Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 22: 11-08-21: https://youtu.be/eBcKGUSB_vI (slides)

Lecture 23: 11/01/19: Markov and Chebyshev's inequalities, Divide and Conquer vs Newton's Method: https://youtu.be/vuKCrS2on9Q

Plan for the day: Lecture 22: November 8, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Markov’s Inequality
- Chebyshev’s Inequality
- Divide and Conquer
- Newton’s Method

General items.

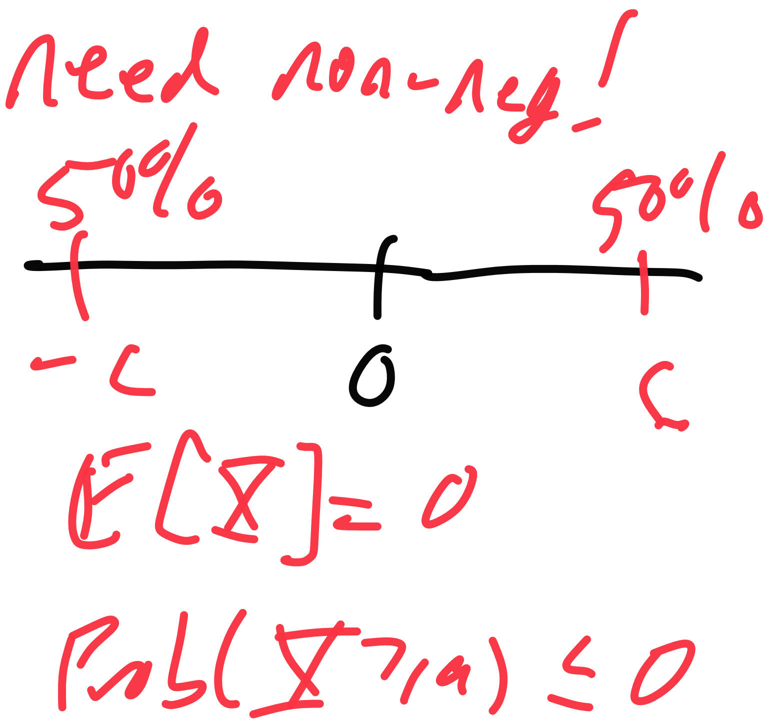
- The more you assume, the more you can deduce…

Markov's inequality. Let X be a non-negative random variable with finite mean $\mathbb{E}[X]$ (this means $\mathrm{Prob}(X < 0) = 0$). Then for any positive a we have

$$\mathrm{Prob}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

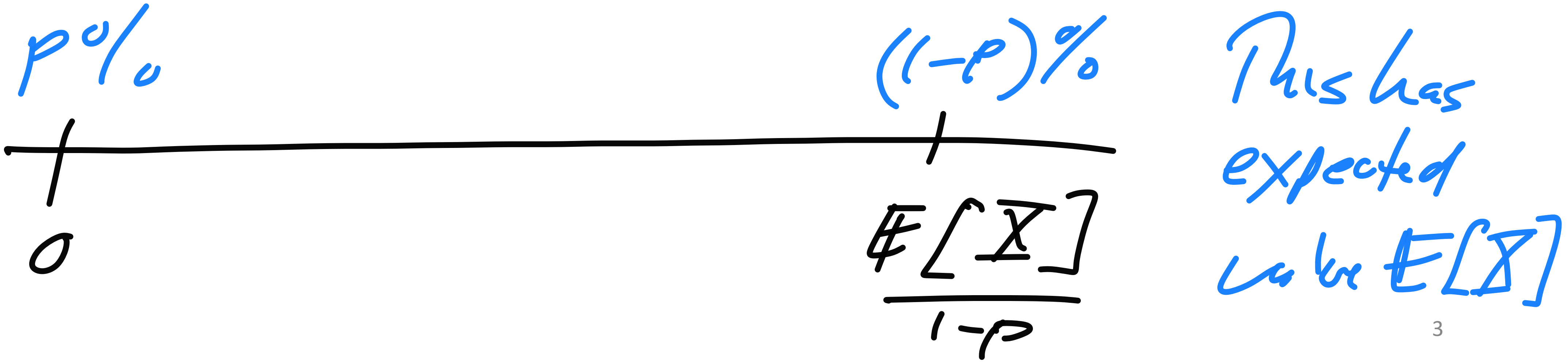
Some authors write μ_X for $\mathbb{E}[X]$. An alternative formulation is

$$\mathrm{Prob}(X < a) \geq 1 - \frac{\mathbb{E}[X]}{a}.$$



Markov's inequality: Sanity Checks:

- Units
- Choices of a
- Special cases / Extreme cases



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If $X \geq a$ then $X/a \geq 1$

use non-neg when throw away integral over some region

Proof: $\text{Prob}(X \geq a) = \int_{x=a}^{\infty} f_X(x)\,dx$

$\mathbb{E}[X] = \int_0^{\infty} x\, f_X(x)\,dx$ as non-neg

$\geq \int_{x=a}^{\infty} x\, f_X(x)\,dx = \int_{x=a}^{\infty} a \cdot \frac{x}{a} f_X(x)\,dx$

$\geq a \int_{x=a}^{\infty} f_X(x)\,dx = a \cdot \text{Prob}(X \geq a) \Rightarrow \text{Prob}(X \geq a) \leq \mathbb{E}[X]/a$

Now that we've seen a proof, let's do an example. *Imagine the mean US income is $60,000. What's the probability a household chosen at random has an income of at least $120,000? Of at least $1,000,000?*

As stated, we don't have enough information to solve this problem. Maybe there's a few very rich people and everyone else earns essentially nothing. Or, the opposite extreme, maybe everyone makes close to the average. Without knowing more about how incomes are distributed, we can't get an exact answer. We can, however, get some bounds on the answer by using Markov's inequality. To use this, we need a non-negative random variable with finite mean. If we assume that no household has a negative income then we're fine, as the other condition is met (the mean is $60,000, which is finite).

Thus the probability of an income of at least $120,000 is at most $60000/120000 = 1/2$; or, at most half the population makes twice the mean. What about the millionaire's club? The probability of being a millionaire is at most $60000/1000000 = .06$, or at most 6% of the households.

Let X be a non-negative random variable with finite mean $\mathbb{E}[X]$. Then the probability of being at least ℓ times the mean is at most $1/\ell$:

$$\text{Prob}(X \geq \ell\mathbb{E}[X]) \ \leq \ \frac{1}{\ell}.$$

Unfortunately this is the best we can do with our limited information. So long as our random variable has finite mean and is non-negative, the probability of being 100 or more times the mean is at most $1/100$ or 1%. Of course, in many problems the true probability is *magnitudes* less than this. This is an excessively high over-estimate at times. This suggests, of course, the next step: incorporate more information and get a better bound! We do this in the next section.

Theorem 17.3.1 (Chebyshev's Inequality) *Let X be a random variable with finite mean μ_X and finite variance σ_X^2. Then for any $k > 0$ we have*

$$\text{Prob}(|X - \mu_X| \geq k\sigma_X) \;\leq\; \frac{1}{k^2}.$$

Some authors write $\mathbb{E}[X]$ for μ_X. This means that the probability of obtaining a value at least k standard deviations from the mean is at most $1/k^2$. A useful, alternative formulation is

$$\text{Prob}(|X - \mu_X| < k\sigma_X) \;>\; 1 - \frac{1}{k^2}.$$

Chebyshev's inequality: Sanity Checks:

- Units
- Choices of k
- Special cases / Extreme cases
- Better than Markov for large deviations (reciprocal of quadratic vs linear)

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$$\text{Prob}(|X - \mu_X| < k\sigma_X) \geq 1 - \frac{1}{k^2}.$$

$Y = |X - E[X]|$

non-neg

$Var(X) = E[X^2] - E[X]^2$

$= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\,dx$

non-neg

Chebyshev's Inequality: Proof from Markov:

$W = (X - E[X])^2 \geq 0$

$E[W] = Var(X) = \sigma_X^2$

Markov: $\text{Prob}(W \geq a) \leq \frac{E[W]}{a}$

$\text{Prob}(|X - \mu_X| \geq a^{\frac{1}{2}}) \leq \frac{\sigma_X^2}{a}$

take square-root

want $a^{1/2} = k\sigma_X$

so $\sigma_X^2 / a = 1/k^2$

substitute!

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$$\text{Prob}(|X - \mu_X| \geq k\sigma_X) \ \leq \ \frac{1}{k^2}.$$

Direct proof of Chebyshev's inequality. Let f_X be the probability density function of X. We assume X is a continuous random variable, though a similar proof holds in the discrete case. We have

$$\begin{aligned}
\text{Prob}(|X - \mu_X| \geq k\sigma_X) &= \int_{x:|x-\mu_X|\geq k\sigma_X} 1 \cdot f_X(x)dx \\
&\leq \int_{x:|x-\mu_X|\geq k\sigma_X} \left(\frac{x-\mu_X}{k\sigma_X}\right)^2 \cdot f_X(x)dx \\
&= \frac{1}{k^2\sigma_X^2} \int_{x:|x-\mu_X|\geq k\sigma_X} (x-\mu_X)^2 f_X(x)dx \\
&\leq \frac{1}{k^2\sigma_X^2} \int_{x=-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx \\
&= \frac{1}{k^2\sigma_X^2} \cdot \sigma_X^2 \ = \ \frac{1}{k^2},
\end{aligned}$$

completing the proof. □

From ℂ to Shining Sea: ℂomplex Dynamics from ℂombinatorics to ℂoastlines

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http://web.williams.edu/Mathematics/sjmiller/public_html/

Michigan Math ℂlub, April 30, 2020

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/CToShiningSeaMichigan2020.pdf
https://youtu.be/TMILk79N_Bs

Finding roots

Much of math is about solving equations.

Example: polynomials:

- $ax + b = 0$, root $x = -b/a$.
- $ax^2 + bx + c = 0$, roots $(-b \pm \sqrt{b^2 - 4ac})/2a$.
- Cubic, quartic: formulas exist in terms of coefficients; not for quintic and higher.

In general cannot find exact solution, how to estimate?

Cubic: For fun, here's the solution to $ax^3 + bx^2 + cx + d = 0$

```
Solve[a x^3 + b x^2 + c x + d == 0, x]
```

$$\left\{\left\{x \to -\frac{b}{3a} - \frac{2^{1/3}\left(-b^2+3ac\right)}{3a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} + \frac{\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{3\times 2^{1/3}a}\right\},\right.$$

$$\left\{x \to -\frac{b}{3a} + \frac{\left(1+i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1-i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\},$$

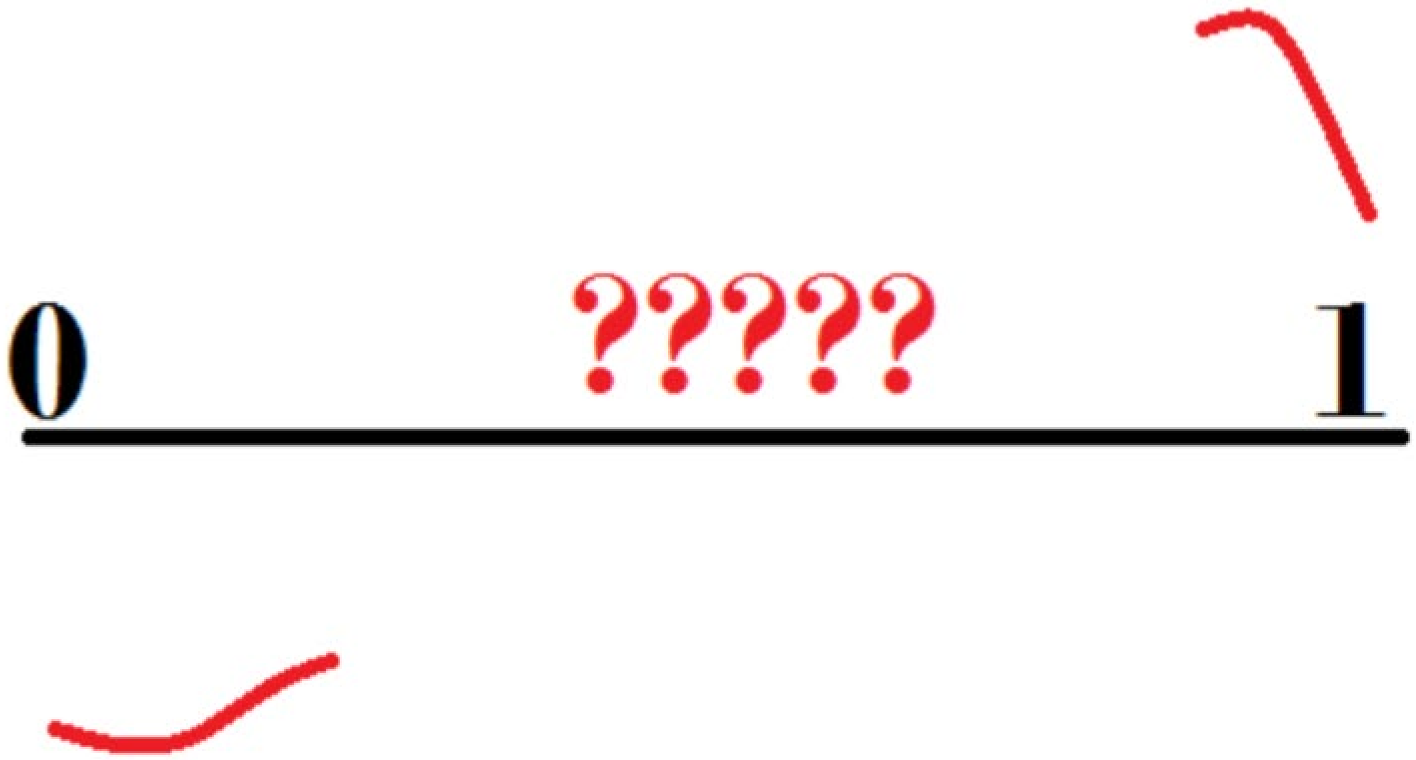
$$\left.\left\{x \to -\frac{b}{3a} + \frac{\left(1-i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1+i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\}\right\}$$

One of four solutions to quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

```
Solve[a x^4 + b x^3 + c x^2 + d x + e == 0, x]
```

$$
\begin{aligned}
\Bigg\{\Bigg\{x \to & -\frac{b}{4a} - \frac{1}{2}\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \right.} \\
& \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right) \Big/ \left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) + \\
& \frac{1}{3\times 2^{1/3}a}\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\Bigg) - \frac{1}{2}\sqrt{\left(\frac{b^2}{2a^2} - \frac{4c}{3a} - \right.} \\
& \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right) \Big/ \left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) - \\
& \frac{1}{3\times 2^{1/3}a}\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3} - \\
& \left(-\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}\right) \Big/ \left(4\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right) \Big/ \right.}\right. \\
& \quad \left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}a} \\
& \quad \left.\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\Bigg)\Bigg)\Bigg)\Bigg\},
\end{aligned}
$$

Divide and Conquer



Divide and Conquer

Divide and Conquer

Assume f is continuous, $f(a) < 0 < f(b)$. Then f has a root between a and b. To find, look at the sign of f at the midpoint $f\left(\frac{a+b}{2}\right)$; if sign positive look in $[a, \frac{a+b}{2}]$ and if negative look in $[\frac{a+b}{2}, b]$. Lather, rinse, repeat.

Example:

- $f(0) = -1, f(1) = 3$, look at $f(.5)$.
- $f(.5) = 2$, so look at $f(.25)$.
- $f(.25) = -.4$, so look at $f(.375)$.

Divide and Conquer (continued)

How fast? Every 10 iterations uncertainty decreases by a factor of $2^{10} = 1024 \approx 1000$.

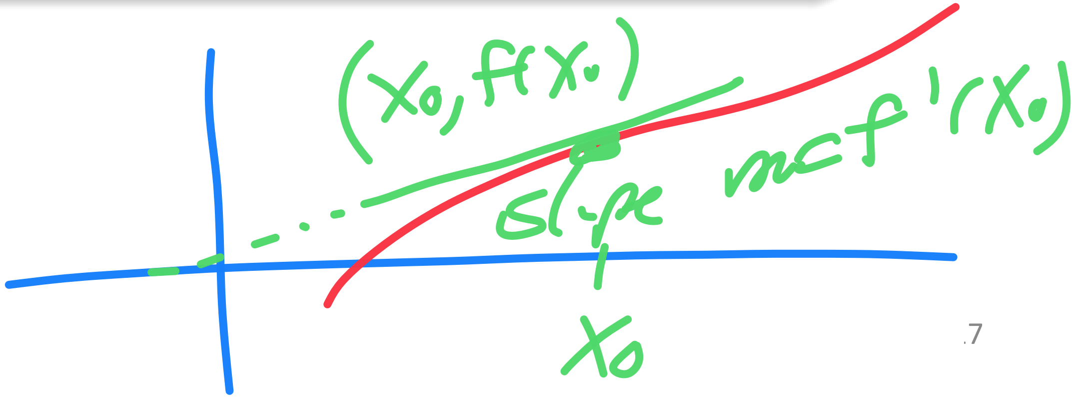
Thus 10 iterations essentially give three decimal digits.

		f(x) = x^2 - 3, sqrt(3)		1.732051		
n	left	right	f(left)	f(right)	left error	right error
1	1	2	-2	1	0.732051	-0.26795
2	1.5	2	-0.75	1	0.232051	-0.26795
3	1.5	1.75	-0.75	0.0625	0.232051	-0.01795
4	1.625	1.75	-0.35938	0.0625	0.107051	-0.01795
5	1.6875	1.75	-0.15234	0.0625	0.044551	-0.01795
6	1.71875	1.75	-0.0459	0.0625	0.013301	-0.01795
7	1.71875	1.734375	-0.0459	0.008057	0.013301	-0.00232
8	1.726563	1.734375	-0.01898	0.008057	0.005488	-0.00232
9	1.730469	1.734375	-0.00548	0.008057	0.001582	-0.00232
10	1.730469	1.732422	-0.00548	0.001286	0.001582	-0.00037

Figure: Approximating $\sqrt{3} \approx 1.7320508075688772935274463415$.

Newton's Method

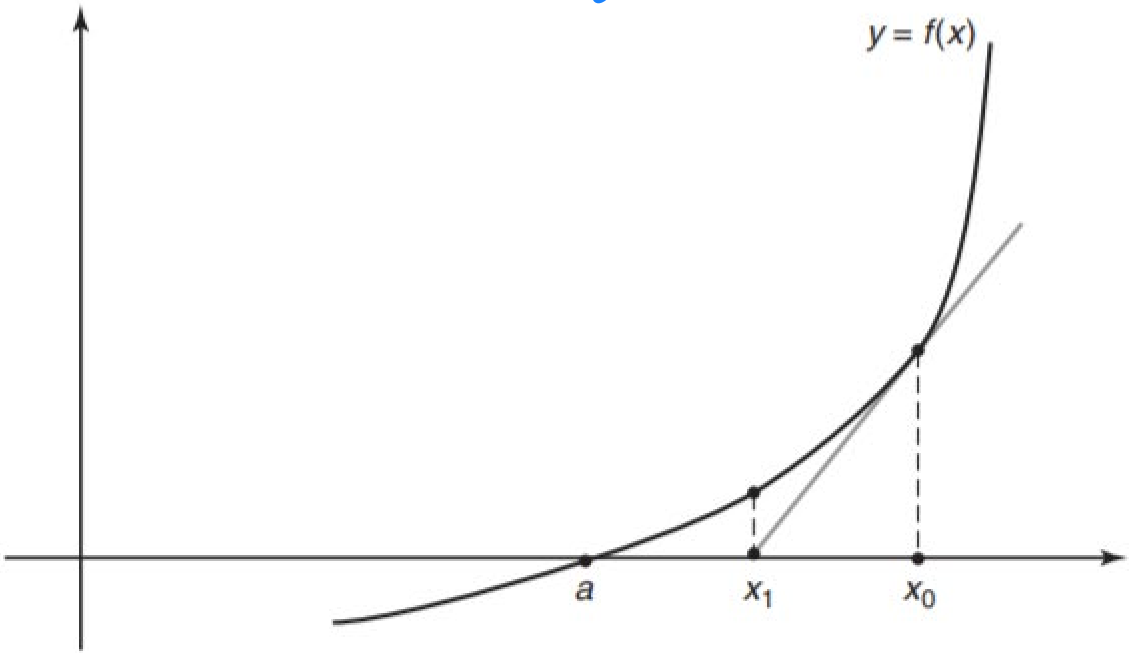
Assume f is continuous and differentiable. We generate a sequence hopefully converging to the root of $f(x) = 0$ as follows. Given x_n, look at the tangent line to the curve $y = f(x)$ at x_n; it has slope $f'(x_n)$ and goes through $(x_n, f(x_n))$ and gives line $y - f(x_n) = f'(x_n)(x - x_n)$. This hits the x-axis at $y = 0, x = x_{n+1}$, and yields $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.



Newton's Method

$y - f(x_0) = f'(x_0)(x - x_0)$

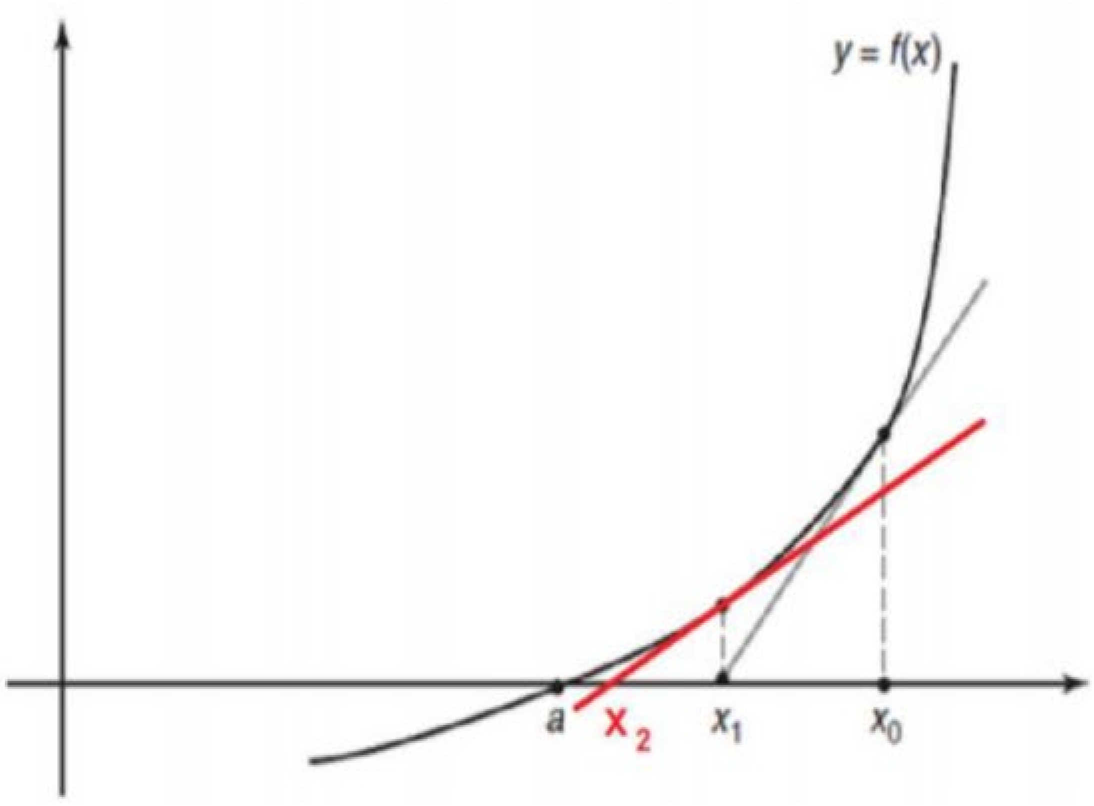
when $y=0$ call that x_1



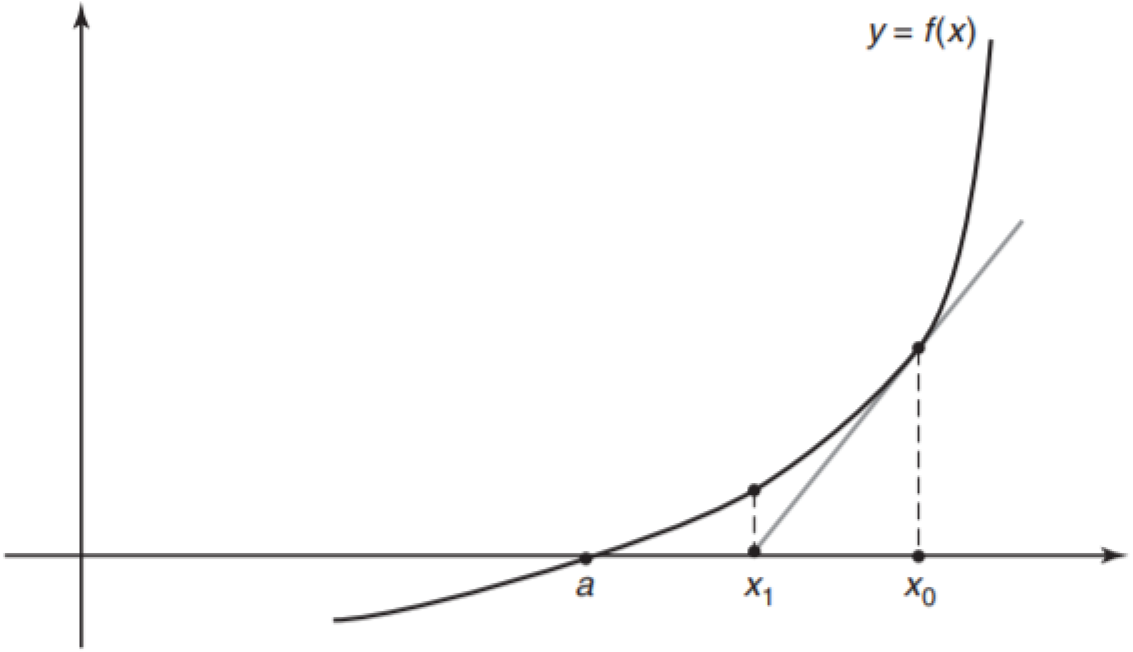
$0 = f(x_0) + f'(x_0)(x_1 - x_0)$

Solve for x_1

Newton's Method



Newton's Method



For example, $f(x) = x^2 - 3$ after algebra get
$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$.

Newton's Method

n	x[n]	1.0 x[n]	Sqrt[3] - x[n]
0	2	2.00000000000000000000000000000000000000	-0.26794919243112270647255365849412763305 7
1	$\frac{7}{4}$	1.75000000000000000000000000000000000000	-0.017949192431122706472553658494127633057
2	$\frac{97}{56}$	1.7321428571428572062984585500089451670 65	-0.000092049573979849329696515636984775914
3	$\frac{18817}{10864}$	1.7320508100147276042690691610914655029774	-2.4458502469732900355191644519 08 × 10^{-9}

Sqrt[3] = 1.7320508075688772935274463415058723669428

x[5] = 1.7320508075688772935274463415058723678037

x[4] = 1.7320508075688772952543539460721719142351

$$\sqrt{3} = 1.73205080756887729352744634150587236\mathbf{6}9428$$

$$x_5 = 1.73205080756887729352744634150587236\mathbf{7}8037$$

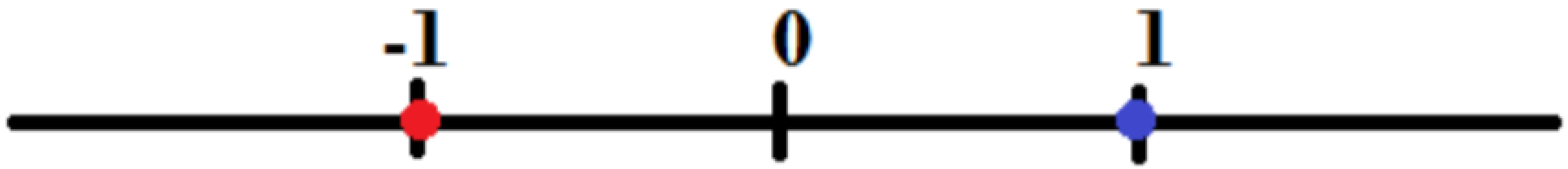
$$x_5 = \frac{10029782734 11373057}{5790697761 45402304}.$$

Newton Method: $x^2 - 3 = 0$

Consider $x^2 - 1 = (x-1)(x+1) = 0$.

Roots are 1, -1; if we start at a point x_0 do we approach a root? If so which?

Recall $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$.



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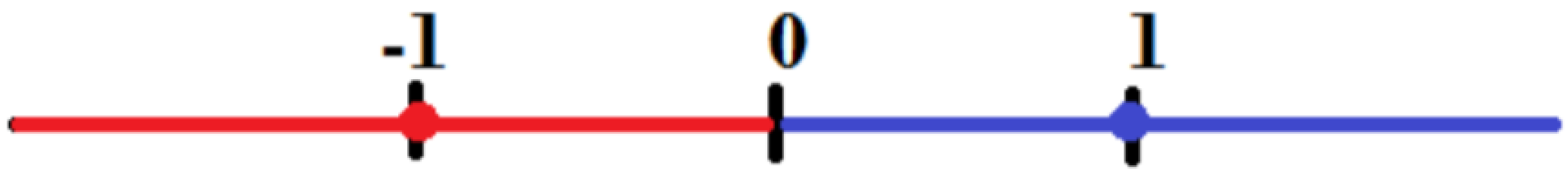


Newton Fractal: $x^3 - 1 = 0$:

https://www.youtube.com/watch?v=ZsFixqGFNRc

What are the roots to $x^3 - 1 = 0$?

Here comes Complex Numbers!
$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}, i = \sqrt{-1}\}$.

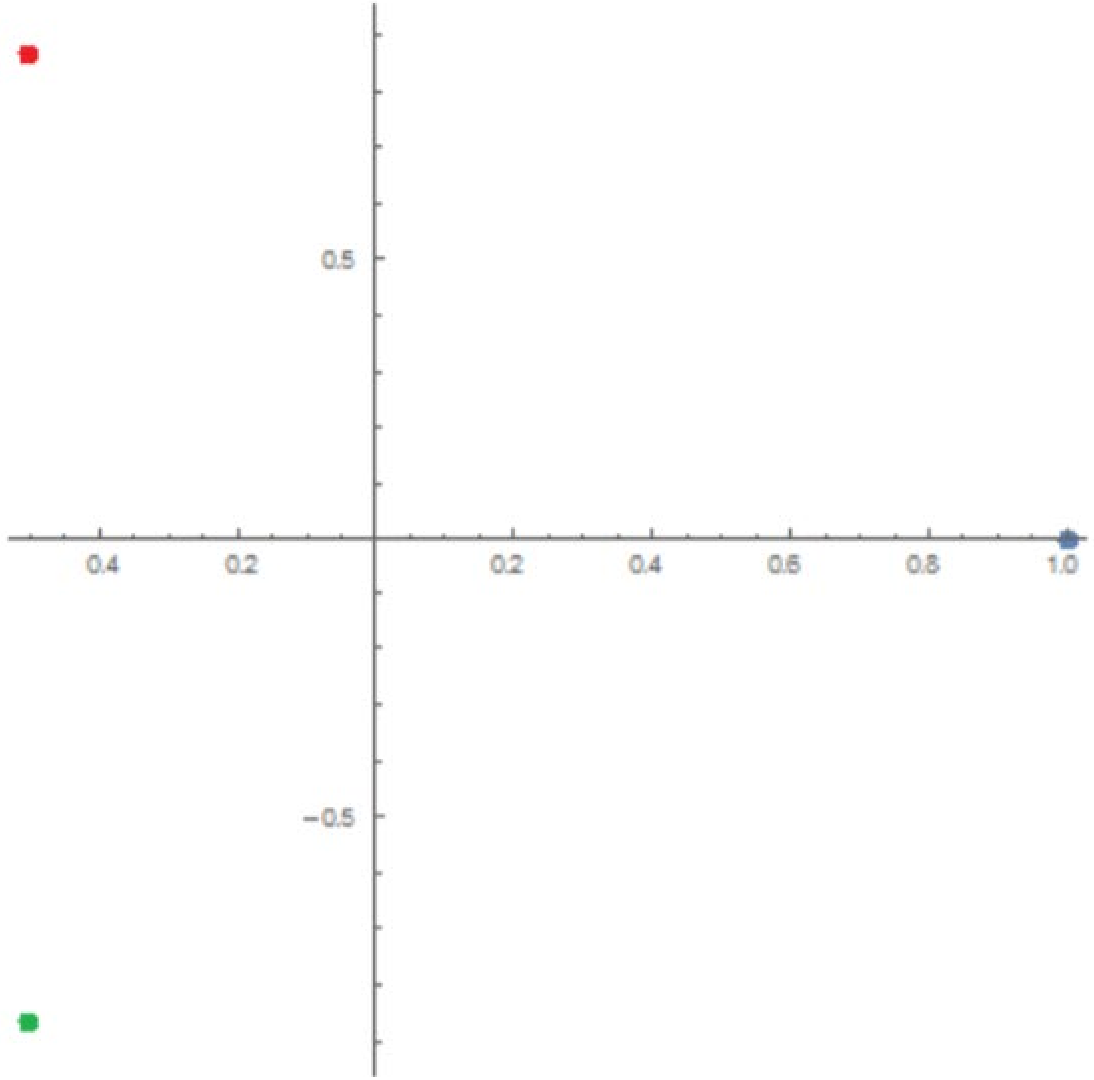
$$
\begin{aligned}
x^3 - 1 &= (x-1)(x^2 + x + 1) \\
&= (x-1)\cdot\left(x - \frac{-1 + \sqrt{1^2 - 4\cdot 1\cdot 1}}{2}\right)\cdot\left(x - \frac{-1 - \sqrt{1^2 - 4\cdot 1\cdot 1}}{2}\right) \\
&= (x-1)\cdot\left(x - \frac{-1 + \sqrt{-3}}{2}\right)\cdot\left(x - \frac{-1 - \sqrt{-3}}{2}\right) \\
&= (x-1)\cdot\left(x - \frac{-1 + i\sqrt{3}}{2}\right)\cdot\left(x - \frac{-1 - i\sqrt{3}}{2}\right).
\end{aligned}
$$

Roots are 1, $-1/2 + i\sqrt{3}/2$, $-1/2 - i\sqrt{3}/2$.

Newton Fractal: $x^3 - 1 = 0$:

https://www.youtube.com/watch?v=ZsFixqGFNRc

If start at (x, y), what root do you iterate to?



Newton Fractal: $x^3 - 1 = 0$:

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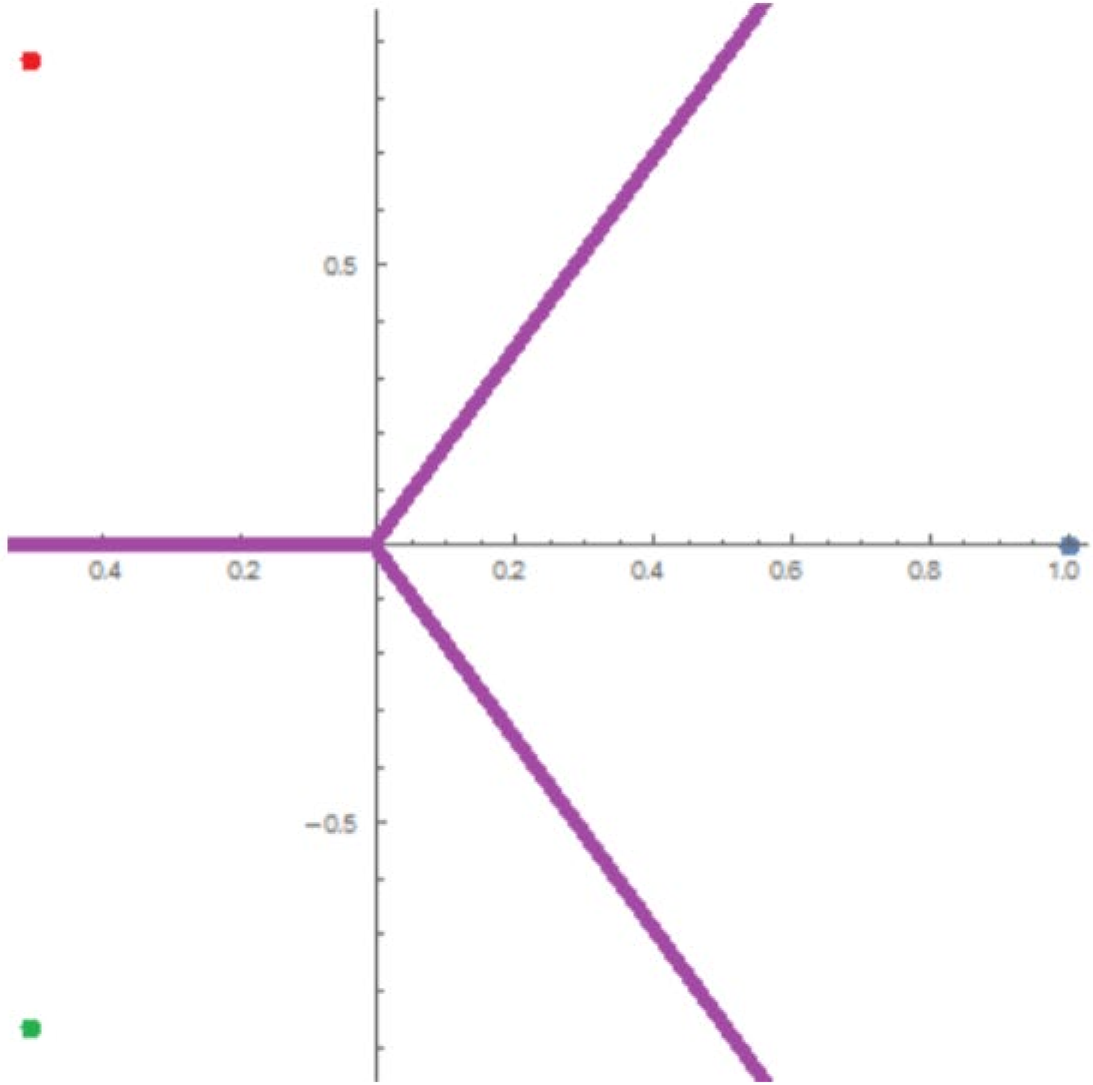
If start at (x, y), what root do you iterate to? Guess



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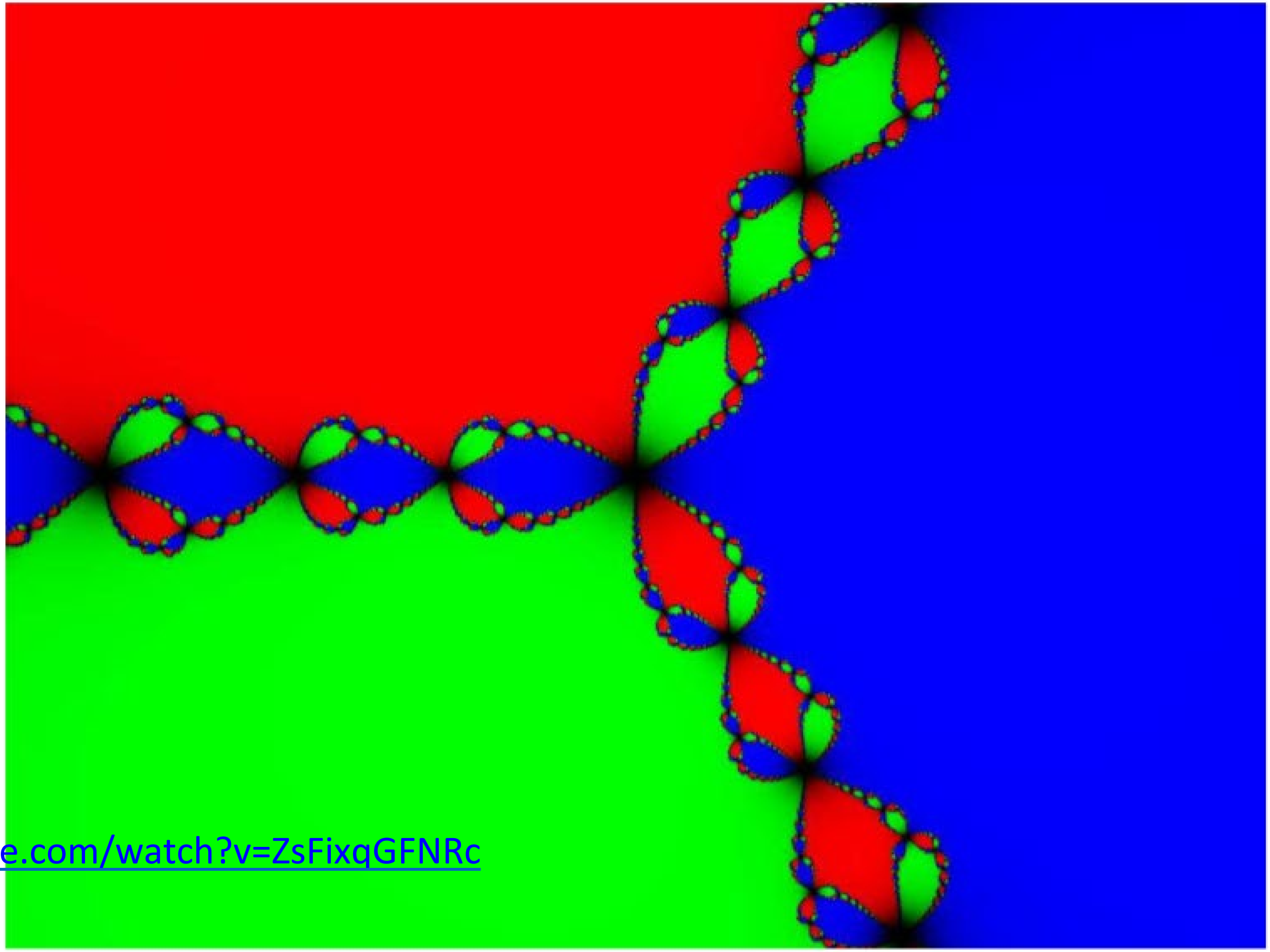

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