Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 02: 9-13-21:
https://youtu.be/_K0kOFa3x1g

Plan for the day: Lecture 2: September 13, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Review factorials, binomials, choosing….
- Birthday problem.
- QWERTY, Babylonian Mathematics

General items.

- Difference between a formula and a useful formula.
- Institutionalization: think hard on notation, actions.

Birthday problem: N people, how many so have a 50% chance at least two share a birthday.

Assumptions: all bdays = likely (no Feb 29)
all indep

$P(n)$ = prob that n people do not share a bday

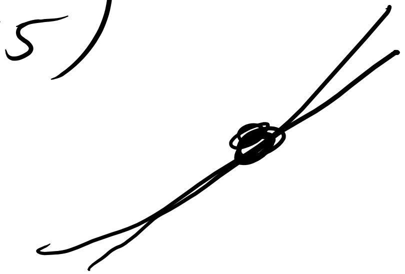
$$P(n) = \frac{365-0}{365} * \frac{365-1}{365} * \cdots * \frac{365-(n-1)}{365} = \frac{365!}{(365-n)!\,365^n}$$

$$= \prod_{k=1}^{n-1} \left(1 - \frac{k}{365}\right)$$ Find n^* st $P(n^*) = 1/2$

Solve $\prod_{k=1}^{n-1} \left(1 - \frac{k}{365}\right) = \frac{1}{2}$

Pavlovian: Take log when see product

$$\log(1/2) = \log \prod_{k=1}^{n-1}\left(1-\frac{k}{365}\right) = \sum_{k=1}^{n-1}\log\left(1-\frac{k}{365}\right)$$



Expect $n <<< 365$, so $1-\frac{k}{365} \approx 1$

Taylor-Series: $(Tf)(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m$

so $\log(1-x) = -x - x^2/2 - x^3/3 \cdots \approx -x$

$f(x) = \log(1-x)$

$f'(x) = \frac{-1}{1-x}$ $\quad f'(0) = -1$

Is this reasonable? If x is small and pos, $\log(1-x) \approx -x$
negative

$$\log(1/2) \approx \sum_{k=0}^{n-1} -\frac{k}{365}$$

(using $\log(1-x) \approx -x$)
note $\log(1/2) = -\log 2$

$$365 \log 2 = \sum_{k=0}^{n-1} k$$

(1) Use Induction

(2) Divine Inspiration: it is $\dfrac{(n-1)(n-1+1)}{2}$

Gauss: $S(n) = 1 + 2 + \cdots + n$

$\underline{S(n)} = \underline{n + (n-1) + \cdots + 1}$

$2S(n) = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = n(n+1)$

$$S(n) = \frac{n(n+1)}{2}$$

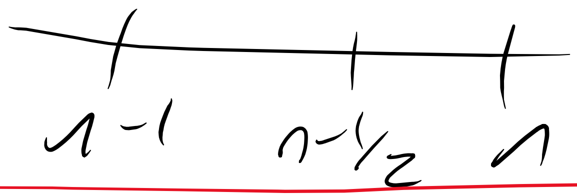
$$365 \log 2 \approx \sum_{k=0}^{n-1} k = \frac{(n-1)n}{2}$$

So $n(n-1) \approx 365 \log 4$

Solve for n, but how?

Use the Quadratic Formula: $ax^2+bx+c=0$, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Estimate: $n(n-1) \approx (n-\frac{1}{2})(n-\frac{1}{2})$



$$(n-1/2)^2 \approx 365 \log 4$$

$$n - 1/2 \approx \sqrt{365 \log 4}$$

$$n \approx \frac{1}{2} + \sqrt{365 \log 4}$$

(This is our n^*)

What if D days in a year?

$$\boxed{n \approx \frac{1}{2} + \sqrt{D \log 4}}$$

Combinatorics

Factorial: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$: # arrangements when order matters

$0! = 1$ (convenient, $n! = n(n-1)!$, Seinfeld)
Functional Eq

Choose 3 from 7, order matters: $7 \cdot 6 \cdot 5 * \frac{4!}{4!} = \frac{7!}{4!} = {}_7P_3$

Choose 3 from 7, order doesn't matter: $\frac{{}_7P_3}{3!} = \frac{7!}{3!\,4!} = {}_7C_3 = \binom{7}{3}$

Binomial Coeff: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Thm: $\binom{n}{k} = \binom{n}{n-k}$

Proof: Brute force algebra!

Proof: Story: Choosing k is the same as excluding $n-k$

Pascal: $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

n students from Williams
1 Amherst
} want $k+1$ people

$$\binom{n+1}{k+1} = \binom{1}{1}\binom{n}{k} + \binom{1}{0}\binom{n}{k+1}$$

Institutionalizing and Good Choices

QWERTY

Babylonian Math: Base 60

mult tables: base 10 have $\approx \frac{1}{2} 10 \cdot 10 \approx 50$

base 60 have $\approx \frac{1}{2} 60 \cdot 60 \approx 1800$

↳ tablets: look up table

$$xy = \frac{(x+y)^2 - (x-y)^2}{4}$$

$$= \frac{(x+y)^2 - x^2 - y^2}{2}$$

only need to know the squares: 60 or 120