Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 33: 12-08-21: https://youtu.be/GNKIJUZrb_A (slides)

Lecture 30: 11/26/18: Monte Carlo Integration: https://youtu.be/eOyGU8dDbVo (2016 has Buffon's needle: http://youtu.be/xdtA0F2NKb0): Mathematica Program here (pdf here)

Plan for the day: Lecture 33: December 8, 2021:

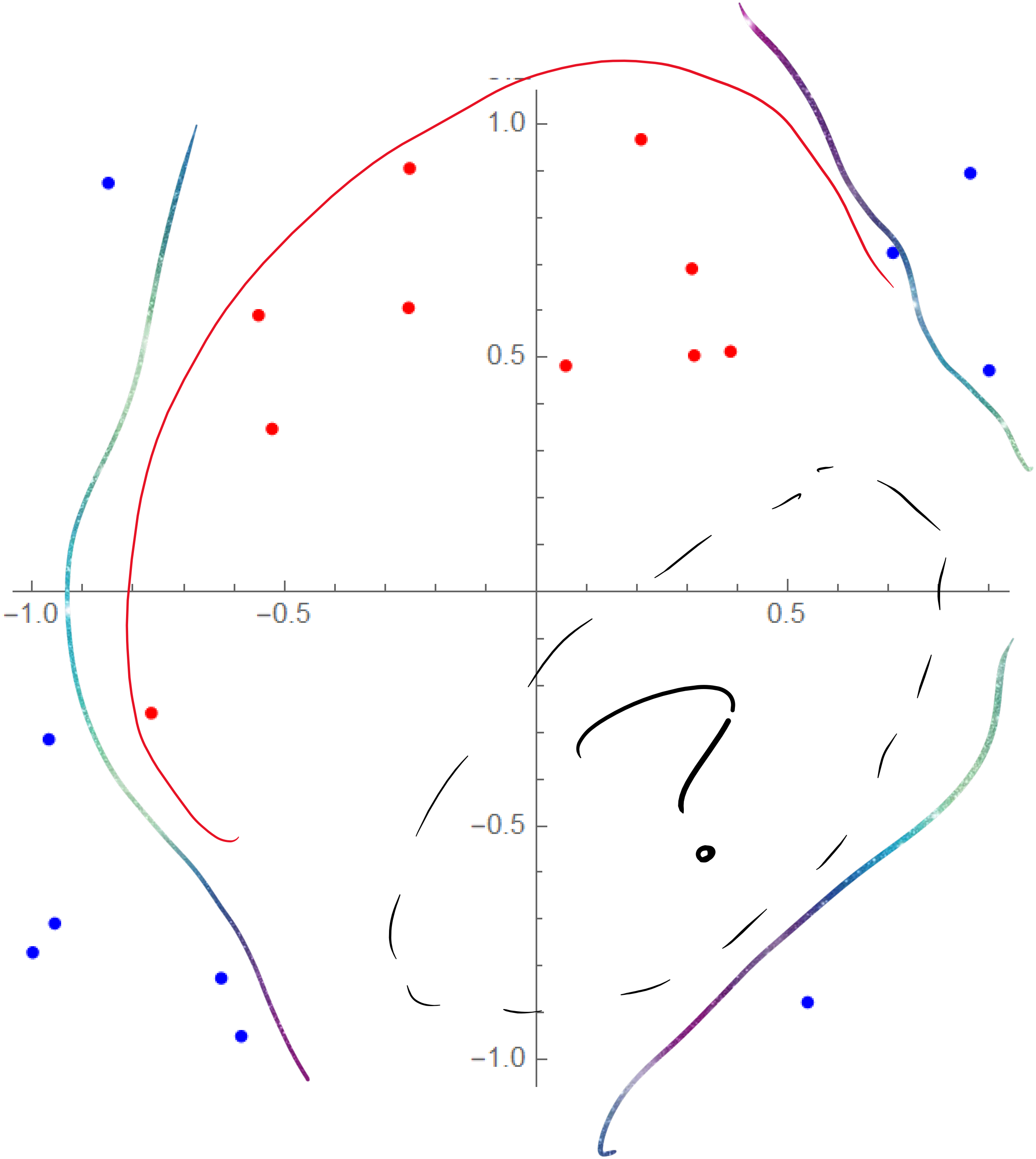
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/

- Monte Carlo Integration

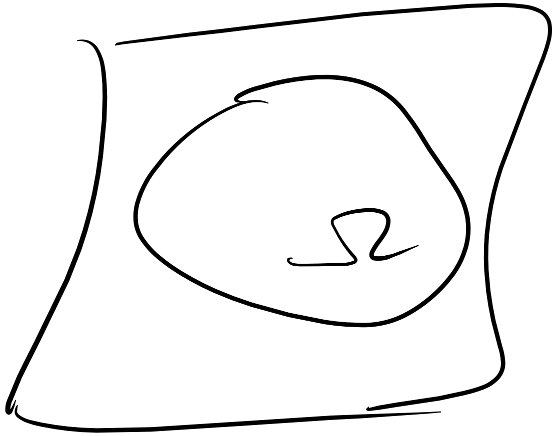
- Egg Drop Problem: Comparing solutions

General items.

- Optimization Issues
- Dynamic Decisions
- Choice of Metric
- Lot of further reading:
  - https://www.johndcook.com/blog/2021/09/02/qmc-integration/
  - https://arxiv.org/pdf/1409.5894.pdf
  - https://drna.padovauniversitypress.it/system/files/papers/Cools_etal_10YPDPTS.pdf



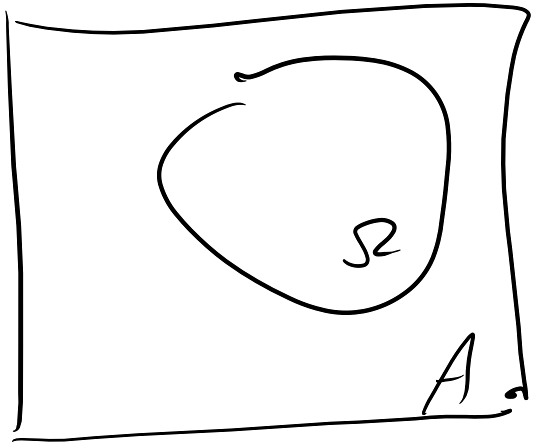
Monte Carlo Integration



Choose N points

See what fraction are in your set

$$\text{Area}(\Omega) \approx \frac{\#\{n \in A\}}{N} \text{Area}(A)$$



Take N iid observations

$X_1, X_2, \ldots, X_N$ $\qquad X_k = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$

$X = X_1 + \cdots + X_N$

Note X_k is 1 with prob $\frac{\text{Area}(\Omega)}{\text{Area}(A)} = p$, 0 otherwise

Bernoulli, sum is Binomial

$$E[X] = N\, E[X_k] = N \cdot \frac{\text{Area}(\Omega)}{\text{Area}(A)}$$

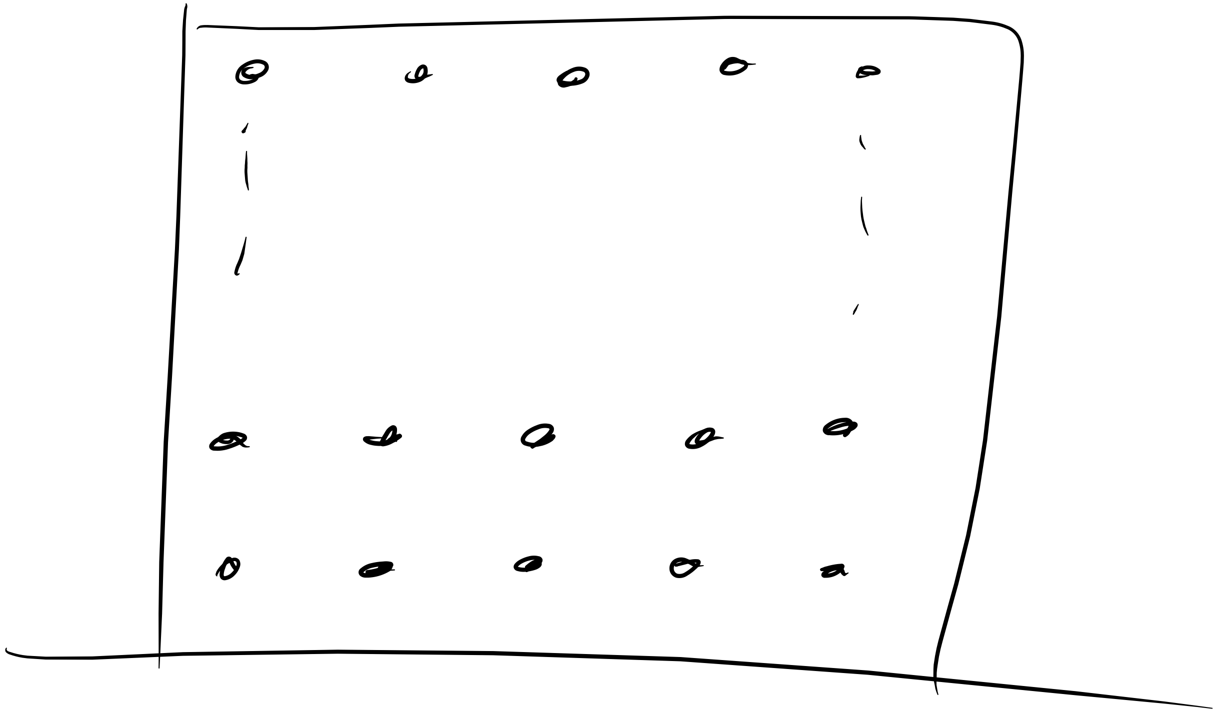
Area Estimate is $E[X]/N = \text{Area}(\Omega)/\text{Area}(A)$

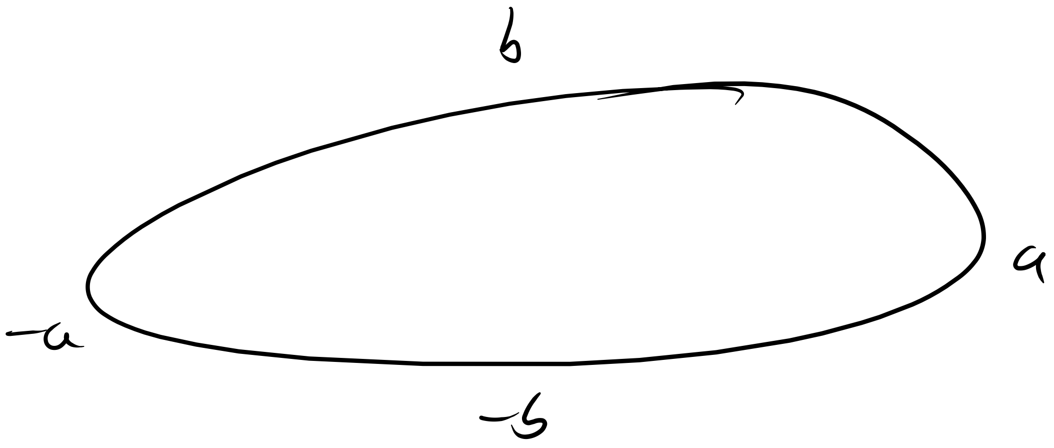
$$\text{Var}(X) = N\, \text{Var}(X_1) = N\, p(1-p)$$

$Y = X/N \qquad E[Y] = \frac{\text{Area}(\Omega)}{\text{Area}(A)} \qquad \text{Var}(Y) = \frac{\text{Var}(X)}{N^2} = \frac{p(1-p)}{N}$

error $\sim 1/\sqrt{N}$

Lattice





Guess area is πab

reduces to πr^2 if $a=b=r$

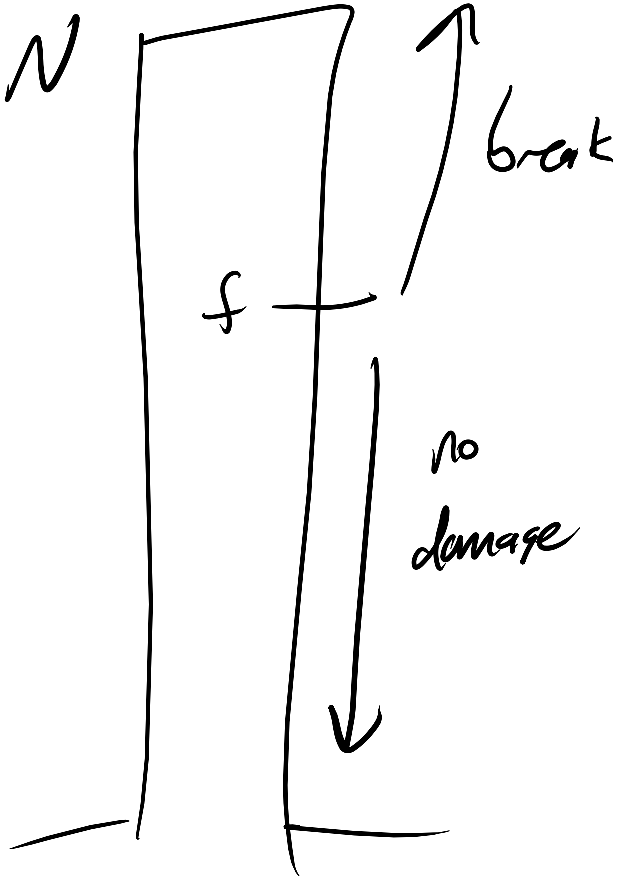
$u = x/a \quad v = y/b \quad du = \frac{dx}{a} \quad dv = \frac{dy}{b}$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\iint_{\text{ellipse}} 1 \, dx\,dy = \iint_{\substack{\text{circle} \\ \text{(unit)}}} 1 \, ab \, du\,dv$$

$$= \pi ab$$

Egg Drop Problem



1 egg: start bottom, move up

2 egg: ① start in middle

Worse case: $1 + \frac{N}{2}$ total

② go up 2 floors

Worse case: $\frac{N}{2} + 1$ total

maybe have a minus 1

Meet in middle: go up X floors

Worse case: $\frac{N}{X} + X - 1$

did $X = N/2$, $X = 2$

$g(x) = \frac{N}{x} + x$: Where is this smallest?

$g'(x) = -\frac{N}{x^2} + 1$ Critical point is $x = \sqrt{N}$

OB: Equalize: try $\frac{N}{x} = x \Rightarrow x = \sqrt{N}$

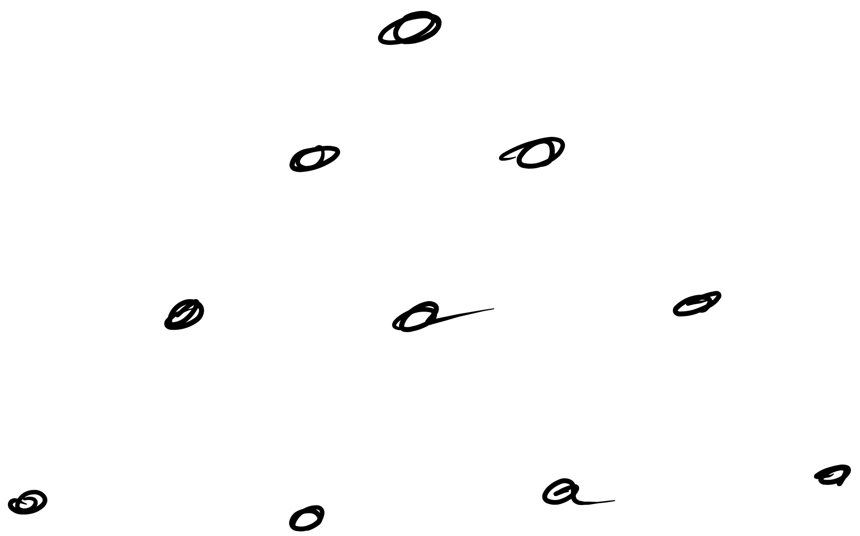
If first egg doesn't break, starting new game with $N-x$ floors

"like" memoryless hoops game

$g(\sqrt{N}) = 2\sqrt{N}$

Crack after 1 drop: $1 + x-1$ drops total

Crack after 2 drops $2 + x-2$ drops total



Triangle Numbers

3 eggs:

Group by X

Worse case: $\frac{N}{X} + \left(\begin{array}{l} X \text{ floors with} \\ 2 \text{ eggs} \end{array} \right)$

# eggs	$X(N)$
2	$N^{1/2}$
3	$N^{2/3}$
n	$N^{\frac{n-1}{n}}$

$$= \frac{N}{X} + 2\sqrt{X}$$ minimize

(1) $\frac{N}{X} = 2\sqrt{X} \Rightarrow \frac{N}{2} = X^{3/2}$ so $X \approx \text{Const} \cdot N^{2/3}$

(2) C.P. $0 = -\frac{N}{X^2} + X^{-\frac{1}{2}} \Rightarrow X = N^{2/3}$

$N = X^{3/2}$