Math/Stat 341: Probability: Fall '21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 26: 11-17-21: https://youtu.be/pk6BwJsDRjM (slides)

Lecture 28: 11/13/19: Generating Functions III: Properties of MGF, Poisson and Normal Example, Poisson to CLT: video didn't record, last year didn't record, 24 minute version: http://youtu.be/Y7NppJoRoxQ

Plan for the day: Lecture 26: November , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Properties of Moment Generating Functions
- Poisson and Normal Examples
- Poisson to Central Limit Theorem
- Change of Base Formula

General items.

- Find the path through the algebra
- Some distributions easier to work with than others

Definition 19.6.2 (Moment generating function) *Let X be a random variable with density f. The moment generating function of X, denoted $M_X(t)$, is given by $M_X(t) = \mathbb{E}[e^{tX}]$. Explicitly, if X is discrete then*

$$M_X(t) = \sum_{m=-\infty}^{\infty} e^{tx_m} f(x_m),$$

while if X is continuous then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Note $M_X(t) = G_X(e^t)$, or equivalently $G_X(s) = M_X(\log s)$.

Theorem 19.6.3 *Let X be a random variable with moments μ'_k.*

1. *We have*

$$M_X(t) \;=\; 1+\mu'_1 t+\frac{\mu'_2 t^2}{2!}+\frac{\mu'_3 t^3}{3!}+\cdots;$$

   in particular, $\mu'_k = d^k M_X(t)/dt^k \Big|_{t=0}$.

2. *Let α and β be constants. Then*

$$M_{\alpha X+\beta}(t) \;=\; e^{\beta t} M_X(\alpha t).$$

   Useful special cases are $M_{X+\beta}(t) = e^{\beta t} M_X(t)$ and $M_{\alpha X}(t) = M_X(\alpha t)$; when proving the central limit theorem, it's also useful to have $M_{(X+\beta)/\alpha}(t) = e^{\beta t/\alpha} M_X(t/\alpha)$.

3. *Let X_1 and X_2 be independent random variables with moment generating functions $M_{X_1}(t)$ and $M_{X_2}(t)$ which converge for $|t| < \delta$. Then*

$$M_{X_1+X_2}(t) \;=\; M_{X_1}(t) M_{X_2}(t).$$

   More generally, if $X_1,\dots,X_N$ are independent random variables with moment generating functions $M_{X_i}(t)$ which converge for $|t| < \delta$, then

$$M_{X_1+\cdots+X_N}(t) = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_N}(t).$$

   If the random variables all have the same moment generating function $M_X(t)$, then the right hand side becomes $M_X(t)^N$.

Proof of (2)

$$G_X(t) = E[e^{tX}]$$

$$Y = \alpha X + \beta$$

$$G_Y(t) = E[e^{tY}]$$

$$= E[e^{t(\alpha X+\beta)}]$$

$$= E[e^{t\alpha X} e^{t\beta}]$$

$$= e^{\beta t} E[e^{(\alpha t) X}]$$

$$= e^{\beta t} M_X(\alpha t) \quad \blacksquare$$

Rescale to have mean 0, st dev 1

Theorem 19.6.5 ***(Uniqueness of moment generating functions for discrete random variables.)*** *Let X and Y be discrete random variables taking on non-negative integer values (i.e., they're non-zero only in $\{0, 1, 2, \ldots\}$) with moment generating functions $M_X(t)$ and $M_Y(t)$, each of which converges for $|t| < \delta$. Then X and Y have the same distribution if and only if there is an $r > 0$ such that $M_X(t) = M_Y(t)$ for $|t| < r$.*

There exist distinct probability distributions which have the same moments. In other words, knowing all the moments doesn't always uniquely determine the probability distribution.

Example 19.6.6 *The standard examples given are the following two densities, defined for $x \geq 0$ by*

$$f_1(x) = \frac{1}{\sqrt{2\pi x^2}} e^{-(\log^2 x)/2}$$
$$f_2(x) = f_1(x)\left[1 + \sin(2\pi \log x)\right]. \quad (19.2)$$

It's a nice calculation to show that these two densities have the same moments; they're clearly different (see Figure 19.1).

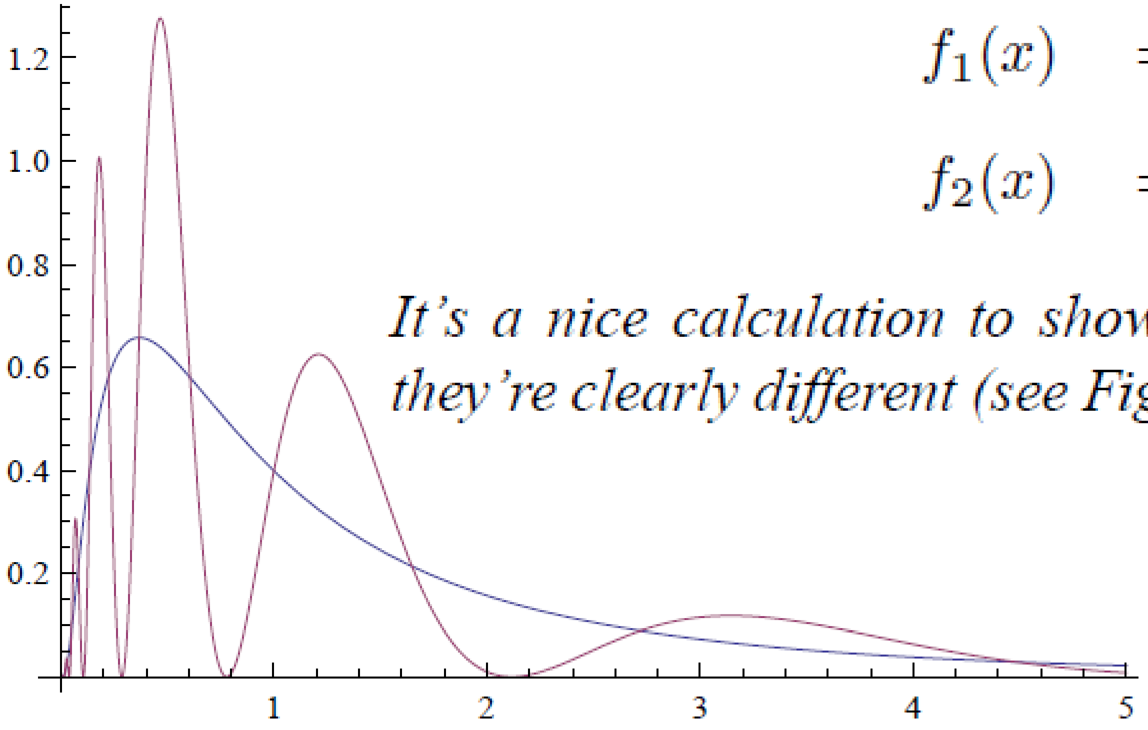

Figure 19.1: Plot of $f_1(x)$ and $f_2(x)$ from (19.2).

$$g(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{19.3}$$

Taylor Series is identically zero as

$g^{(n)}(0) = 0$

by L'Hopital

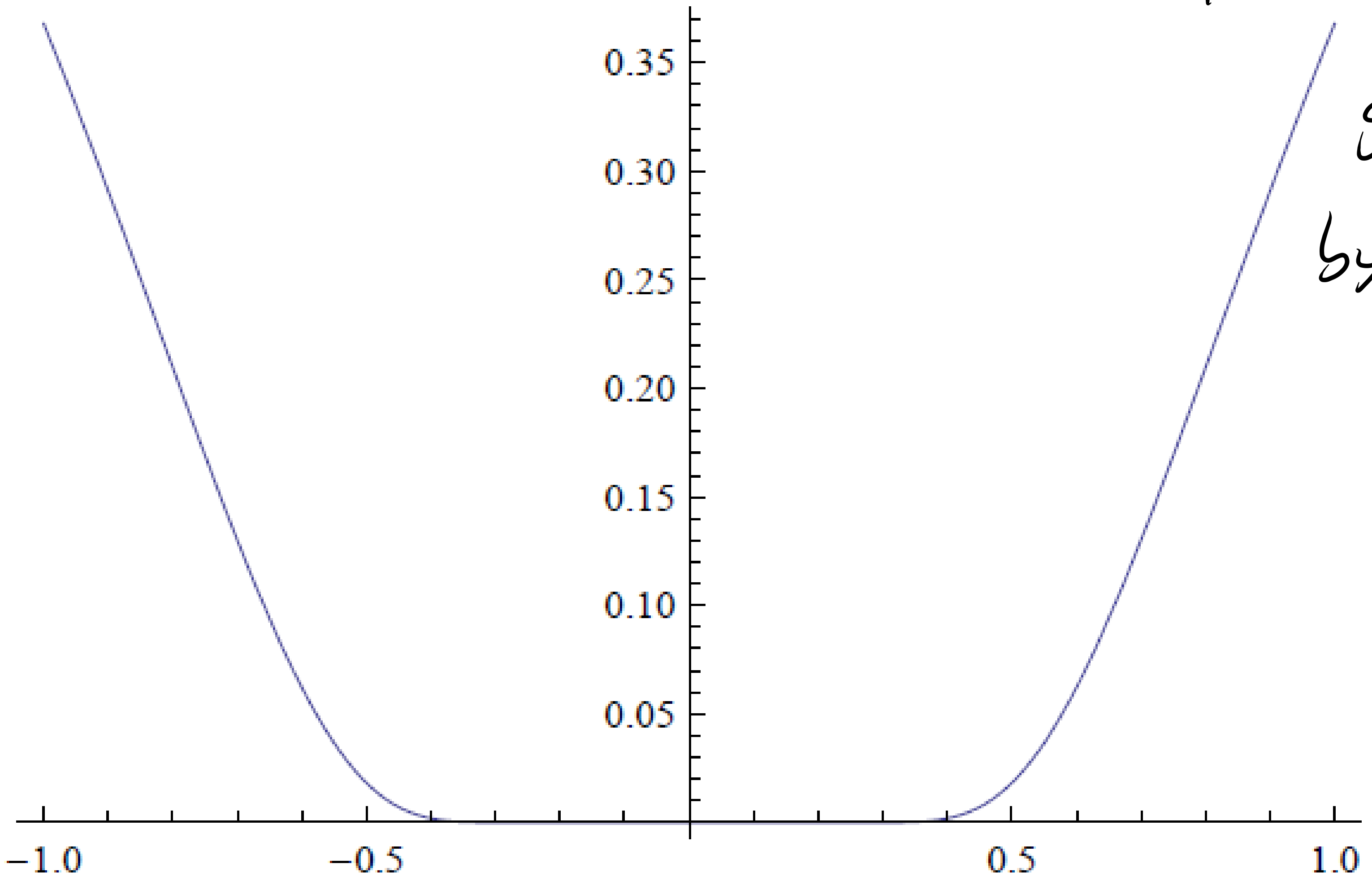

Figure 19.2: Plot of $g(x)$ from (19.3).

Poisson Generating Functions

$$M_X(t) = e^{\lambda(e^t-1)} \qquad \mu = \frac{d}{dt}M_X(t)\Big|_{t=0} \qquad \mu_2' = \frac{d^2}{dt^2}M_X(t)\Big|_{t=0}$$

$$M_X(t) = 1 + \mu_1' t + \frac{\mu_2' t^2}{2!} + \frac{\mu_3' t^3}{3!} + \cdots$$

$$M_X'(t) = e^{\lambda(e^t-1)} \left(\lambda(e^t-1)\right)' = e^{\lambda(e^t-1)} \cdot \lambda e^t$$

$$M_X'(0) = e^{\lambda(e^0-1)} \cdot \lambda e^0 = \lambda$$

$M_X''(t)$ uses product rule

Can avoid product! $M_X'(t) = \lambda e^{\lambda(e^t-1)+t}$

OR: $M_X(t) = e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} \frac{\lambda^n (e^t-1)^n}{n!}$

$$= 1 + \lambda(e^t-1) + \frac{1}{2}\lambda^2(e^t-1)^2 + \text{at least } t^3$$

(n=0) (n=1) (n=2)

$$= 1 + \lambda\left(1+t+\frac{t^2}{2}+\cdots-1\right) + \frac{\lambda^2}{2}\left(1+t+\frac{t^2}{2}+\cdots-1\right)^2 + \cdots$$

$$= 1 + \lambda t + \frac{\lambda}{2}t^2 + \frac{\lambda^2}{2}t^2 = 1 + \lambda t + \frac{\lambda^2+\lambda}{2}t^2$$

Second moment is $\lambda^2+\lambda$, var is $\lambda^2+\lambda-\lambda^2 = \lambda$

$$M_X'(t) = \lambda e^{\lambda(e^t-1)+t}$$

$$M_X''(t) = \lambda e^{\lambda(e^t-1)+t} \cdot (\lambda(e^t-1)+t)'$$

$$= \lambda e^{\lambda(e^t-1)+t} \cdot (\lambda e^t + 1)$$

$$M_X''(0) = \lambda e^{\lambda(e^0-1)+0} \cdot (\lambda e^0 + 1)$$

$$= \lambda(\lambda+1) = \lambda^2 + \lambda$$

Poiss (λ): mean is λ

var is λ

stdev is $\sqrt{\lambda}$

Moment Generating Function for the Standard Normal

$X \sim N(0,1)$

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2tx + t^2 - t^2)}\, dx$$

Complete the square by doing nothing (add zero)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((x-t)^2 - t^2)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} e^{t^2/2}\, dx$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2}}_{\text{density of } N(t,1)}\, dx = e^{t^2/2}$$

Sums of independent Poisson random variables converge to being normally distributed….

Poisson: $M_X(t) = e^{\lambda(e^t-1)}$ mean λ, var λ, stdev $\sqrt{\lambda}$

$X = X_1 + \cdots + X_n$, each $X_k \sim \text{Poiss}(1)$ so $X \sim \text{Poiss}(n)$

$$M_X(t) = \prod_{k=1}^{n} M_{X_k}(t) = (M_{X_1}(t))^n = \left(e^{1\cdot(e^t-1)}\right)^n = e^{n(e^t-1)}$$

Standardize: $Z = \frac{X - \mu_X}{\sigma_X} = \frac{X - n}{\sqrt{n}} = \frac{1}{\sqrt{n}} X - \sqrt{n}$

Use $M_{\alpha X+\beta}(t) = e^{\beta t} M_X(\alpha t)$ with $\alpha = 1/\sqrt{n}$ and $\beta = -\sqrt{n}$

$$M_Z(t) = e^{-\sqrt{n}t} M_X(t/\sqrt{n}) = e^{-\sqrt{n}t} e^{n(e^{t/\sqrt{n}}-1)}$$

Use $M_{\alpha X+\beta}(t) = e^{\beta t} M_X(\alpha t)$ with $\alpha = 1/\sqrt{n}$ and $\beta = -\sqrt{n}$

$$M_Z(t) = e^{-\sqrt{n}t} M_X(t/\sqrt{n}) = e^{-\sqrt{n}t} e^{n(e^{t/\sqrt{n}}-1)}$$

$$\log M_Z(t) = -\sqrt{n}t + n(e^{t/\sqrt{n}} - 1)$$

$$= -\sqrt{n}t + n\left(1 + \frac{t}{\sqrt{n}} + \frac{1}{2!}\left(\frac{t}{\sqrt{n}}\right)^2 + \frac{1}{3!}\left(\frac{t}{\sqrt{n}}\right)^3 + \cdots - 1\right)$$

$$= \frac{t^2}{2} + \underbrace{\frac{1}{3!}\frac{t^3}{\sqrt{n}} + \frac{1}{4!}\frac{t^4}{n} + \frac{1}{5!}\frac{t^5}{n\sqrt{n}} + \cdots}_{\text{as } n\to\infty,\text{ goes to zero}}$$

as $n \to \infty$ $\log M_Z(t)$ converges to $t^2/2$

so $M_Z(t)$ converges to $e^{t^2/2}$

This is the MGF of the standard normal!

Definition 20.4.1 (Standardization of a random variable) *Let X be a random variable with mean μ and standard deviation σ, both of which are finite. The standardization, Z, is defined by*

$$Z := \frac{X - \mathbb{E}[X]}{\mathrm{StDev}(X)} = \frac{X - \mu}{\sigma}.$$

Note that

$$\mathbb{E}[Z] = 0 \text{ and } \mathrm{StDev}(Z) = 1.$$

Theorem 20.5.1 (Moment generating function of normal distributions) *Let X be a normal random variable with mean μ and variance σ^2. Its moment generating function is*

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

In particular, if Z has the standard normal distribution, its moment generating function is

$$M_Z(t) = e^{t^2/2}.$$

Theorem 20.5.3 *Assume the moment generating functions $M_X(t)$ and $M_Y(t)$ exist in a neighborhood of zero (i.e., there's some δ such that both functions exist for $|t| < \delta$). If $M_X(t) = M_Y(t)$ in this neighborhood, then $F_X(u) = F_Y(u)$ for all u. As the densities are the derivatives of the cumulative distribution functions, we have $f = g$.*

Theorem 20.5.4 *Let $\{X_i\}_{i \in I}$ be a sequence of random variables with moment generating functions $M_{X_i}(t)$. Assume there's a $\delta > 0$ such that when $|t| < \delta$ we have $\lim_{i \to \infty} M_{X_i}(t) = M_X(t)$ for some moment generating function $M_X(t)$, and all moment generating functions converge for $|t| < \delta$. Then there exists a unique cumulative distribution function F whose moments are determined from $M_X(t)$, and for all x where $F_X(x)$ is continuous, $\lim_{n \to \infty} F_{X_i}(x) = F_X(x)$.*

$M_X(t) = \mathbb{E}[e^{tX}]$ doesn't always exist

Characteristic function:

$\Phi_X(t) = \mathbb{E}[e^{itX}]$ where $i = \sqrt{-1}$

$= \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx$ where $e^{itx} = \cos(tx) + i\sin(tx)$

$|e^{itx}| = 1$

$|\Phi_X(t)| \leq \int_{-\infty}^{\infty} |e^{itx}| f_X(x)\,dx = 1$

Change of Base Formula for Logarithms

$\log_b x = y$ means $x = b^y$

$\log_b x = \dfrac{\log_c x}{\log_c b}$ only need to know logs in one base

$\log_b x = y \Rightarrow x = b^y$

$\log_c x = w_1 \Rightarrow x = c^{w_1}$

$\log_c b = w_2 \Rightarrow b = c^{w_2}$

us $x = b^y = (c^{w_2})^y = c^{w_1}$

So $w_2 y = w_1$

so $y = \dfrac{w_1}{w_2}$

or $\log_b x = \dfrac{\log_c x}{\log_c b}$ ∎