# Math/Stat 341 and Math 433
# Probability and Mathematical Modeling I: Discrete Systems

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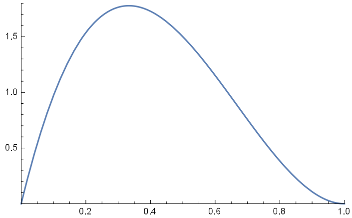
## Goal

- Quickly review some probability.
- Introduction to Difference Equations.
- Solving Difference Equations.
- Roulette.

# Background

## Probability Review




- Let $X$ be random variable with density $p(x)$:
  - ◇ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
  - ◇ $\mathrm{Prob}\,(a \leq X \leq b) = \int_a^b p(x)\mathrm{d}x$.

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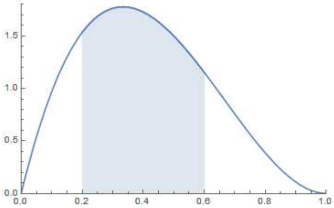


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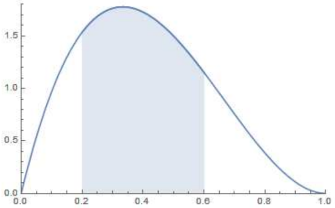


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- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.

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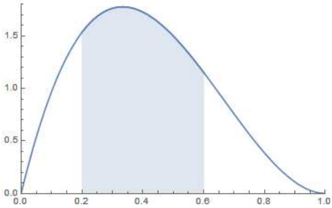


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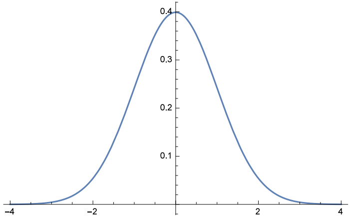
- Mean $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.
- Variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.
- Independence: knowledge of one random variable gives no knowledge of the other.

## Central Limit Theorem

$$\text{Normal } N(\mu, \sigma^2): \; p(x) \;=\; e^{-(x-\mu)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}.$$



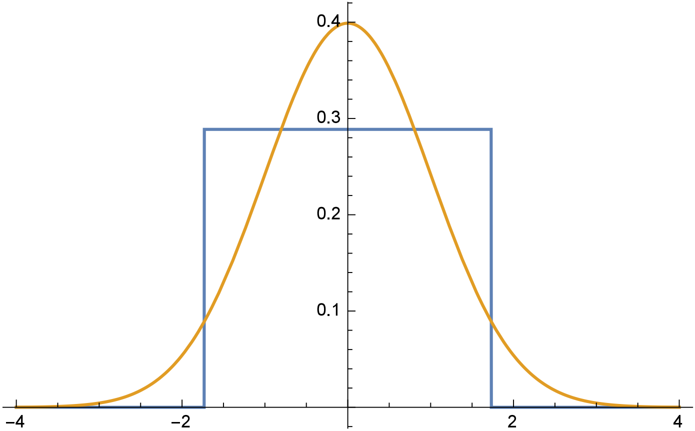

### Theorem

If $X_1, X_2, \dots$ independent, identically distributed random variables (mean $\mu$, variance $\sigma^2$, finite moments) then

$$S_N \;:=\; \frac{X_1 + \cdots + X_N - N\mu}{\sigma\sqrt{N}} \text{ converges to } N(0,1).$$

# Central Limit Theorem: Sums of Uniform Random Variables
## $X_i \sim \mathrm{Unif}(-1/2, 1/2)$

$$Y_1 = X_1/\sigma_{X_1} \text{ vs } N(0,1).$$



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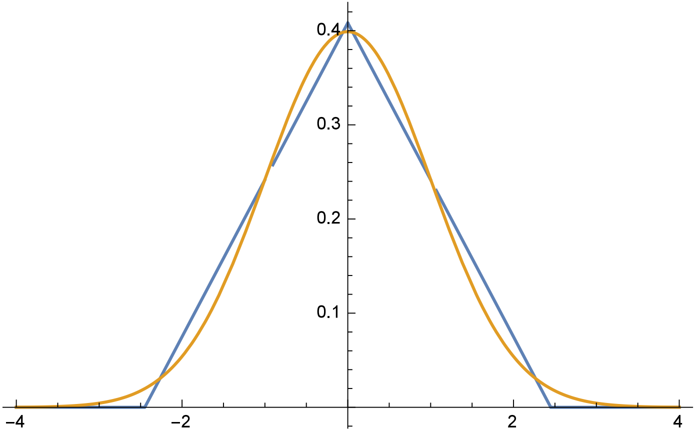
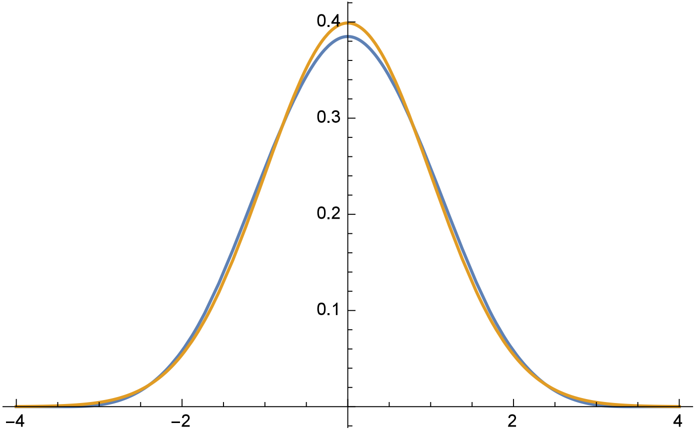


# Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$

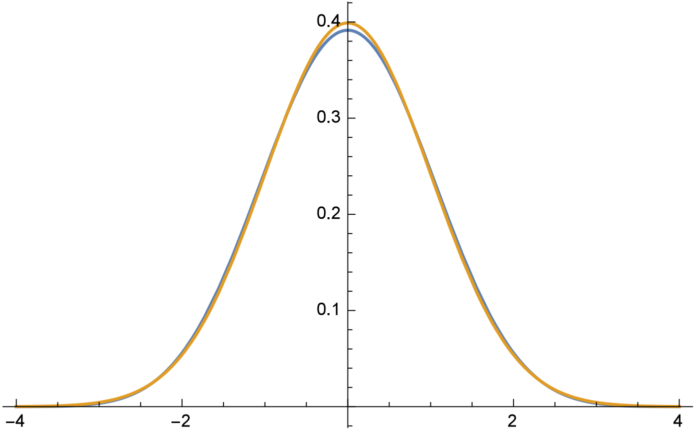
$Y_4 = (X_1 + X_2 + X_3 + X_4)/\sigma_{X_1+X_2+X_3+X_4}$ vs $N(0,1)$.



# Central Limit Theorem: Sums of Uniform Random Variables

$X_i \sim \mathrm{Unif}(-1/2, 1/2)$

$$Y_8 = (X_1 + \cdots + X_8)/\sigma_{X_1+\cdots+X_8} \text{ vs } N(0,1).$$



## Central Limit Theorem: Sums of Uniform Random Variables $X_i \sim \text{Unif}(-1/2, 1/2)$

Density of $Y_4 = (X_1 + \cdots + X_4)/\sigma_{X_1+\cdots+X_4}$.

$$\frac{\begin{cases} \frac{1}{27}\left(18 + 9\sqrt{3}\,y - \sqrt{3}\,y^3\right) & y == 0 \\ \frac{1}{18}\left(12 - 6\,y^2 - \sqrt{3}\,y^3\right) & -\sqrt{3} < y < 0 \\ \frac{1}{54}\left(72 - 36\sqrt{3}\,y + 18\,y^2 - \sqrt{3}\,y^3\right) & \sqrt{3} < y < 2\sqrt{3} \\ \frac{1}{54}\left(18\sqrt{3}\,y - 18\,y^2 + \sqrt{3}\,y^3\right) & y == \sqrt{3} \\ \frac{1}{18}\left(12 - 6\,y^2 + \sqrt{3}\,y^3\right) & 0 < y < \sqrt{3} \\ \frac{1}{54}\left(72 + 36\sqrt{3}\,y + 18\,y^2 + \sqrt{3}\,y^3\right) & -2\sqrt{3} < y \le -\sqrt{3} \\ 0 & \text{True} \end{cases}}{\sqrt{3}}$$

(Don't even think of asking to see $Y_8$'s!)

# Introduction to Difference Equations

Discrete version of differential equations (discrete time step).

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- Can compute but expensive....

Consider Fibonacci numbers:

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}.$$

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Leads to matrix formulation:

$$\overrightarrow{v}_{n+1} = A\overrightarrow{v}_n$$

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Can now use linear algebra to solve. In general if matrix is diagonalizable with eigenvalues $\lambda_i$ and eigenvectors $\overrightarrow{u}_i$, there are $c_i$ such that

$$\overrightarrow{v}_{n+1} = c_1\lambda_1^n\overrightarrow{u}_1 + \cdots + c_L\lambda_L^n\overrightarrow{u}_2.$$

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Binet's Formula:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

**Imagine population of whales with following assumptions:**

- Always die when turn four, never earlier.
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Can we figure out how many whales of each age at each moment? Yes: Deterministic!

Will set up a system to describe population.

- $a_n$: number of pairs born in year $n$.
- $b_n$: number of pairs of 1 year olds in year $n$.
- $c_n$: number of pairs of 2 year olds in year $n$.
- $d_n$: number of pairs of 3 year olds in year $n$.

## Leslie Matrices: III

Use information to set up system:

- $a_{n+1} = 2b_n + 1c_n$.
- $b_{n+1} = a_n$.
- $c_{n+1} = b_n$.
- $d_{n+1} = c_n$.

Use information to set up system:

- $a_{n+1} = 0a_n + 2b_n + 1c_n + 0d_n$.
- $b_{n+1} = 1a_n + 0b_n + 0c_n + 0d_n$.
- $c_{n+1} = 0a_n + 1b_n + 0c_n + 0d_n$.
- $d_{n+1} = 0a_n + 0b_n + 1c_n + 0d_n$.

Use information to set up system:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} = A^{n+1} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}.$$

Can solve exactly! Call above a Leslie matrix:
http://en.wikipedia.org/wiki/Leslie_matrix

What properties does $A$ have? Is this form reasonable?

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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```
In[1]:= A = {{0, 2, 1, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}};
        Eigenvalues[A]
        Eigenvectors[A]
```

Out[2]= $\left\{\frac{1}{2}\left(1+\sqrt{5}\right), -1, \frac{1}{2}\left(1-\sqrt{5}\right), 0\right\}$

Out[3]= $\left\{\left\{2+\sqrt{5}, \frac{1}{2}\left(3+\sqrt{5}\right), \frac{1}{2}\left(1+\sqrt{5}\right), 1\right\}, \{-1, 1, -1, 1\}, \left\{2-\sqrt{5}, \frac{1}{2}\left(3-\sqrt{5}\right), \frac{1}{2}\left(1-\sqrt{5}\right), 1\right\}, \{0, 0, 0, 1\}\right\}$

**Figure:** Mathematica code

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What is the solution? Random variables for entries!

$$\begin{pmatrix} 0 & 2r_1 & r_2 & r_3 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{pmatrix}$$

Multiply many matrices of this form with different choices.

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Use $S_{n,1}, S_{n_2}, S_{n,3}$ for three survival rates at time $n$.

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$$\begin{pmatrix} 0 & 2r_{n,1} & r_{n,2} & r_{n,3} \\ s_{n,1} & 0 & 0 & 0 \\ 0 & s_{n,2} & 0 & 0 \\ 0 & 0 & s_{n,3} & 0 \end{pmatrix} \begin{pmatrix} 0 & 2r_{n-1,1} & r_{n-1,2} & r_{n-1,3} \\ s_{n-1,1} & 0 & 0 & 0 \\ 0 & s_{n-1,2} & 0 & 0 \\ 0 & 0 & s_{n-1,3} & 0 \end{pmatrix}$$
$$\cdots \begin{pmatrix} 0 & 2r_{1,1} & r_{1,2} & r_{1,3} \\ s_{1,1} & 0 & 0 & 0 \\ 0 & s_{1,2} & 0 & 0 \\ 0 & 0 & s_{1,3} & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}.$$

Central Limit Theorem: *sums* of random variables; here have products.

## Products of Matrices: I

Products of Matrices very difficult.

Define $[A, B] = AB - BA$ (the commutator).

Define matrix exponential (for square matrices) by

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

http://en.wikipedia.org/wiki/Matrix_exponential

## Products of Matrices: II

### Baker–Campbell–Hausdorff

Consider $n \times n$ square matrices $A, B$. Then $e^{A+B}$ is

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$$e^A e^B e^{-[A,B]/2} e^{(2[B,[A,B]]+[A,[A,B]])/6} \cdots$$

http://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula.

See also fast multiplication / exponentiation for $A^n$, and Strassen algorithm for $AB$.

# Efficient Computation

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Find $2F_{n-2} \leq F_n \leq 2F_{n-1}$.

Thus $\sqrt{2}^n \leq F_n \leq 2^n$, *suggests* exponential growth!
Try $F_n = r^n$.

# Generating Function (Example: Binet's Formula)

## Binet's Formula

$$F_1 = F_2 = 1; \; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

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$$\Rightarrow g(x) = x/(1 - x - x^2).$$

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$$\Rightarrow \ g(x) \ = \ \frac{x}{1-x-x^2} \ = \ \frac{1}{\sqrt{5}} \left( \frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x} \right).$$

## Partial Fraction Expansion (Example: Binet's Formula)

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x}\right).$$

**Coefficient of** $x^n$ (power series expansion):

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

## Power of Generating Functions

Extremely important, bundle information in usable manner.

Allow us to deduce numerous properties.

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Allow us to deduce numerous properties.

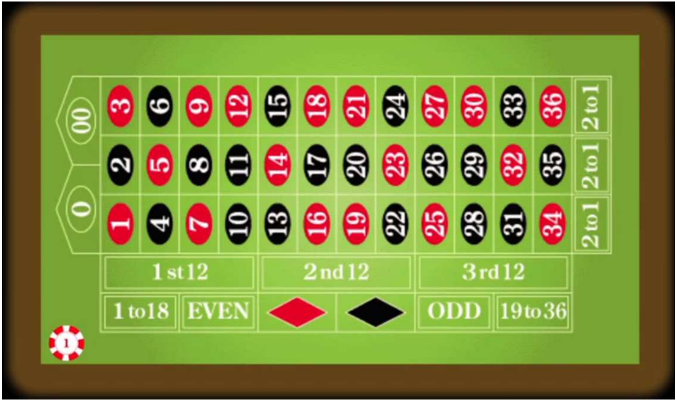
Example: $\sum_{n=0}^{\infty} F_n/3^n \ = \ 3/5!$

```
In[10]:= f[x_] := {Sum[Fibonacci[n] x^n, {n, 0, Infinity}],
                   x/(1 - x - x^2)};
```

Application: Roulette

YouTube: http://youtu.be/Esa2TYwDmwA

## Roulette




Probability $p$ of red, $1-p$ of **not red** (assume $p = .5$).

## Strategy: Double Plus One

- Bet $1 on red, if win up $1 else down $1.

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## Strategy: Double Plus One

- Bet $1 on red, if win up $1 else down $1.
- Bet $2 on red, if win up $1 else down $3.
- Bet $4 on red, if win up $1 else down $7.

## Strategy: Double Plus One

- Bet $1 on red, if win up $1 else down $1.
- Bet $2 on red, if win up $1 else down $3.
- Bet $4 on red, if win up $1 else down $7.
- Bet $8 on red, if win up $1 else down $15.
  Lather, rinse, repeat.

## Strategy: Double Plus One

- Bet $1 on red, if win up $1 else down $1.
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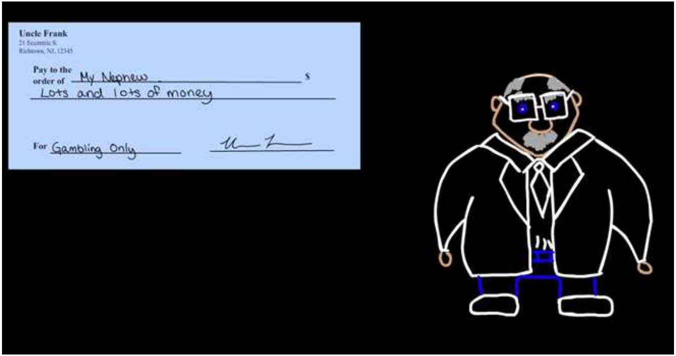
- Bet $4 on red, if win up $1 else down $7.
- Bet $8 on red, if win up $1 else down $15.
  Lather, rinse, repeat.

Eventually up $1. Why am I not at Vegas?

## Issue 1: Bankroll

# Issue 1: Bankroll: Eccentric Rich Aunt / Uncle Hypothesis



# Issue 2: Table Limits

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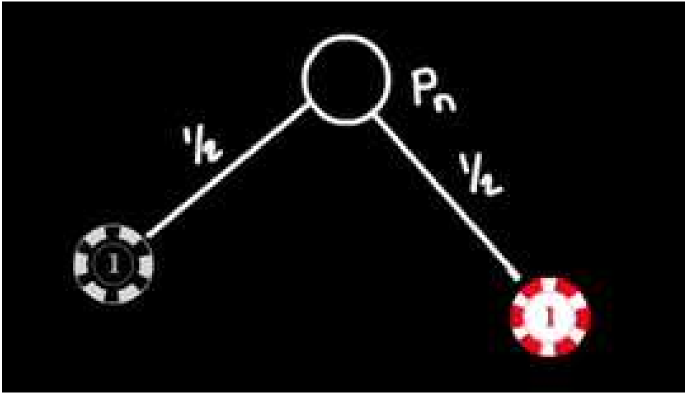
# Analysis of Double Plus One Strategy

- 5 blacks will make us bankrupt
- $P_n$ = probability we will not get 5 consecutive blacks in N spins
- $Q_n$ = probability we will get 5 consecutive blacks in N spins

$$P_n + Q_n = 1$$

# Analysis of Double Plus One Strategy



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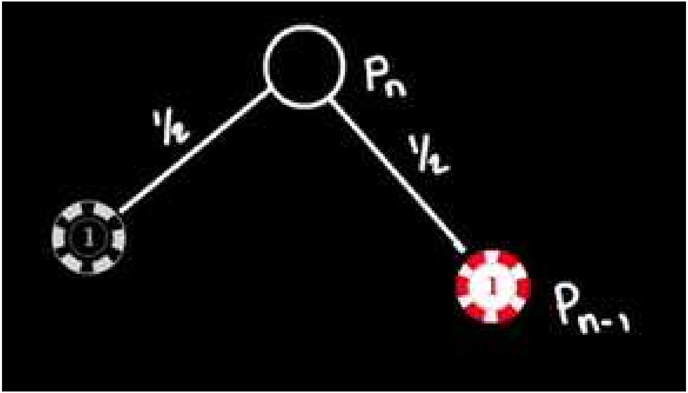

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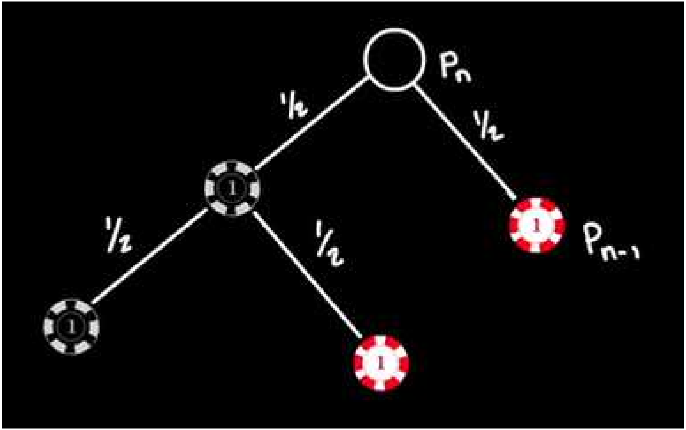

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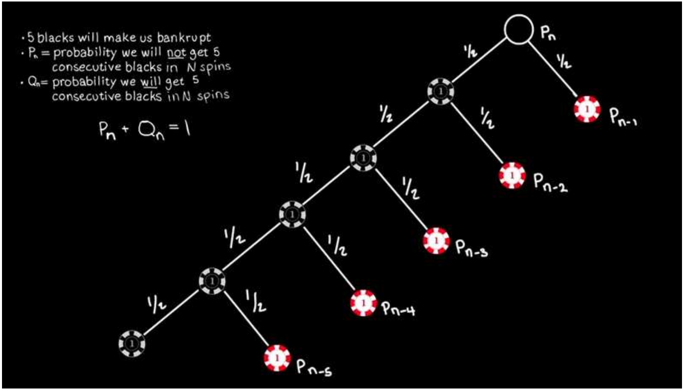

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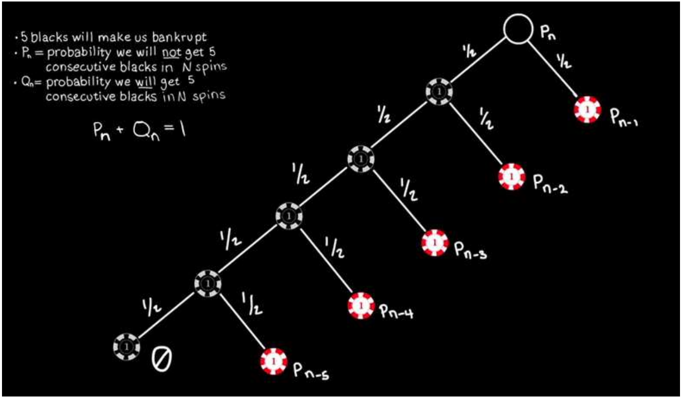
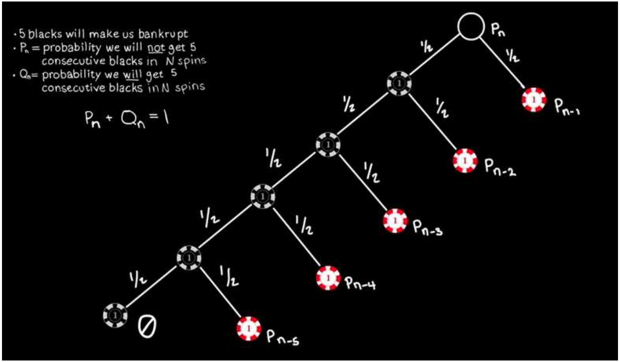
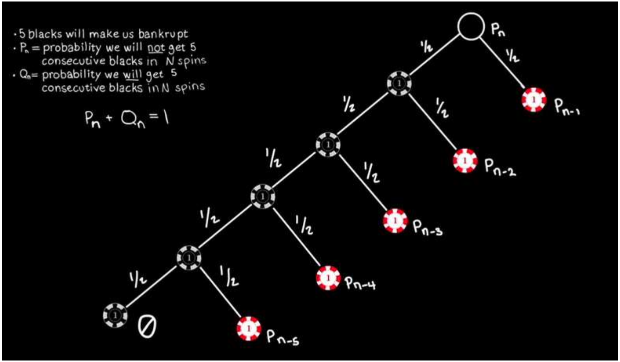


# Solving the Recurrence: I



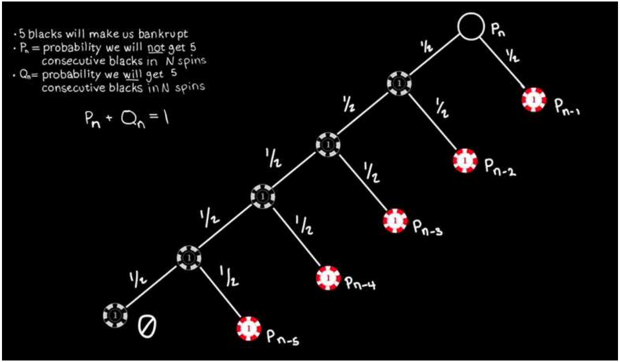

$$P_n \;=$$

# Solving the Recurrence: I



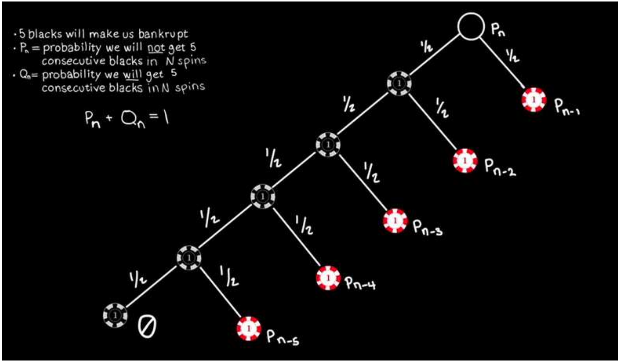

$$P_n = \frac{1}{2}P_{n-1}$$

## Solving the Recurrence: I




$$P_n \;=\; \frac{1}{2}P_{n-1} + \left(\frac{1}{2}\right)^2 P_{n-2}$$

## Solving the Recurrence: I




$$P_n \ = \ \frac{1}{2}P_{n-1} + \left(\frac{1}{2}\right)^2 P_{n-2} + \cdots + \left(\frac{1}{2}\right)^5 P_{n-5},$$

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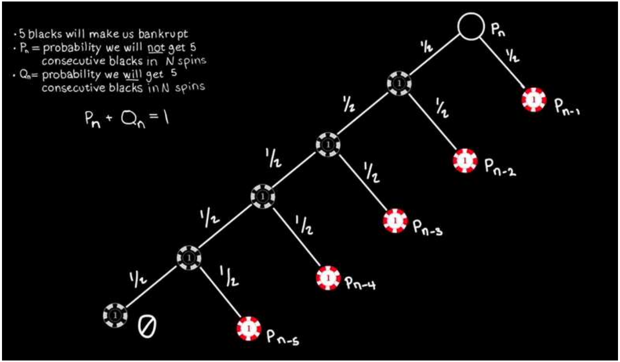


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and initial conditions are $P_0 =$

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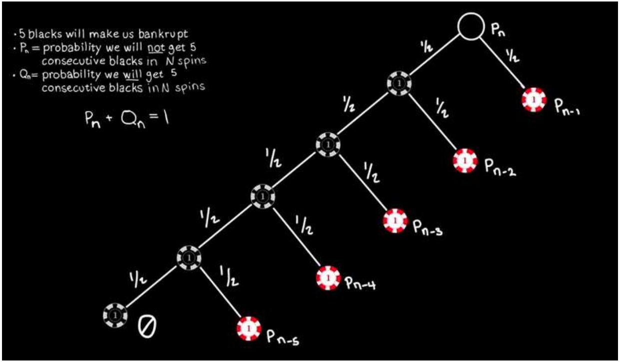
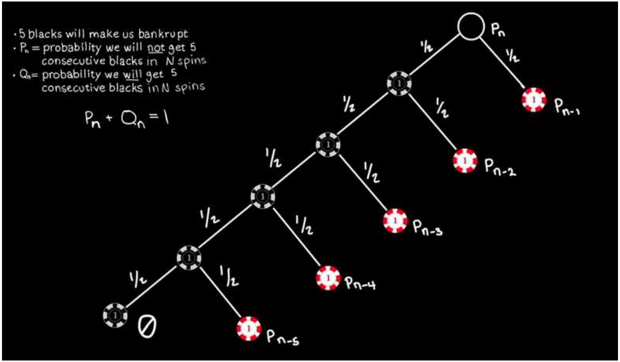



$$P_n \;=\; \frac{1}{2}P_{n-1} + \left(\frac{1}{2}\right)^2 P_{n-2} + \cdots + \left(\frac{1}{2}\right)^5 P_{n-5},$$

and initial conditions are $P_0 = 1$

# Solving the Recurrence: I




$$P_n = \frac{1}{2}P_{n-1} + \left(\frac{1}{2}\right)^2 P_{n-2} + \cdots + \left(\frac{1}{2}\right)^5 P_{n-5},$$

and initial conditions are $P_0 = 1 = P_1 = P_2 = P_3 = P_4$.

## Solving the Recurrence: II

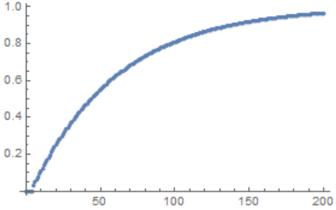
Can use Mathematica (as quintic need to numerically approximate roots, use 1./2).

```
In[24]:= RSolve[{a[n] == (1./2) a[n-1] + (1/4) a[n-2] + (1/8) a[n-3]
         + (1/16) a[n-4] + (1/32) a[n-5], a[0] == 1, a[1] == 1,
         a[2] == 1, a[3] == 1, a[4] == 1}, a[n], n]

Out[24]= {{a[n] → (-0.0780088 - 0.0615499 i) ((1. + 0. i) (-0.339175 - 0.229268 i)^n +
            (0.232635 - 0.972564 i) (-0.339175 + 0.229268 i)^n +
            (0.153251 + 0.994255 i) (0.0976883 - 0.424427 i)^n - (0.931325 + 0.380345 i)
             (0.0976883 + 0.424427 i)^n - (8.35517 - 6.59234 i) 0.982974^n)}}
```

## Solving the Recurrence: III

```
a[0] = a[1] = a[2] = a[3] = a[4] = 1;
For[n = 5, n ≤ 2000, n++, a[n] = a[n - 1] / 2
    + a[n - 2] / 4 + a[n - 3] / 8 + a[n - 4] / 16
    + a[n - 5] / 32];
list = {};
For[n = 0, n ≤ 200, n++,
  list = AppendTo[list, {n, 1 - 1.0 a[n]}]];
Print[ListPlot[list]];
```



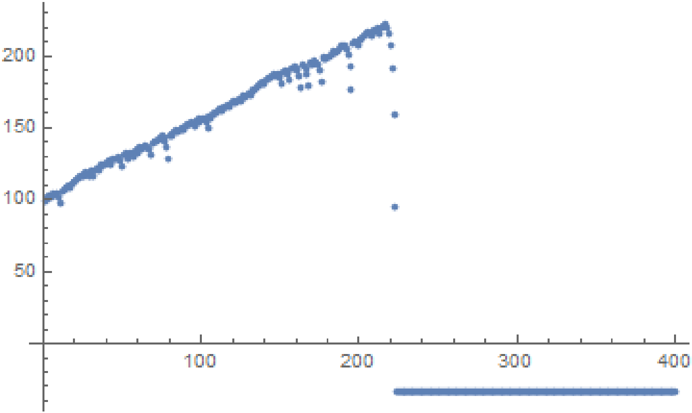

Probability 5 consecutive blacks in 100 spins is 81.01%, in 200 spins is 96.59%.

# Simulations

```
In[90]:= doubleplusone[capital_, spins_] := Module[{},
    loss = 0;
    money = capital;
    results = {};
    For[n = 1, n ≤ spins, n++,
     {
      If[money > 0, bet = loss + 1, bet = 0];
      If[Random[] ≤ .5, win = 1, win = 0];
      If[win == 1,
       {
        money = money + bet;
        loss = 0;
       },
       {
        money = money - bet;
        loss = loss + bet;
       }];
      results = AppendTo[results, {n, money}];
     }]; (* end of n loop *)
    Print[ListPlot[results]];
   ];
```

# Simulations

In[121]:= doubleplusone[100, 400]



# Introduction to Zeckendorf Decompositions

## Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

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Example: $51 = ?$

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Example: $51 = 34 + 17 = F_8 + 17$.

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**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$.

## Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . . .

**Zeckendorf's Theorem**

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + 1$.

## Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
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Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
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### Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.
Example: $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
Observe: 51 miles $\approx$ 82.1 kilometers.

## Old Results

### Central Limit Type Theorem

As $n \to \infty$, the distribution of number of summands in Zeckendorf decomposition for $m \in [F_n, F_{n+1})$ is Gaussian.

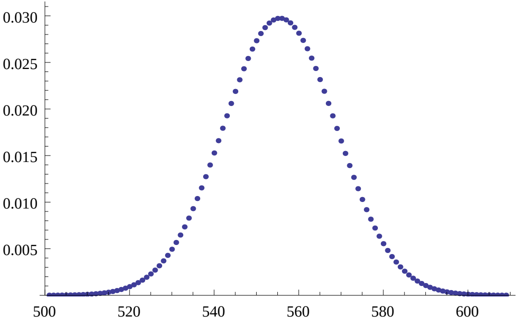


**Figure:** Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1,$$

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$$1, \quad 2,$$

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$$1, \;\; 2, \;\; 3,$$

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3, \quad 5,$$

## Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3, \quad 5, \quad 8,$$

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$$1,\ \ 2,\ \ 3,\ \ 5,\ \ 8,\ \ 13 \ldots.$$

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13 \dots .$$

- Key to entire analysis: $F_{n+1} = F_n + F_{n-1}$.
- View as bins of size 1, cannot use two adjacent bins:

$$[1] \;\; [2] \;\; [3] \;\; [5] \;\; [8] \;\; [13] \quad \cdots .$$

- Goal: How does the notion of legal decomposition affect the sequence and results?

## Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the $a_i$'s (e.g. cannot use the recurrence relation to remove any summand).

- Central Limit Type Theorem

## Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \dots, 9\}$ $(1 \le i < m)$, $a_m \in \{1, \dots, 9\}$.

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- For large $n$, the contribution of $A_n$ is immaterial. $A_i$ $(1 \le i < n)$ are identically distributed random variables with mean 4.5 and variance 8.25.
- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \to$ Gaussian with mean $4.5n + O(1)$ and variance $8.25n + O(1)$.

## Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

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Individual: Similar questions about gaps for a fixed $m \in [F_n, F_{n+1})$: distribution of gaps, longest gap.

## New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m = \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

**Theorem (Zeckendorf Gap Distribution)**

*Gap measures $\nu_{m;n}$ converge to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.*

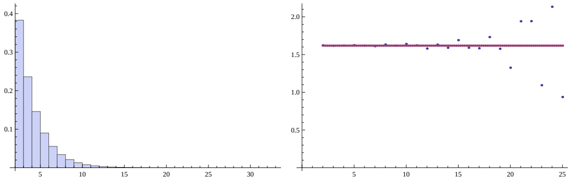


**Figure:** Distribution of gaps in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

## New Results: Longest Gap

Fair coin: largest gap tightly concentrated around $\log n / \log 2$.

### Theorem (Longest Gap)

*As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to*

$$\mathrm{Prob}\left(L_n(m) \le f(n)\right) \approx e^{-e^{\log n - f(n)\cdot \log \phi}}$$

- $\mu_n = \frac{\log\left(\frac{\phi^2}{\phi^2+1)} n\right)}{\log \phi} + \frac{\gamma}{\log \phi} - \frac{1}{2} +$ Small Error.

- If $f(n)$ grows ***slower*** (resp. ***faster***) than $\log n / \log \phi$, then $\mathrm{Prob}(L_n(m) \le f(n))$ goes to **0** (resp. ***1***).