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- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).

Plot of 100 most populous cities





Definition of Logarithms

- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and *e* for calculus; many sources write ln *x* for the natural logarithm of *x*, which is its logarithm base *e* (*e* is approximately 2.71828).

• Examples: $\log_b x = y$ means we need y powers of b to get x.

- $100 = 10^2$ becomes $\log_{10} 100 = 2$. In base *e* it is about 4.6.
- $1 = 10^{0}$ becomes $\log_{10} 1 = 0$. In base *e* it is still 0.
- $.001 = 10^{-3}$ becomes $\log_{10} .001 = -3$. In base *e* it is about -6.9.

Order of Magnitude of some			Length of Lake Erie	Length of Lake Erie			
LENGTH	meters		Diameter of red blood	NI			galaxy
radius of proton	10 ⁻¹⁵	Diameter of nuclear	corpuscie		Radius of	first star beyond	
radius of atom	10 ⁻¹⁰	particles	9	AND A STREET STREET	6	sun 4	ana da interna
radius of virus	10-7	Diamotor	Trans 1977	n n n n n n n n n n n n n n n n n n n	9	r	Vin Longo
radius of amoeba	10 ⁻⁴	of atom		M	the the des	un sam	Allenzy C al-
height of human being	10 ⁰	-15 -13 -11 -9	-7 -5 -3 -	10+1 3 5 7	9 11 13	15 17 19	21 23 25 27
radius of earth	107						
radius of sun	109	A Card Date (Salary)	1 or	1.		Man most in	damin's license
earth-sun distance	10 ¹¹	Wavelength	1 mm	Y NO	1	light your	Margine and
radius of solar system	10 ¹³	of X ray	1µm	Length at	e solari dadi		and bottom
distance of sun to nearest star	10 ¹⁶	and some set of the	VVVC Navelength	whate (D)	1	15	Y
radius of milky way galaxy	10 ²¹		of light	De la		Diam	eter of
radius of visible Universe	10 ²⁶			Radius		our M Gi	liky Way Ilaxy 4

Earthquake frequency and destructive power

The left side of the chart shows the magnitude of the earthquake and the right side represents the amount of high explosive required to produce the energy released by the earthquake. The middle of the chart shows the relative frequencies.

Mag	nitude	Notable earthquakes	Fnergy equivalents	(equivalent of explosive)
10000000		notable cal inquinco	Life By equivalence	 123 trillion lb.
10 -		Chile (1960)		(56 trillion kg)
1000		Alaska (1964)		4 trillion lb.
8 -	Great earthquake: near total	Japan (2011) 🖤		(1.8 trillion kg)
	doctruction monohio loss of life	Many Modeld May (4040)	Krakatoa volcanic eruption	
	destruction, massive loss of me	New Madrid, Mo. (1812)	World's largest nuclear fact (11990)	123 billion lb.
8 -	Major earthquake; severe eco-	San Francisco (1906) 🛉 °	Mount St. Helens eruption	(56 billion kg)
	nonne impact, large loss of me	Loma Prieta, Calif. (1989)		4 billion lb.
(-	Strong earthquake; damage	Kobe, Japan (1995) 2 Northridge, Calif. (1994)	7	(1.8 billion kg)
10000	(\$ Dillions), loss of life		🖕 Hiroshima atomic bomb	123 million lb.
6 -	Moderate earthquake;	Long Island, N.Y. (1884)		(56 million kg)
2003	property damage	00	00	4 million lb.
0	Light earthquake;	- 40	Average tornado	(1.8 million kg)
13272	some property damage	107		12,300 lb.
4 -	Minor earthquake;	12,		(56,000 kg)
Sec.	felt by humans		Large lightning bolt	4 000 lb
3 -	-	100,	000 Oklahoma City bombing	- (1 800 ka)
3283			Moderate lightning	holt (1,000 kg)
			inductorie inglianing	122 lb
2 -	-	1,000	000	- (56 km)
Sector Sector				(30 kg)
	5	Number of earthquakes	s per year (worldwide)	
Source	a-115 Cantonical Survey			MCT



The pH Scale



shutterstock.com · 1216954318

Recall: Definition of Logarithms

- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and *e* for calculus; many sources write ln *x* for the natural logarithm of *x*, which is its logarithm base *e* (*e* is approximately 2.71828).

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Plots of Exponentiation and Logarithms

- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y.



- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).







• Linearize many non-linear functions (calculus becomes available).



Notice that even on a small range, from 1 to 10, the polynomial of highest degree drowns out the others and can barely see.

• Linearize many non-linear functions (calculus becomes available).



Left: Semi-log plot: $y = \log x^r$. Right: log-log plot: $\log y = \log x^r$. Note that we can now see the four functions on one plot, and the log-log plot now has linear relations.

Review: Exponent Laws

Laws

- $b^m b^n = b^{m+n}$
- $b^m / b^n = b^{m-n}$
- $(b^m)^n = b^{mn}$

Examples

- $10^3 10^2 = (10 * 10 * 10) * (10 * 10) = 10^5$
- $10^3/10^2 = (10 * 10 * 10)/(10 * 10) = 10^1$
- $(10^3)^2 = 10^3 * 10^3 = (10 * 10 * 10) * (10 * 10 * 10) = 10^6$

Logarithm Laws

Parts of a Slide Rule



- Remember if $x = b^y$ then $\log_b x = y$.
- Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.
- These allow us to simplify computations with logarithms.

THEOREM

• $\log_b(x^n) = n \log_b x$. • $\log_b(x_1 x_2) = \log_b(x_1) + \log_b(x_2)$. Log of a product is the sum of the logs. • $\log_b(x_1/x_2) = \log_b(x_1) - \log_b(x_2)$. Log of a quotient is the difference of the logs. • $\log_b x = \log_c x/\log_c b$. If know logs in one base, know in all.

OPTIONAL – PROOFS OF THE LOG LAWS



Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

• $\log_b(x^n) = n \log_b x$, Log of a power is that power times the log.

•
$$\log_b x = y$$
 means $x = b^y$.

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

• $\log_b(x^n) = n \log_b x$, Log of a power is that power times the log.

Proof:

•
$$\log_b x = y$$
 means $x = b^y$.

• Thus $x^n = (b^y)^n = b^{ny}$.

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

• $\log_b(x^n) = n \log_b x$, Log of a power is that power times the log.

•
$$\log_b x = y$$
 means $x = b^y$.

- Thus $x^n = (b^y)^n = b^{ny}$.
- Taking logarithms: $\log_b(xn) = ny = n \log_b x$.

- Remember if $x = b^{y}$ then $\log_{b} x = y$.
- Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.
- $\log_b(x_1 x_2) = \log_b(x_1) + \log_b(x_2)$. Log of a product is the sum of the logs.

Proof:

• As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.

- Remember if $x = b^{y}$ then $\log_{b} x = y$.
- Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.
- $\log_b(x_1 x_2) = \log_b(x_1) + \log_b(x_2)$. Log of a product is the sum of the logs.

Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.
- Thus $x_1 x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.

$$\label{eq:second} \begin{split} & (s,t)^{2} \operatorname{Implication}_{\mathcal{D}_{1}}(s,t), \\ & \operatorname{Implication}_{\mathcal{D}_{2}}(s,t), \\ & \operatorname{Implication}_{\mathcal{D}_{2}}(s,t),$$

- Remember if $x = b^{y}$ then $\log_{b} x = y$.
- Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.
- $\log_b(x_1 x_2) = \log_b(x_1) + \log_b(x_2)$. Log of a product is the sum of the logs.

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.
- Thus $x_1 x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.
- Therefore $\log_b(x_1 x_2) = y_1 + y_2 = \log_b x_1 + \log_b x_2$.

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

• $\log_b x = \log_c x / \log_c b$. Know logs in one base, know in all.

• As
$$\log_b x = y$$
 have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

• $\log_b x = \log_c x / \log_c b$. Know logs in one base, know in all.

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^y = (c^v)^y = cv^y$.

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

• $\log_b x = \log_c x / \log_c b$. Know logs in one base, know in all.

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^{y} = (c^{v})^{y} = cv^{y}$.
- As also have $x = c^u$ we have u = vy or y = u/v.

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

• $\log_b x = \log_c x / \log_c b$. Know logs in one base, know in all.

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^{y} = (c^{v})^{y} = cv^{y}$.
- As also have $x = c^u$ we have u = vy or y = u/v.
- Substituting gives $\log_b x = \log_c x / \log_c b$.

```
Example: Factorial Function:
Number ways to order n objects when order matters:
n! = n * (n - 1) * \cdots * 3 * 2 * 1.
```

```
list = {}; semiloglist = {}; logloglist = {};
For[n = 1, n <= 200, n++,</pre>
```

```
list = AppendTo[list, {n, n!}];
```

```
semiloglist = AppendTo[semiloglist, {n, Log[n!]}];
```

```
logloglist = AppendTo[logloglist, {Log[n], Log[n!]}];
```

}];

```
Print[ListPlot[list]]; Print[ListPlot[semiloglist]]; Print[ListPlot[logloglist]];
```

Example: Factorial Function: Number ways to order *n* objects when order matters: $n! = n * (n - 1) * \cdots * 3 * 2 * 1.$



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Steven J. Miller, Williams College Steven.J.Miller@williams.edu

http://web.williams.edu/Mathematics/sjmiller/public_html/

ATMIM Spring Conference Assabet Valley Regional Technical High School, 3/23/13



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These slides are from the keynote address at the 2013 Spring Conference of ATMIM. If you are interested in using any of these topics (or anything from the math riddles page) in your class, please email me at sjm1@williams.edu, and I am happy to talk with you about implementation.

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Than	۲S				

Wanted to thank many people who encouraged me and provided opportunities.

- Parents and brother.
- Math teachers from preschool to graduate school.
- Colleagues and students for many discussions.
- Henry Bolton (henry.bolter@gmail.com) from Teachers As Scholars: http://www.teachersasscholars.org/
- Mr. Anthony for stepping up so many times.



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Some	Some Issues for the Future								

- World is rapidly changing powerful computing cheaply and readily available.
- What skills are we teaching? What skills should we be teaching?
- One of hardest skills: how to think / attack a new problem, how to see connections, what data to gather.

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Goals of the Talk: Opportunities Everywhere!

- Ask Questions! Often simple questions lead to good math.
- Gather data: observe, program and simulate.
- Use games to get to mathematics.
- Discuss implementation: Please interrupt!

Joint work with Cameron (age 6) and Kayla (age $4 - 2\epsilon$) Miller

My math riddles page: http://mathriddles.williams.edu/

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The M&M Game

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Motiv	ating Quest	ion			

Cam (4 years): If you're born on the same day, do you die on the same day?

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M&M Game Rules

Cam (4 years): If you're born on the same day, do you die on the same day?



(1) Everyone starts off with *k* M&Ms (we did 5).(2) All toss fair coins, eat an M&M if and only if head.



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Be ac	tive – ask o	questions!			

What are natural questions to ask?
Intro 0000	M&M Game: I ○●○○○○	Hoops Game	M&M Game: II 000000000000000	Takeaways o	Appendix: Generating Fns
Be ac	tive – ask q	uestions!			

What are natural questions to ask?

Question 1: How likely is a tie (as a function of *k*)?

Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiousity is good and to be encouraged! Value to the journey and not knowing the answer.

Let's gather some data!

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Probability of a tie in the M&M game (2 players)



 $Prob(tie) \approx 33\%$ (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

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Probability of a tie in the M&M game (2 players)



But we're celebrating 110 years of service, so....

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Probability of a tie in the M&M game (2 players)



... where will the next 110 bring us? Never too early to lay foundations for future classes.

Intro 0000	M&M Game: I ○○○●○○	Hoops Game	M&M Game: II ০০০০০০০০০০০০০০০০	Takeaways o	Appendix: Generating Fns

Welcome to Statistics and Inference!

- ◊ Goal: Gather data, see pattern, extrapolate.
- Methods: Simulation, analysis of special cases.
- Presentation: It matters how we show data, and which data we show.

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Viewi	na M&M Pla	te			



Hard to predict what comes next.

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Viewing M&M Plots: Log-Log Plot



Not just sadistic teachers: logarithms useful!

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Viewing M&M Plots: Log-Log Plot



Best fit line:

 $\log (\text{Prob(tie)}) = -1.42022 - 0.545568 \log (\#\text{M\&Ms}) \text{ or } \text{Prob}(k) \approx 0.2412/k^{.5456}.$

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Viewing M&M Plots: Log-Log Plot



Best fit line:

 $\log (\text{Prob(tie)}) = -1.42022 - 0.545568 \log (\#\text{M\&Ms}) \text{ or } \text{Prob}(k) \approx 0.2412/k^{.5456}.$

Predicts probability of a tie when k = 220 is 0.01274, but answer is 0.0137. What gives?

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Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from k = 50 to 110.



Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from k = 50 to 110.



Best fit line:

 $\log (\text{Prob}(\text{tie})) = -1.58261 - 0.50553 \log (\#\text{M}\&\text{Ms}) \text{ or }$ $\operatorname{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).



Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from k = 50 to 110.



Best fit line:

 $\log (\text{Prob}(\text{tie})) = -1.58261 - 0.50553 \log (\#\text{M}\&\text{Ms}) \text{ or }$ $\operatorname{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Get 0.01344 for *k* = 220 (answer 0.01347); much better!

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From Shooting Hoops to the Geometric Series Formula

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Simpl	er Game: H	loops			

Game of hoops: first basket wins, alternate shooting.



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Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- Bird always gets basket with probability *p*.
- Magic always gets basket with probability q.

Let *x* be the probability **Bird** wins – what is *x*?

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Solvii	ng the Hoo	p Game			

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Solvii	ng the Hoop	o Game			

Break into cases:

• **Bird** wins on 1st shot: *p*.

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- **Bird** wins on 1st shot: *p*.
- Bird wins on 2^{nd} shot: $(1 p)(1 q) \cdot p$.

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Solving the Hoop Game

Classic solution involves the geometric series.

- **Bird** wins on 1st shot: *p*.
- Bird wins on 2^{nd} shot: $(1 p)(1 q) \cdot p$.
- Bird wins on 3^{rd} shot: $(1-p)(1-q) \cdot (1-p)(1-q) \cdot p$.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Solving the Hoop Game

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- **Bird** wins on 1st shot: *p*.
- Bird wins on 2^{nd} shot: $(1 p)(1 q) \cdot p$.
- Bird wins on 3^{rd} shot: $(1 p)(1 q) \cdot (1 p)(1 q) \cdot p$.
- Bird wins on nth shot:

$$(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p.$$

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Solvi	ng the Hoon	Game			

Break into cases:

- **Bird** wins on 1st shot: *p*.
- Bird wins on 2^{nd} shot: $(1-p)(1-q) \cdot p$.
- Bird wins on 3^{rd} shot: $(1-p)(1-q) \cdot (1-p)(1-q) \cdot p$.

• Bird wins on nth shot:

$$(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p.$$

Let r = (1 - p)(1 - q). Then

$$= \operatorname{Prob}(\operatorname{Bird wins})$$

$$= \rho + r\rho + r^2\rho + r^3\rho + \cdots$$

$$= \rho \left(1 + r + r^2 + r^3 + \cdots\right),$$

the geometric series.

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Showed

x = Prob(**Bird** wins) =
$$p(1 + r + r^2 + r^3 + \cdots)$$
;

will solve without the geometric series formula.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Showed

x = Prob(**Bird** wins) =
$$p(1 + r + r^2 + r^3 + \cdots)$$
;

will solve without the geometric series formula.

Have

 $\mathbf{X} = \text{Prob}(\text{Bird wins}) = \mathbf{p} + \mathbf{Prob}(\mathbf{Prob})$

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Showed

x = Prob(**Bird** wins) =
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Have

$$\mathbf{x} = \operatorname{Prob}(\operatorname{Bird} \operatorname{wins}) = \mathbf{p} + (1 - \mathbf{p})(1 - q)$$

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Showed

x = Prob(**Bird** wins) =
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Have

$$\mathbf{x} = \operatorname{Prob}(\operatorname{Bird} \operatorname{wins}) = \mathbf{p} + (1 - \mathbf{p})(1 - q)\mathbf{x}$$

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Showed

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Have

$$\mathbf{x} = \operatorname{Prob}(\operatorname{Bird} \operatorname{wins}) = \mathbf{p} + (1 - \mathbf{p})(1 - q)\mathbf{x} = \mathbf{p} + r\mathbf{x}.$$

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Showed

$$\mathbf{x} = \operatorname{Prob}(\operatorname{Bird} \operatorname{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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$$\mathbf{x} = \operatorname{Prob}(\operatorname{Bird} \operatorname{wins}) = \mathbf{p} + (1 - \mathbf{p})(1 - q)\mathbf{x} = \mathbf{p} + r\mathbf{x}.$$

Thus

$$(1-r)\mathbf{x} = \mathbf{p}$$
 or $\mathbf{x} = \frac{\mathbf{p}}{1-r}$.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Showed

$$x = Prob(Bird wins) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$\mathbf{x} = \operatorname{Prob}(\operatorname{Bird} \operatorname{wins}) = \mathbf{p} + (1 - \mathbf{p})(1 - q)\mathbf{x} = \mathbf{p} + r\mathbf{x}.$$

Thus

$$(1-r)\mathbf{x} = \mathbf{p}$$
 or $\mathbf{x} = \frac{\mathbf{p}}{1-r}$.

As
$$\mathbf{x} = p(1 + r + r^2 + r^3 + \cdots)$$
, find
 $1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}$.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Lesso	ons from Ho	op Problen	n		

- o Power of Perspective: Memoryless process.
- Can circumvent algebra with deeper understanding! (Hard)
- Output of a problem not always what expect.
- Importance of knowing more than the minimum: connections.
- Math is fun!

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns

The M&M Game

Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II ●○○○○○○○○○○○○	Takeaways o	Appendix: Generating Fns

Solving the M&M Game

Overpower with algebra: Assume *k* M&Ms, two people, fair coins:

Prob(tie) =
$$\sum_{n=k}^{\infty} {\binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}} \cdot {\binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II ●○○○○○○○○○○○○	Takeaways o	Appendix: Generating Fns

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is a binomial coefficient.

"Simplifies" to $4^{-k} {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function! (Look up / write report.)

Obviously way beyond the classroom - is there a better way?

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.

Each person has exactly k - 1 heads in first n - 1 tosses, then ends with a head.

Prob(tie) =
$$\sum_{n=k}^{\infty} {\binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}} \cdot {\binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}}.$$



Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Use the lesson from the Hoops Game: Memoryless process!

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Use the lesson from the Hoops Game: Memoryless process!

If neither eat, as if toss didn't happen. Now game is finite.
Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

If neither eat, as if toss didn't happen. Now game is finite.

Much better perspective: each "turn" one of three equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is 1/3 or about 33%

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^n \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}^{\frac{1}{3}}.$$



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Solving the M&M Game (cont)

Interpretation: Let Cam have c M&Ms and Kayla have k; write as (c, k).

Then each of the following happens 1/3 of the time after a 'turn':

•
$$(c,k) \longrightarrow (c-1,k-1).$$

• $(c,k) \longrightarrow (c-1,k).$
• $(c,k) \longrightarrow (c-1,k).$

•
$$(c,k) \longrightarrow (c,k-1).$$



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Figure: The M&M game when k = 4. Count the paths! Answer 1/3 of probability hit (1,1).

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Figure: The M&M game when k = 4, going down one level.

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Figure: The M&M game when k = 4, removing probability from the second level.

Intro 0000	M&M Game: I 000000	Hoops	Game	M&M Game: II	T 0 000	akeaways	Appendix: G	enerating Fns
Solvi	ng the M&	M Gam	ne (cont	:): Assum	ne <i>k</i> =	4		
				(4,4)				
				\cap	\cap			
			(4,3)	(3,3)	(3,4)			
		• -	\sim	\circ	$\bigcirc \blacktriangleleft$			
		(4,2)	(3,2)	(2,2) 5 (2,7	(2,3) 0/27	(2,4)		
	1	1 I	9/2/	5/27	9/2/	1	4	
		¥.					•	
	\circ	\bigcirc	\bigcirc	\circ	\bigcirc	\circ	\circ	
	(4,1)	(3,1)	(2,1)	(1,1)	(1,2)	(1,3)	(1,4)	

Figure: Removing probability from two outer on third level.

1/27

1/27

1/27

1/27

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Figure: Removing probability from the (3,2) and (2,3) vertices.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Figure: Removing probability from the (2,2) vertex.

Intro 0000	M&M Game: I 000000	Hoops G 00000	Game N	//&M Game: II	Tak 000 0	eaways	Appendix: Generating Fns
Solvi	ng the M&I	M Gam	e (cont)): Assum	e <i>k</i> = 4		
				(4,4)			
			(4,3)	(<u>3.</u> 3)	(3,4)		
	((4,2)	(3,2)	(2,2)	(2,3)	(2,4)	
	(4,1)	(3,1)	(2,1)	(1,1)	(1,2)	(1,3)	• • (1.4)

Figure: Removing probability from the (4,1) and (1,4) vertices.

33/243

60/243

13/81

13/81

60/243



Figure: Removing probability from the (3,1) and (1,3) vertices.

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Solvii	ng the M&N	/I Game (co	ont): Assume k	= 4	



Figure: Removing probability from (2,1) and (1,2) vertices. Answer is 1/3 of (1,1) vertex, or 245/2187 (about 11%).

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis:
$$F_{n+2} = F_{n+1} + F_n$$
 with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, http://www.youtube.com/watch?v=kkGeOWYOFoA.

Binet's Formula (can prove via 'generating functions'):

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Interpreting Proof: Connections to the Fibonacci Numbers!

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M&Ms: For $c, k \ge 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \ge 1$:

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Interpreting Proof: Finding the Recurrence

What if we didn't see the 'simple' recurrence?

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}$$

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The following recurrence is 'natural':

$$x_{c,k} = \frac{1}{4}x_{c,k} + \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}.$$

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The following recurrence is 'natural':

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Obtain 'simple' recurrence by algebra: subtract $\frac{1}{4}x_{c,k}$:

$$\frac{3}{4}x_{c,k} = \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}$$

therefore $x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}$.

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}$$

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}$$

•
$$x_{0,0} = 1$$
.

Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II ○○○○○○●○○○○○○	Takeaways o	Appendix: Generating Fns

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}$$

• *x*_{0,0} = 1.

•
$$x_{1,0} = x_{0,1} = 0.$$

• $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%.$

Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II ○○○○○○●○○○○○○	Takeaways o	Appendix: Generating Fns

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}$$

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$$x_{1,0} = x_{0,1} = 0.$$

• $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%.$

•
$$x_{2,0} = x_{0,2} = 0.$$

• $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} = x_{1,2}.$
• $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} = \frac{5}{27} \approx 18.5\%.$

Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II ○○○○○○○●○○○○○	Takeaways ○	Appendix: Generating Fns
Try Si	mpler Case	s!!!			

Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II ○○○○○○○●○○○○○	Takeaways o	Appendix: Generating Fns
Try Si	mpler Case	s!!!			

Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.



Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.

Generalization of the Catalan problem. There don't have (1,1) and stay on or below the main diagonal.



Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.

Generalization of the Catalan problem. There don't have (1,1) and stay on or below the main diagonal.



Interpretation: Catalan numbers are valid placings of (and).

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Aside: Fun Riddle Related to Catalan Numbers

Young Saul, a budding mathematician and printer, is making himself a fake ID. He needs it to say he's 21. The problem is he's not using a computer, but rather he has some symbols he's bought from the store, and that's it. He has one 1, one 5, one 6, one 7, and an unlimited supply of + - * / (the operations addition, subtraction, multiplication and division). Using each number exactly once (but you can use any number of +, any number of -, ...) how, oh how, can he get 21 from 1,5, 6,7? Note: you can't do things like 15+6 = 21. You have to use the four operations as 'binary' operations: ((1+5)*6) + 7. Problem submitted by ohadbp@infolink.net.il, phrasing by yours truly.

Solution involves valid sentences: $((w + x) + y) + z, w + ((x + y) + z), \dots$

For more riddles see my riddles page: http://mathriddles.williams.edu/.

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Examining Probabilities of a Tie

When
$$k = 1$$
, Prob(tie) = 1/3.

```
When k = 2, Prob(tie) = 5/27.
```

```
When k = 3, Prob(tie) = 11/81.
```

When k = 4, Prob(tie) = 245/2187.

When k = 5, Prob(tie) = 1921/19683.

When k = 6, Prob(tie) = 575/6561.

When k = 7, Prob(tie) = 42635/531441.

When k = 8, Prob(tie) = 355975/4782969.

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Examining Ties: Multiply by 3^{2k-1} to clear denominators.

When k = 1, get 1.

When k = 2, get 5.

When k = 3, get 33.

When k = 4, get 245.

When k = 5, get 1921.

When k = 6, get 15525.

When k = 7, get 127905.

When k = 8, get 1067925.

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OEIS					

Get sequence of integers: 1, 5, 33, 245, 1921, 15525,

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OEIS					

Get sequence of integers: 1, 5, 33, 245, 1921, 15525,

```
OEIS: http://oeis.org/.
```

Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II ○○○○○○○○○○○●○	Takeaways o	Appendix: Generating Fns
OEIS					

```
Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....
```

```
OEIS: http://oeis.org/.
```

Our sequence: http://oeis.org/A084771.

The web exists! Use it to build conjectures, suggest proofs....

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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OEIS (continued)

4084771	Coefficients of $1/sort(1, 10*v+0*v^2)$; also, $a(n)$ is the central coefficient of $(1+5*v+4*v^2)^n$
1 5 00	245 1921 15525 122005 1062925 0004545 76490525 653808673 5614005265
484164545	29, 118895174885, 3634723102113, 31616937184725, 275621102802945,
240733194	1640325,21061836725455905,184550106298084725 (<u>list; graph; refs; listen; history; text; internal</u>
format)	
OFFSET	0,2
COMMENTS	Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. N= \underline{N} - \underline{E} - \underline{E} - \underline{R} - \underline{hass}_{1} , Mar 30 2008 Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and
	<pre>three kinds of steps (1,1). [obsry Arnot, out of 2014] Sums of squares of coefficients of (1-2*x)^n. [Jeerg Arnot, Jul 06 2011] The Hankel transform of this sequence gives <u>Al03488</u> <u>Philippe DELEHAM</u>, Dec 02 2007</pre>
REFERENCES	Faul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Faths, Journal of Integer Sequences, Vol. 15, 2012, 412.4.8 From M. J. A. Sloane, Oct 08 2012 Michael Z. Spirey and Laura L. Steil, The k-Binomial Transforms and the Michael Z. Spirey and Laura L. Steil, The k-Binomial Transforms and the Michael Z. Spirey and Laura L. Steil, The k-Binomial Transforms and the Michael Z. Spirey and Laura L. Steil, The k-Binomial Transforms and the Michael Z. Spirey and Laura L. Steil, The k-Binomial Transforms and the Michael Z. Spirey and Laura L. Steil, The k-Binomial Transforms and the Michael Schuler Schul
	HARKEI FRANSFORM, JOURNAI OF INTEGER SEQUENCES, VOL. 9 (2006), Article 06.1.1.
LINKS	Table of n, a(n) for n=019.
	Tony D. Noe, <u>On the Divisibility of Generalized Central Trinomial</u> <u>Coefficients</u> , Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.
FORMULA	G.f.: 1/sqrt(1-10*x+9*x^2).
	$ \begin{array}{l} \texttt{Binomial transform of \underline{\texttt{A059301}}, G.f.: \texttt{Sum} (k \!\!>\!\! \texttt{0}) \texttt{ binomial}(2^{\star}k, k)^{\star} \\ (2^{\star}x)^{\star}/(1\!\!-\!x)^{\star}(k\!\!+\!\!1), \texttt{ E.g.f.: } \texttt{sup}(5^{\star}x)^{\star}\texttt{Bessell}(0, 4^{\star}x). & - \texttt{Vladeta Jovovic} \\ (\texttt{vladeta}(\texttt{AT})\texttt{eunet.rs}), \texttt{ Aug 20 2003} \end{array} $
	a(n) = sum(k=0n, sum(j=0n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j))) Paul Barry, May 19 2006
	<pre>a(n) = sum(k=0n, 4^k*(C(n,k))^2) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]</pre>
	<pre>Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [<u>Vaclav Kotesovec</u>, Sep 11 2012]</pre>
	Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0 R. J. Mathar,

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Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns

Takeaways



Intro 0000	M&M Game: I 000000	Hoops Game	M&M Game: II 00000000000000	Takeaways ●	Appendix: Generating Fns
Lesso	ons				

- Always ask questions.
- Many ways to solve a problem.
- Sector Sector
- Need to look at the data the right way.
- Often don't know where the math will take you.
- Value of continuing education: more math is better.

Ocnnections: My favorite quote: If all you have is a hammer, pretty soon every problem looks like a nail.

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Generating Functions

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Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1; \ F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{-1+\sqrt{5}}{2} \right)^n \right].$$

(1)

• Recurrence relation:
$$\boldsymbol{F}_{n+1} = \boldsymbol{F}_n + \boldsymbol{F}_{n-1}$$

• Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$(1) \Rightarrow \sum_{n\geq 2} \mathbf{F}_{n+1} \mathbf{x}^{n+1} = \sum_{n\geq 2} \mathbf{F}_n \mathbf{x}^{n+1} + \sum_{n\geq 2} \mathbf{F}_{n-1} \mathbf{x}^{n+1}$$
$$\Rightarrow \sum_{n\geq 3} \mathbf{F}_n \mathbf{x}^n = \sum_{n\geq 2} \mathbf{F}_n \mathbf{x}^{n+1} + \sum_{n\geq 1} \mathbf{F}_n \mathbf{x}^{n+2}$$
$$\Rightarrow \sum_{n\geq 3} \mathbf{F}_n \mathbf{x}^n = \mathbf{x} \sum_{n\geq 2} \mathbf{F}_n \mathbf{x}^n + \mathbf{x}^2 \sum_{n\geq 1} \mathbf{F}_n \mathbf{x}^n$$
$$\Rightarrow g(\mathbf{x}) - \mathbf{F}_1 \mathbf{x} - \mathbf{F}_2 \mathbf{x}^2 = \mathbf{x}(g(\mathbf{x}) - \mathbf{F}_1 \mathbf{x}) + \mathbf{x}^2 g(\mathbf{x})$$
$$\Rightarrow g(\mathbf{x}) = \mathbf{x}/(1 - \mathbf{x} - \mathbf{x}^2).$$

Intro	M&M Game: I	Hoops Game	M&M Game: II	Takeaways	Appendix: Generating Fns
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Partial Fraction Expansion (Example: Binet's Formula)

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x} \right).$$

Coefficient of *x*^{*n*} (power series expansion):

$$\boldsymbol{F}_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{-1+\sqrt{5}}{2} \right)^n \right] \text{ - Binet's Formula!}$$
(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).