Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 09: 9-29-21: https://youtu.be/jbXlsYsqH34

Lecture 09: 9/25/19: Trump Splits, Conditional Probability, Bayes' Theorem: https://youtu.be/Id0h1Nawh9Q

Plan for the day: Lecture 09: September 29, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Trump Splits
- Conditional Probability (sniffing out formula)
- Inclusion/Exclusion
- Bayes' Theorem

General items.

- Run simulations!
- Importance of phrasing.
- Explore extreme cases.

Probability of having a 5–0 trump split in bridge

4 hands of 13, partner and I have 8 trumps missing 5

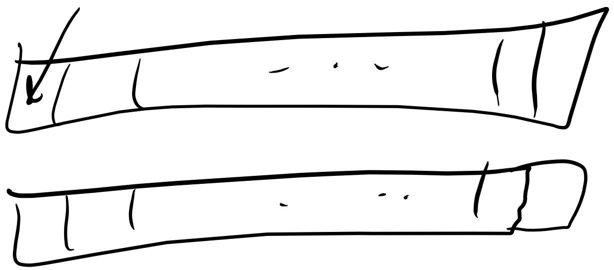
$$\frac{\binom{2}{1}\binom{5}{5}\binom{21}{8}\binom{13}{13}}{\binom{26}{13} * \binom{13}{13}}$$

$\binom{2}{1}$ which opponent; $\binom{5}{5}$ Get all missing; $\binom{21}{8}$ fill up hand; $\binom{13}{13}$ other hand; $\binom{26}{13}$ first hand; $\binom{13}{13}$ second hand

equal to $\frac{9}{230} \approx 3.91\%$

$$\binom{2}{1} \quad \frac{13}{26} * \frac{12}{25} * \frac{11}{24} * \frac{10}{23} * \frac{9}{22}$$

$\binom{2}{1}$ choose who gets



OR $\binom{2}{1} * \left(\frac{1}{2}\right)^5 = \frac{1}{16} \approx 6.25\%$

$\binom{2}{1}$ choose person

Probabilty of a 5-0 split: 9/230, or about 3.91% or is it 2/32 or about 6.25%?

```
badtrumpsplit[numbad_, numiter_] := Module[{},
   deck = {};
   For[c = 1, c ≤ numbad, c++, deck = AppendTo[deck, 1]];
   For[c = numbad + 1, c ≤ 26, c++, deck = AppendTo[deck, 0]];
   badsplits = 0;
   For[n = 1, n ≤ numiter, n++,
    {
     hand = RandomSample[deck, 13];
     If[Mod[Sum[hand[[i]], {i, 1, 13}], numbad] == 0, badsplits = badsplits + 1];
    }];
   Print["Observed badsplits is ", SetAccuracy[100.0 badsplits / numiter, 4], "%."];
  ];

Timing[badtrumpsplit[5, 1000000]]

Observed badsplits is 3.942%.

{15.5625, Null}

Timing[badtrumpsplit[5, 10000000]]

Observed badsplits is 3.913%.

{188.891, Null}
```

Are these two items equivalent:

Each person is equally likely to be chosen, form a group of two people from four.

Chose any group of two people, all groups equally likely to be chosen.

Not Equivalent

A_1 B_1 B_2 A_2

$A_1 B_2$
$A_1 A_2$
$B_1 B_2$
$B_1 A_2$

} each person is 2 of 4 or 50%

never have $A_1 B_1$ or $A_2 B_2$

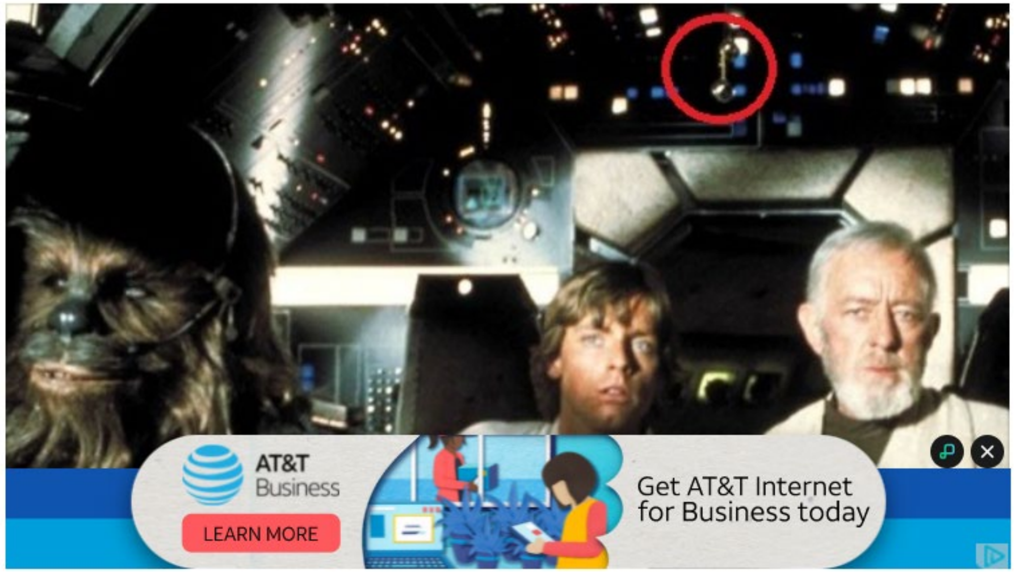
Rolling two fair independent die….

What is the probability that

1. The sum is an 11: $\frac{2}{36} = \frac{1}{18}$
2. The sum is a 7: $\frac{6}{36} = \frac{1}{6}$
3. Given first die is a 3, the sum is an 11: 0
4. Given first die is a 3, the sum is a 7: $\frac{1}{6}$

Conditional probability can be the same or different as the probability.

20. Han's Dice



When Luke boards the Millennium Falcon in The Last Jedi, he grabs a pair of gold dice which belonged to Han Solo, and though you may not have ever noticed them before, they were hanging up in the Falcon in A New Hope and also reappeared in The Force Awakens.

Is it possible that there's some nice function F such that

$$\Pr(A|B) = F(\Pr(A), \Pr(B))?$$

(Assumption $P(B) > 0$)

Conditional prob of A given B happens

Try: $B = \Omega$ the entire space, so $P(B) = 1 \Rightarrow P(A|\Omega) = P(A)$

- $A = B$ Then $P(A|A) = 1$
- $A = B^c$ Then $P(A|A^c) = 0$

If $A = B$ $(\Pr(A), \Pr(B))$ same as if $A = B^c$ $(\Pr(A), \Pr(A^c))$

Take $\Pr(A) = \Pr(B) = \Pr(B^c) = 1/2$

$F(1/2, 1/2) = 1$ and $F(1/2, 1/2) = 0$

NO SUCH F EXISTS!

$$\Pr(A|B) = G(\Pr(A), \Pr(B), \Pr(A \cap B))$$

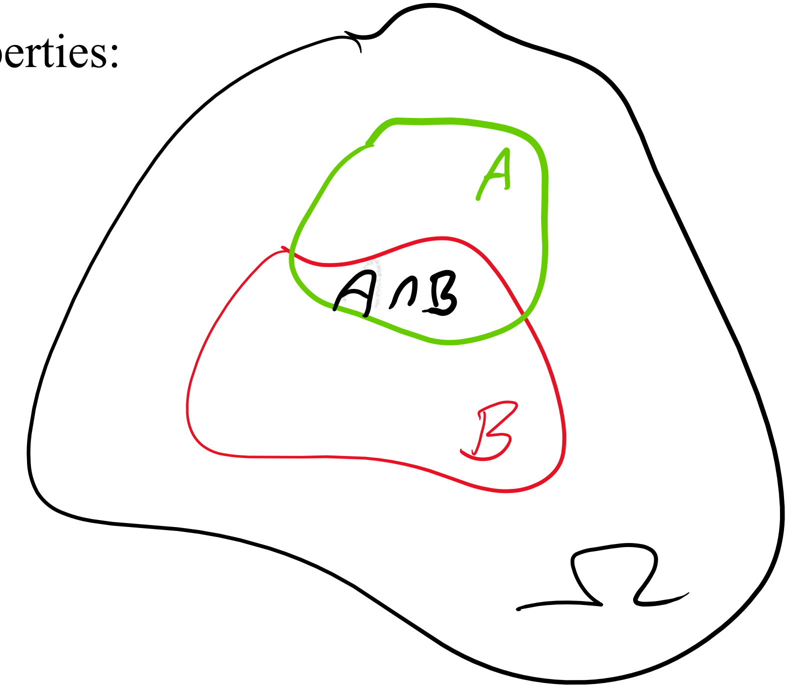
$$\Pr(A|B) = G(\Pr(A), \Pr(B), \underbrace{\Pr(A \cap B)}_{\text{A and B happen}})$$

- $\Pr(B|B) = 1$,
- $\Pr(B^c|B) = 0$, and
- $0 \le \Pr(A|B) \le 1$.

There's a simple expression using our three building blocks that has these three properties:

Try $$P(A|B) = \frac{\Pr(A \cap B)}{P(B)}$$

Satisfies the three points above



Expected Counts Approach

Suppose that you go out fishing one day, and you have the following set of rules: you stop fishing once you catch a fish, or after you've been on the water for four hours (whichever comes first). Let's also imagine that there's a 40% chance that you catch a trout, a 25% chance you catch a bass, and a 35% chance you don't catch anything. Notice that the percentages sum to 100%, and that you never catch more than one fish in a day. Now, if we know that you caught a fish one day, what are the odds that fish was a trout? Suppose that you went fishing 1000. Then….

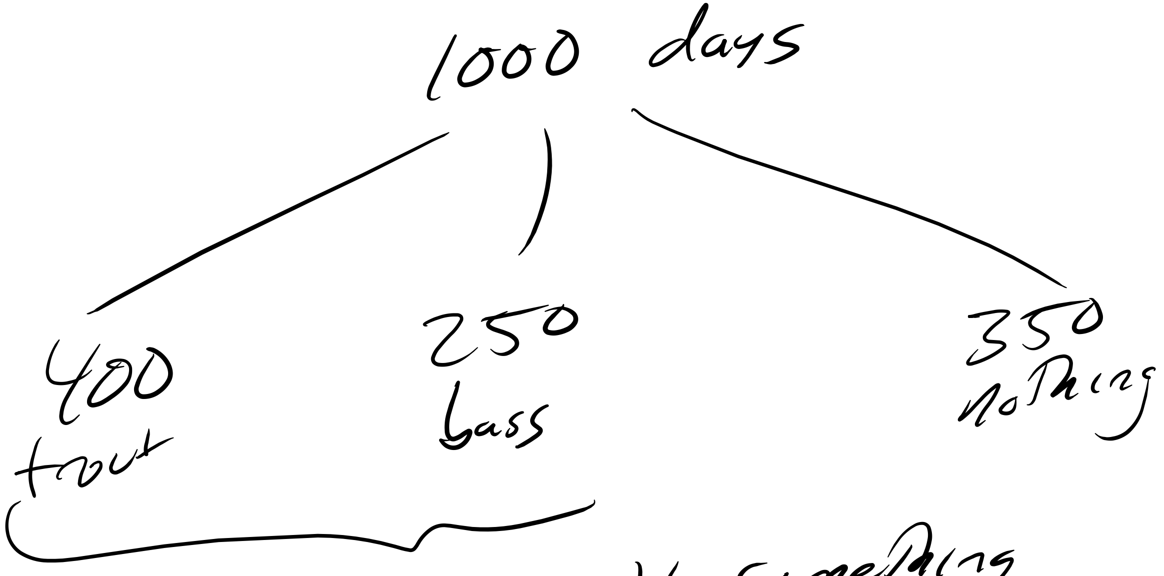
40% chance catch trout

25% chance catch a bass

35% chance catch nothing

P(trout | caught fish)

$$= \frac{.40}{.65}$$



650 times caught something

400 of those are trout

So $\frac{400}{650}$

	A	A^c
B	$A \cap B$	$A^c \cap B$
B^c	$A \cap B^c$	$A^c \cap B^c$

Table 4.1: These are the possible outcomes for events A and B. If we know that event B has happened, we need only worry about the events in B's row.

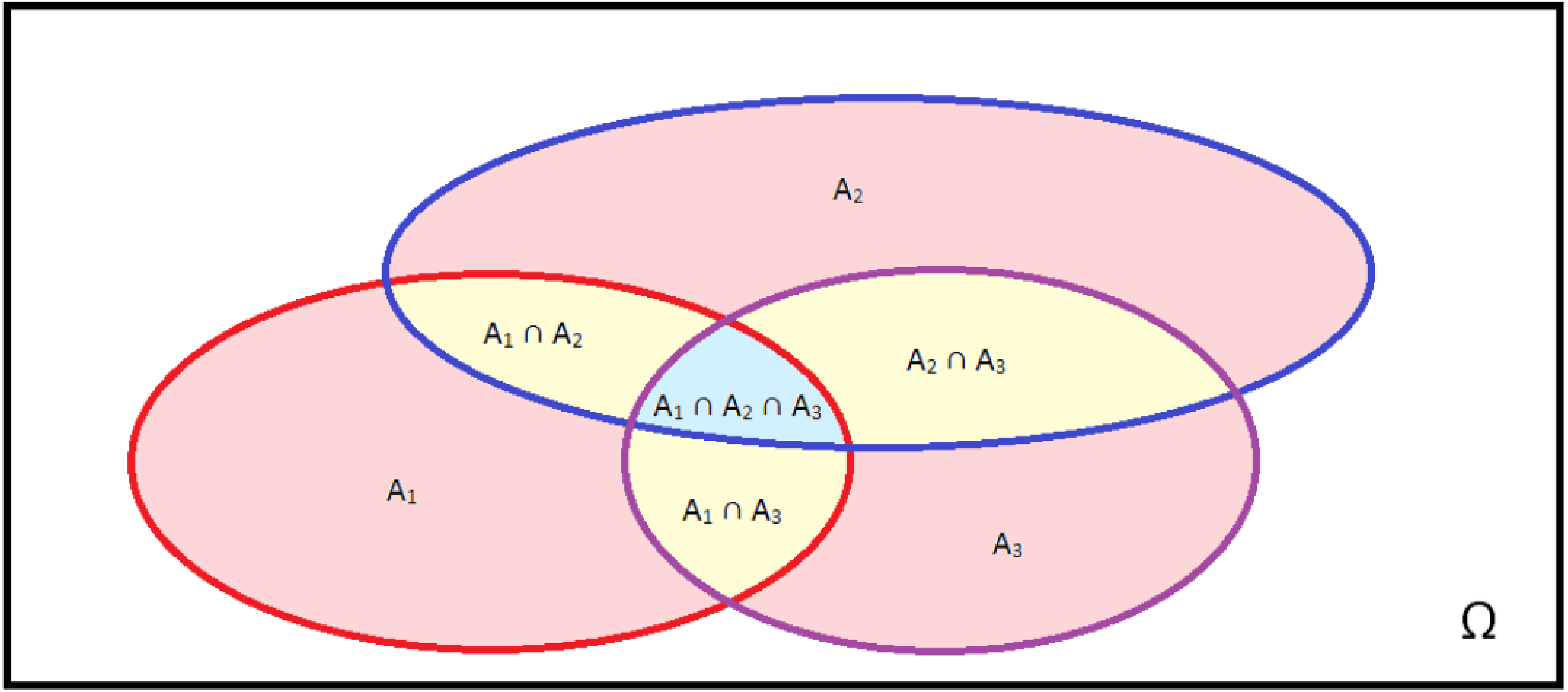
Conditional Probability: Let B be an event such that $\Pr(B) > 0$. Then the conditional probability of A given B is

$$\Pr(A|B) = \Pr(A \cap B) / \Pr(B).$$

Rewrite: $Pr(A \cap B) = Pr(A|B) * Pr(B)$

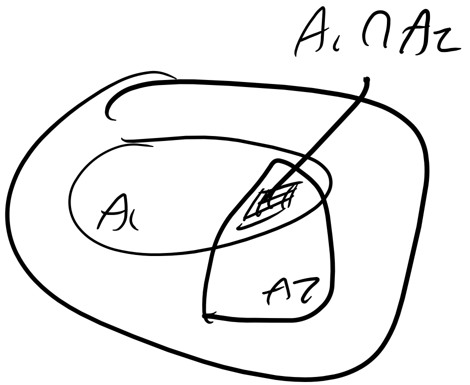
If you were really reading carefully, you might've noticed a new condition snuck into the box above: $\Pr(B) > 0$. If $\Pr(B) = 0$, then B cannot happen. If B cannot happen, it doesn't make sense to talk about the probability A happens given B happens! Fortunately if $\Pr(B) = 0$ then $\Pr(A \cap B)$ is also 0, and we have the indeterminate ratio 0/0, which warns us that we are in dangerous waters.

Illustration of inclusion-exclusion with three sets.



$$\Pr(A_1 \cup A_2 \cup A_3) = \Pr(A_1) + \Pr(A_2) + \Pr(A_3) - \Pr(A_1 \cap A_2) - \Pr(A_1 \cap A_3) - \Pr(A_2 \cap A_3) + \Pr(A_1 \cap A_2 \cap A_3)$$

$$Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2) - Pr(A_1 \cap A_2)$$



Inclusion-Exclusion Principle: Consider sets $A_1, A_2, \ldots, A_n$. Denote the number of elements of a set S by $|S|$ and the probability of a set S by $\Pr(S)$. Then

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \\ &\quad \cdots + (-1)^{n-2} \sum_{1<\ell_1<\ell_2<\cdots<\ell_{n-1}\le n} |A_{\ell_1} \cap A_{\ell_2} \cap \cdots \cap A_{\ell_{n-1}}| \\ &\quad +(-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|;\end{aligned}$$

this also holds if we replace the size of all the sets above with their probabilities. We may write this more concisely. Let $A_{\ell_1\ell_2\ldots\ell_k} = A_{\ell_1} \cap A_{\ell_2} \cap \cdots \cap A_{\ell_k}$ (so $A_{12} = A_1 \cap A_2$ and $A_{489} = A_4 \cap A_8 \cap A_9$). Then

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_{ij}| + \sum_{1\le i<j<k\le n} |A_{ijk}| - \cdots \\ &\quad +(-1)^{n-2} \sum_{1<\ell_1<\ell_2<\cdots<\ell_{n-1}\le n} |A_{\ell_1\ell_2\cdots\ell_{n-1}}| + (-1)^{n-1}|A_{12\cdots n}|.\end{aligned}$$

If the A_i's live in a finite set and we use the counting measure where each element of our outcome space is equally likely, we may replace all $|S|$ above with $\Pr(S)$.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-2} \sum_{1<\ell_1<\ell_2<\cdots<\ell_{n-1}\le n} |A_{\ell_1} \cap A_{\ell_2} \cap \cdots \cap A_{\ell_{n-1}}| + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|;$$

Special case:

$|A_i|$ = same

$|A_i \cap A_j|$ = same ....

$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} \cdots + (-1)^{n-1}\binom{n}{n} \underbrace{- \binom{n}{0} + \binom{n}{0}}_{\text{adding zero}}$$

$$= \binom{n}{0} - \left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n}\binom{n}{n}\right]$$

Binomial Thm: $x = 1, y = -1$

$$= 1 - (1 - 1)^n$$

$$= 1$$

Simplifies when we assume the equalities above

$$\Pr(A_1 \cup \cdots \cup A_n)$$

$$= n \Pr(A_1)$$

$$- \binom{n}{2} \Pr(A_1 \cap A_2)$$

$$+ \binom{n}{3} \Pr(A_1 \cap A_2 \cap A_3)$$

$$-$$

$$\ldots$$

$$+ (-1)^{n-1} \binom{n}{n} \Pr(A_1 \cap \cdots \cap A_n)$$