Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 29: 11-29-21: https://youtu.be/0t_I5j598vQ (slides)

Lecture 31: 11/20/19: Proof of the CLT: https://youtu.be/4m77G15eINk

Plan for the day: Lecture 2: November 29, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Proof of the Central Limit Theorem (assuming results from Complex Analysis)
- If time permits estimating probabilities / erf….

General items.

- Power of doing simpler cases first
- Power of Taylor Series
- Power of logarithms

Definition 19.6.2 (Moment generating function) *Let X be a random variable with density f. The moment generating function of X, denoted $M_X(t)$, is given by $M_X(t) = \mathbb{E}[e^{tX}]$. Explicitly, if X is discrete then*

$$M_X(t) = \sum_{m=-\infty}^{\infty} e^{tx_m} f(x_m),$$

while if X is continuous then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Note $M_X(t) = G_X(e^t)$, or equivalently $G_X(s) = M_X(\log s)$.

Theorem 19.6.3 *Let X be a random variable with moments μ'_k.*

1. *We have*

$$M_X(t) \;=\; 1 + \mu'_1 t + \frac{\mu'_2 t^2}{2!} + \frac{\mu'_3 t^3}{3!} + \cdots;$$

in particular, $\mu'_k = d^k M_X(t)/dt^k \big|_{t=0}$.

2. *Let α and β be constants. Then*

$$M_{\alpha X+\beta}(t) \;=\; e^{\beta t} M_X(\alpha t).$$

Useful special cases are $M_{X+\beta}(t) = e^{\beta t} M_X(t)$ and $M_{\alpha X}(t) = M_X(\alpha t)$; when proving the central limit theorem, it's also useful to have $M_{(X+\beta)/\alpha}(t) = e^{\beta t/\alpha} M_X(t/\alpha)$.

3. *Let X_1 and X_2 be independent random variables with moment generating functions $M_{X_1}(t)$ and $M_{X_2}(t)$ which converge for $|t| < \delta$. Then*

$$M_{X_1+X_2}(t) \;=\; M_{X_1}(t) M_{X_2}(t).$$

More generally, if $X_1, \dots, X_N$ are independent random variables with moment generating functions $M_{X_i}(t)$ which converge for $|t| < \delta$, then

$$M_{X_1+\cdots+X_N}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_N}(t).$$

If the random variables all have the same moment generating function $M_X(t)$, then the right hand side becomes $M_X(t)^N$.

Proof of (2)

$G_X(t) = E[e^{tX}]$

$Y = \alpha X + \beta$

$G_Y(t) = E[e^{tY}]$

$= E[e^{t(\alpha X+\beta)}]$

$= E[e^{t\alpha X} e^{t\beta}]$

$= e^{\beta t} E[e^{(\alpha t) X}]$

$= e^{\beta t} M_X(\alpha t)$ ∎

Rescale to have mean 0, st dev 1

Definition 20.4.1 (Standardization of a random variable) *Let X be a random variable with mean μ and standard deviation σ, both of which are finite. The standardization, Z, is defined by*

$$Z := \frac{X - \mathbb{E}[X]}{\mathrm{StDev}(X)} = \frac{X - \mu}{\sigma}.$$

Note that

$$\mathbb{E}[Z] = 0 \quad \text{and} \quad \mathrm{StDev}(Z) = 1.$$

Theorem 20.5.1 (Moment generating function of normal distributions) *Let X be a normal random variable with mean μ and variance σ^2. Its moment generating function is*

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

In particular, if Z has the standard normal distribution, its moment generating function is

$$M_Z(t) = e^{t^2/2}.$$

Theorem 20.5.3 *Assume the moment generating functions $M_X(t)$ and $M_Y(t)$ exist in a neighborhood of zero (i.e., there's some δ such that both functions exist for $|t| < \delta$). If $M_X(t) = M_Y(t)$ in this neighborhood, then $F_X(u) = F_Y(u)$ for all u. As the densities are the derivatives of the cumulative distribution functions, we have $f = g$.*

Theorem 20.5.4 *Let $\{X_i\}_{i \in I}$ be a sequence of random variables with moment generating functions $M_{X_i}(t)$. Assume there's a $\delta > 0$ such that when $|t| < \delta$ we have $\lim_{i \to \infty} M_{X_i}(t) = M_X(t)$ for some moment generating function $M_X(t)$, and all moment generating functions converge for $|t| < \delta$. Then there exists a unique cumulative distribution function F whose moments are determined from $M_X(t)$, and for all x where $F_X(x)$ is continuous, $\lim_{n \to \infty} F_{X_i}(x) = F_X(x)$.*

$M_X(t) = \mathbb{E}[e^{tX}]$ doesn't always exist

Characteristic function:

$\Phi_X(t) = \mathbb{E}[e^{itX}]$ where $i = \sqrt{-1}$

$= \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx$ where $e^{itx} = \cos(tx) + i\sin(tx)$

$|e^{itx}| = 1$

$|\Phi_X(t)| \le \int_{-\infty}^{\infty} |e^{itx}| f_X(x)\,dx = 1$

Definition 20.2.1 (Normal distribution) *A random variable X is normally distributed (or has the normal distribution, or is a Gaussian random variable) with mean μ and variance σ^2 if the density of X is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We often write $X \sim N(\mu, \sigma^2)$ to denote this. If $\mu = 0$ and $\sigma^2 = 1$, we say X has the standard normal distribution.

Theorem 20.2.2 (Central Limit Theorem (CLT)) *Let $X_1, \ldots, X_N$ be independent, identically distributed random variables whose moment generating functions converge for $|t| < \delta$ for some $\delta > 0$ (this implies all the moments exist and are finite). Denote the mean by μ and the variance by σ^2, let*

$$\overline{X}_N = \frac{X_1 + \cdots + X_N}{N}$$

and set

$$Z_N = \frac{\overline{X}_N - \mu}{\sigma / \sqrt{N}}.$$

Then as $N \to \infty$, the distribution of Z_N converges to the standard normal (see Definition 20.2.1 for a statement).

$$X = X_1 + \cdots + X_n \qquad \mu_X = n\mu \qquad \sigma_X^2 = n\sigma^2$$

$$Z = \frac{X - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sigma\sqrt{n}} X - \frac{\sqrt{n}\mu}{\sigma} = \alpha X + \beta$$

$$\alpha = \frac{1}{\sigma\sqrt{n}}$$

$$\beta = \frac{\sqrt{n}\mu}{\sigma}$$

$$M_Z(t) = M_{\alpha X + \beta}(t) = e^{\beta t} M_X(\alpha t)$$

$$= e^{\beta t} M_{X_1 + \cdots + X_n}(\alpha t)$$

$$= e^{\beta t} \left[M_{X_1}(\alpha t) \right]^n$$

$$= e^{-t\mu\sqrt{n}/\sigma} \left[M_{X_1}\left(\frac{t}{\sigma\sqrt{n}} \right) \right]^n \qquad (\text{Eq } 4)$$

$$M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right) = ?$$

$$M_{X_1}(w) = 1 + \mu w + \frac{\mu_2' w^2}{2!} + \frac{\mu_3' w^3}{3!} + \frac{\mu_4' w^4}{4!} + \cdots$$

converges if $|w| < \delta$

$$= 1 + \mu w + \left[\sigma^2 + \mu^2\right]\frac{w^2}{2!} + \frac{\mu_3' w^3}{3!} + \cdots$$

$$M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{\mu t}{\sigma\sqrt{n}} + (\sigma^2+\mu^2)\frac{t^2}{2\sigma^2 n} + \text{Order}(1/n^{3/2})$$

$$\left[M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n = \left[\qquad\qquad\right]^n$$

$$\log\left(M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n = n\log\left[1 + \frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2+\mu^2)t^2}{2\sigma^2 n} + O(n^{-3/2})\right]$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(Tf)(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\log \left(M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n = n \log \Big[1 + \overbrace{\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2+\mu^2)t^2}{2\sigma^2 n} + O(n^{-3/2})}^{x} \Big]$$

$$= n\left[\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2+\mu^2)t^2}{2\sigma^2 n} + O(n^{-3/2}) - \frac{\mu^2 t^2}{2\sigma^2 n} + O(n^{-3/2}) + O(n^{-3/2})\right]$$

$$= \frac{\sqrt{n}\,\mu t}{\sigma} + \frac{t^2}{2} + O(n^{-1/2})$$

$$M_Z(t) = e^{-t\mu\sqrt{n}/\sigma}\left[M_{X_1}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

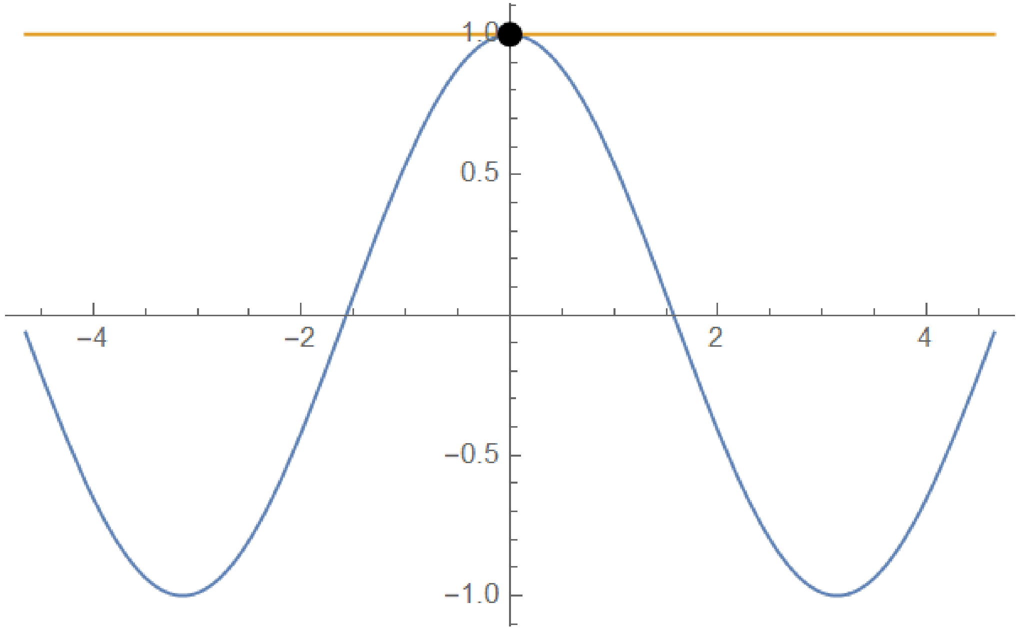
$$\log M_Z(t) = -\frac{t\sqrt{n}}{\sigma} + n \log M_{X_1}(t/\sigma\sqrt{n})$$

$$= -\frac{t\sqrt{n}\mu}{\sigma} + \frac{\sqrt{n}t\mu}{\sigma} + \frac{t^2}{2} + O(n^{-1/2})$$

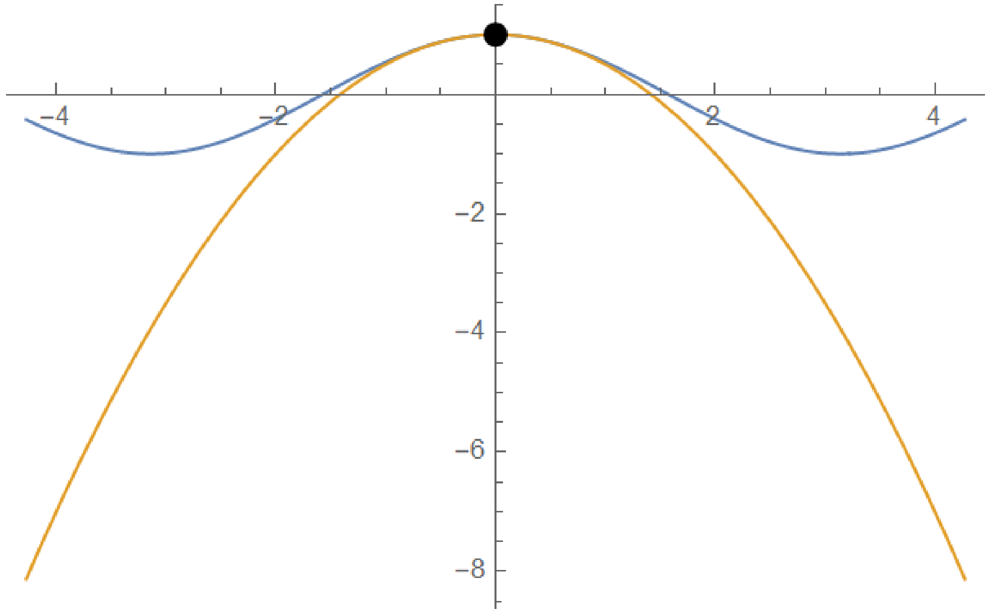
$$= t^2/2 + O(n^{-1/2})$$

$$M_Z(t) = e^{t^2/2}\underbrace{e^{O(n^{-1/2})}}_{\text{as } n\to\infty \text{ goes to } 1} \xrightarrow[n\to\infty]{} e^{t^2/2}$$

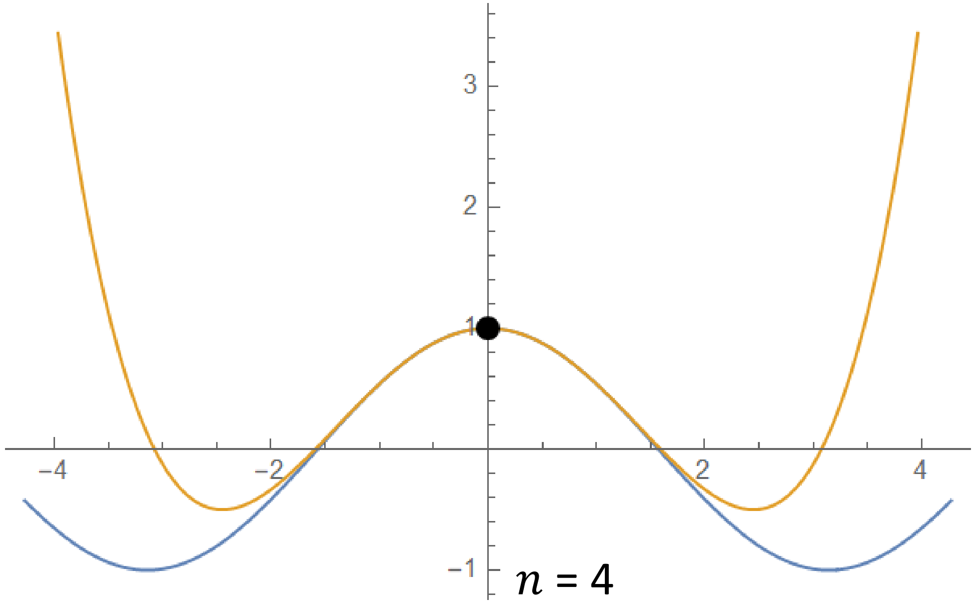
MGF of the Standard Normal!



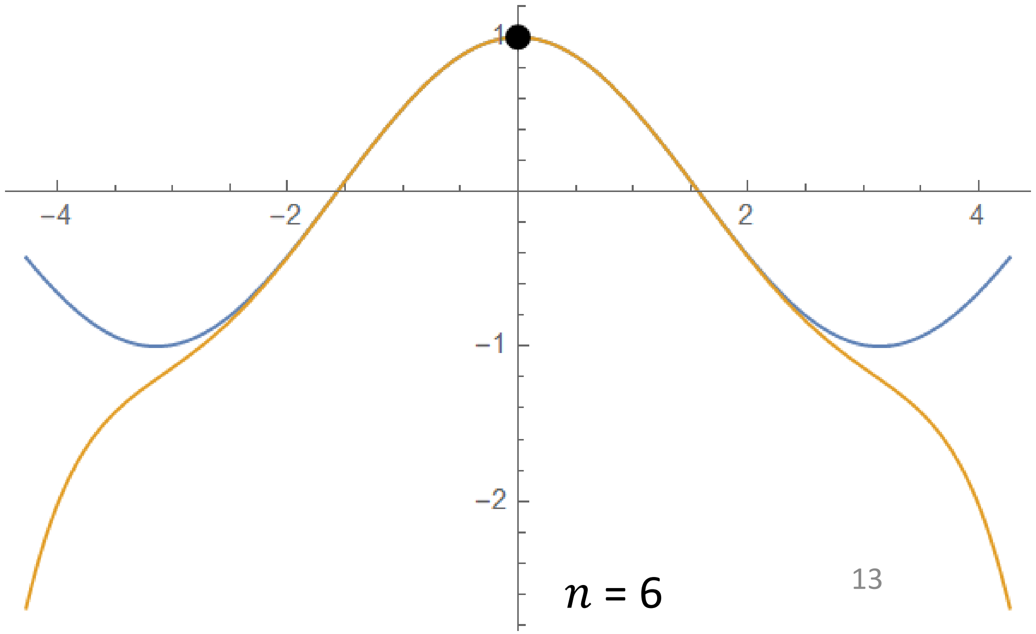
$n = 0$



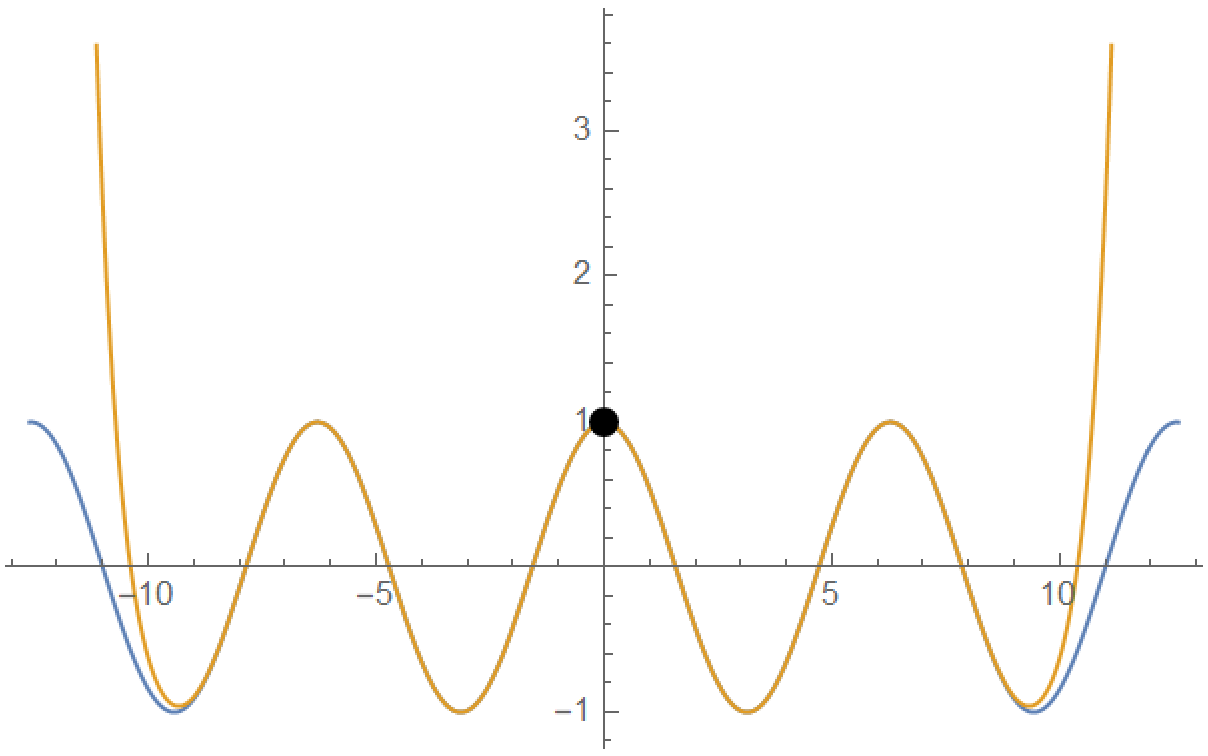
$n = 2$



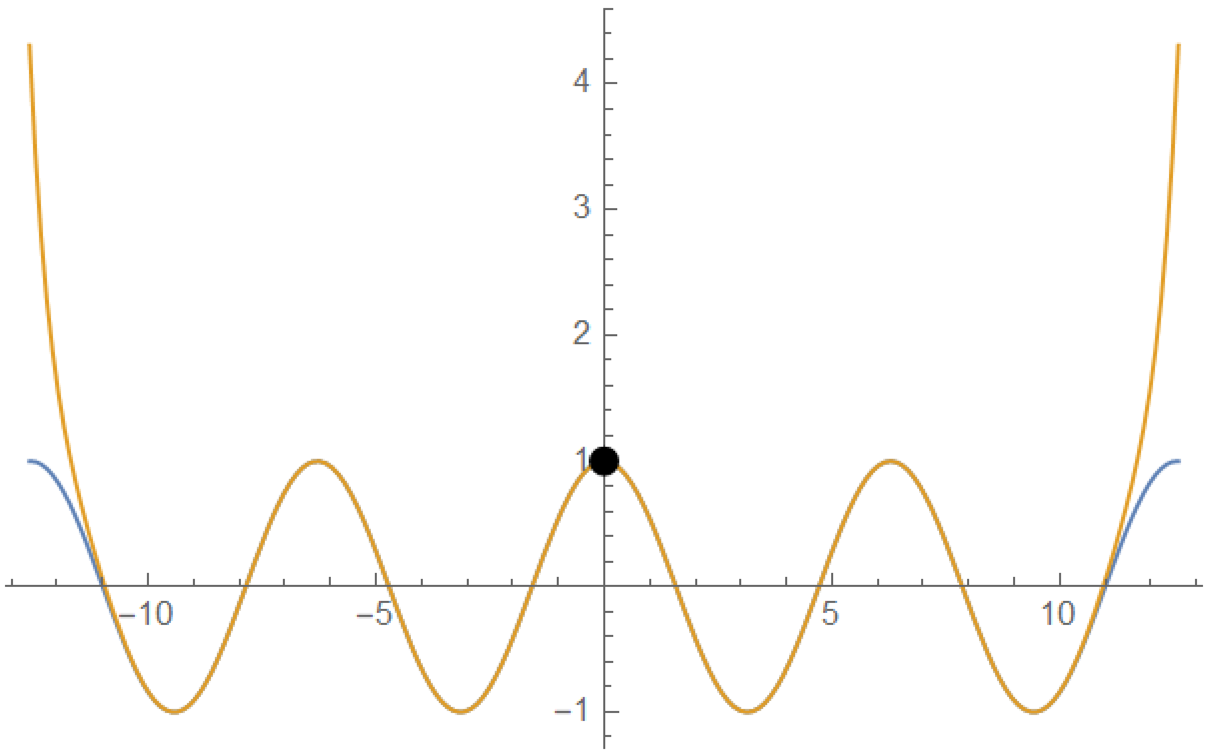
$n = 4$



$n = 6$



$n = 24$



$n = 28$

Estimating: What is Prob($|X| > n$) if $X \sim N(0,1)$?

Is $2 \int_n^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$

$x = u + n \quad x: n \to \infty \quad u: 0 \to \infty \quad dx = du$

Is $2 \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(u+n)^2/2} \, du$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u^2/2} e^{-un} e^{-n^2/2} \, du$

$= 2 e^{-n^2/2} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \underbrace{e^{-un}}_{\text{at most } 1} \, du$

Get $\leq e^{-n^2/2}$

Equal to $2e^{-r^2/2}\int_0^\infty \underbrace{e^{-u^2/2}}_{\text{at most } 1} e^{-ur}\,du * \frac{1}{\sqrt{2\pi}}$

$$\le 2e^{-r^2/2}\int_0^\infty \frac{1}{\sqrt{2\pi}}\, e^{-ur}\, r\,du * \frac{1}{r}$$

$$< \frac{2e^{-r^2/2}}{r\sqrt{2\pi}} = e^{-r^2/2} * \frac{\sqrt{2}}{r\sqrt{\pi}}$$

Estimate: $\int_{-\infty}^{x} e^{-t^2/2}\,dt$ $\qquad e^{u} = \sum_{n=0}^{\infty} \frac{u^n}{n!}$

$$= \int_{-\infty}^{x} \sum_{n=0}^{\infty} \frac{(-t^2/2)^n}{n!}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{2^n} \underbrace{\int_{-\infty}^{x} t^{2n}\,dt}$$

$$\frac{t^{2n+1}}{2n+1}\Big|_{-\infty}^{x} = \underline{\text{Infinity}}!$$

BAD!

$\text{Prob}(X \le x) = \text{Prob}(-\infty < X \le x)$

Compute $\text{Prob}(0 \le X \le x)$

$\text{Prob } X \le 0$ is $1/2$

erf

https://en.wikipedia.org/wiki/Error_function

$$\int_0^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$= \int_0^x \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-t^2/2)^n}{n!}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{(-1)^n}{n!\, 2^n} \frac{x^{2n+1}}{2n+1}$$