C_{12} A_{12}\, e^{-x^2\left(a_2 - \frac{a_2^2}{(a_1+a_2)^2}\right)}$$ Gaussian!

$$P_{\underline{X}}(x) = C_{12} A_{12}\, e^{-x^2\left(a_2 - \frac{a_2^2}{(a_1+a_2)^2}\right)}$$ Gaussian!

mean = 0 StDev is a fn of a_1, a_2
which are $1/2\sigma_1^2$ and $1/2\sigma_2^2$

$\underline{X} = \underline{X}_1 + \underline{X}_2$, indep, $Var(\underline{X}) = Var(\underline{X}_1) + Var(\underline{X}_2)$

$= \sigma_1^2 + \sigma_2^2$

Show $a_2 - \frac{a_2^2}{(a_1+a_2)^2} = \frac{1}{2(\sigma_1^2+\sigma_2^2)}$

$$C_{12} A_{12} = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}$$

Gaussians are Stable!

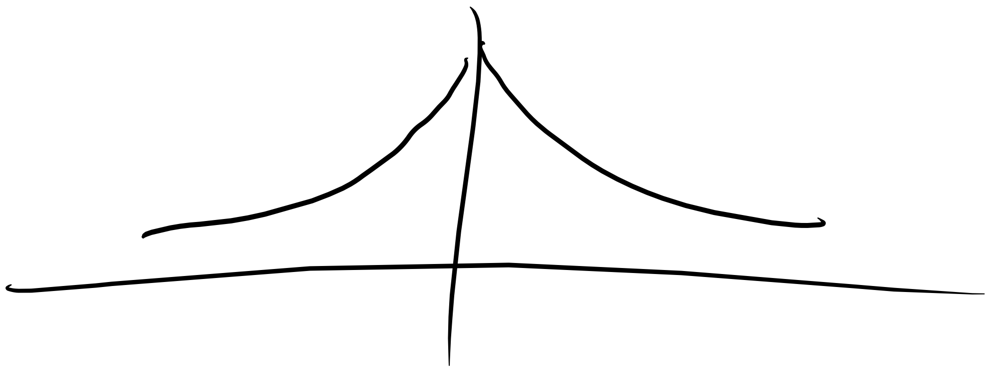
Gaussian + Gaussian = Gaussian

Uniform is not stable!

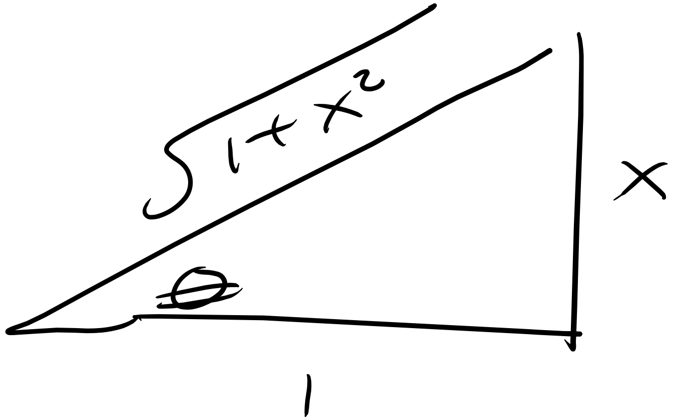
What else is stable?

Cauchy Distribution

$$p(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$



non-neg ✓ $\quad \int$ to 1 ✓



$$\tan(\arctan x) = x$$

$$\tan'(\arctan x) \cdot \arctan'(x) = 1$$

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)}$$

$$= \cos^2(\arctan x) = \frac{1}{1+x^2}$$

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+x^2}\,dx = \frac{1}{\pi} \arctan(x) \Big|_{-\infty}^{\infty}$$

$$= \frac{2}{\pi} \arctan(x) \Big|_{0}^{\infty}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - 0\right] = 1 \checkmark$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$\tan\theta = 0 \Rightarrow \theta = 0$

$\tan\theta = \infty \Rightarrow \theta = \pi/2$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$ converges: never ∞, decays like $\frac{1}{x^2}$

$X \sim$ Cauchy

$$E[X] = \int_{-\infty}^{\infty} \underbrace{x \frac{1}{\pi} \frac{1}{1+x^2}}_{\text{odd function}} dx \text{ “}=\text{” } 0$$

Not quite true: $x/_{1+x^2}$ is not integrable

need $\lim\limits_{A,B\to\infty} \int_{-A}^{B} f(x)\,dx$ = one answer

$B = 2A$ (A BIG)

$B = A$

$$\int_{-A}^{A} x \frac{1}{\pi} \frac{1}{1+x^2} dx = 0$$

$$\int_{-A}^{2A} x \frac{1}{\pi} \frac{1}{1+x^2} dx \simeq \int_{A}^{2A} \frac{1}{\pi} \frac{1}{x} dx$$

$$= \log x \Big|_{A}^{2A} = \log 2$$

$X \sim$ Cauchy

$$\cancel{Var(X)} = \int_{-\infty}^{\infty} x^2 \frac{1}{\pi} \frac{1}{1+x^2}\,dx = \infty$$

$E[X^2]$

no mean

Infinite variance

(median = 0)

Cauchy + Cauchy = Cauchy (Stable)

(add a parameter)

Gregory–Leibniz

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}$$

$$\frac{1}{1+x^2} = \int_0^1 \frac{1}{1-(-x^2)} = \int_0^1 1 + (-x^2) + (-x^2)^2 + \cdots$$

$$= 1 - \frac{x^3}{3}\Big|_0^1 + \frac{x^5}{5}\Big|_0^1 + \cdots$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \pi/4$$

$$\int_0^{1-\varepsilon}$$

take limit as $\varepsilon \to 0$