Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 18: 10-27-21: https://youtu.be/Y6Lvb-ppyHE

Lecture 18: 10/21/19: Marriage Problem, Two Envelope Problem, Buffon's Needle: https://youtu.be/-36UUC_oPUs
(see http://youtu.be/wxOlDaA_cGs for two envelope problem: 18mins to 19:30mins)

Plan for the day: Lecture 18: October 27, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Differentiating Identities
- Marriage / Secretary Problem
- Two envelope problem
- Buffon's Needle

General items.

- Powerful techniques: differentiating identities
- Important of asking the right question
- Power of simple strategies

Method of Differentiating Identities: Let $\alpha, \beta, \gamma, \ldots, \omega$ be some parameters. Assume

$$\sum_{n=n_{\min}}^{n_{\max}} f(n; \alpha, \beta, \ldots, \omega) = g(\alpha, \beta, \ldots, \omega),$$

where f and g are differentiable functions with respect to α. Then

$$\sum_{n=n_{\min}}^{n_{\max}} \frac{\partial f(n; \alpha, \beta, \ldots, \omega)}{\partial \alpha} = \frac{\partial g(\alpha, \beta, \ldots, \omega)}{\partial \alpha},$$

provided that f has sufficient decay to justify the interchange of summation and differentiation.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1} \text{ if } |x|<1$$

apply $x \frac{d}{dx}$

$$x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} (1-x)^{-1}$$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

$$x = 1/2 \Rightarrow \sum_{n=0}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$$

$$\sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{N} x^n + \sum_{n=N+1}^{\infty} x^n$$

$$= \sum_{n=0}^{N} x^n + x^{N+1} \frac{1}{1-x}$$

$X \sim \text{Geometric}(p)$ $\quad$ $\text{Prob}(X = n) = p^n$ $\quad$ $n \in \{1, 2, \ldots\}$

need $p = 1/2$

$$\text{Prob}(X = n) = (1-p)\,p^n \qquad n \in \{0, 1, 2, \ldots\}$$

# of failures before the first head, prob head is $1-p$

$$E[X] = \sum_{n=0}^{\infty} n\,\text{Prob}(X = n)$$

$$= \sum_{n=0}^{\infty} n\,(1-p)\,p^n = (1-p)\underbrace{\sum_{n=0}^{\infty} n\,p^n}$$

$$= (1-p) * \frac{p}{(1-p)^2} = \frac{p}{1-p}$$

$p = 1$: Goes to ∞ good

$p = 0$: Goes to 0, good

$p = 1/2$: Goes to 1

Binomial Thm

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

apply $x \frac{d}{dx}$

$$n(x+y)^{n-1} x = \sum_{k=0}^{n} \binom{n}{k} k \, x^k y^{n-k}$$

Let $x = p \quad y = 1-p$

$$n(p + (1-p))^{n-1} p = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$np = E[X] \qquad X \sim \text{Bin}(n,p)$$

Variance: $E[X^2] - E[X]^2$ apply $x \frac{d}{dx}\left[x \frac{d}{dx}[\sim] \right]$

apply $x\frac{d}{dx}$ to $n\,x\,(x+y)^{n-1}$

write $x = x+y-y$

have $n\,(x+y-y)\,(x+y)^{n-1}$

$n\,(x+y)^{n} - n\,y\,(x+y)^{n-1}$

adding zero bypasses product rule

apply $x\frac{d}{dx}$ get $n^2(x+y)^{n-1}x - n(n-1)\,xy\,(x+y)^{n-2}$

$$\text{Prob}(X = k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{0, 1, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.$$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$\text{Var}(X) = \mathbb{E}[(X - \mu_X)^2] = \sum_{k=0}^{n} (k - np)^2 \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sum_{k=0}^{n} k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} - (np)^2$$

Marriage/Secretary Problem

Know # of people N

can compare people seen

success if get best person, else failure

If always take a specific spot, get best $1/N$ of the time

$\frac{N}{2}$

first person better than the best in first half

$N/2$ $N/2$

2nd Best "1/2"

Best 1/2

25% chance!

Look at first $k = k(N)$ people, choose first person better than the best you've seen

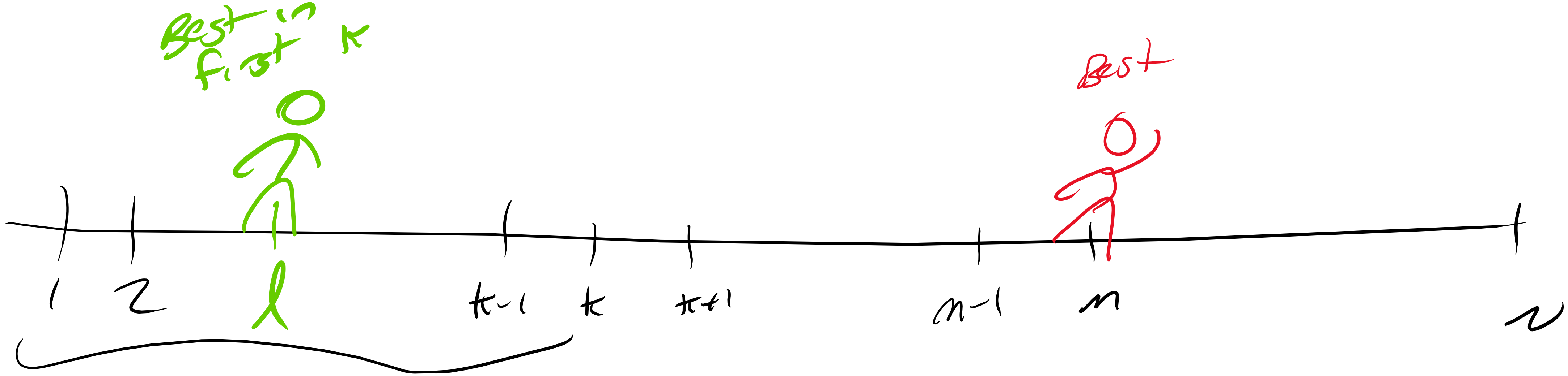
If k large, ↗ chance best seen

If k small, ↗ chance have too low of standards

Prob have best given look at k then choose first better

conditional Prob

$\sum$ Prob(Have best | look at k) ??



If best in first k, always lose

If best b/w k+1 and N have a chance

↳ Best of the first m-1 must be b/w 1 and k

↳ Prob is $\frac{k}{m-1}$

$$\text{Prob}(\text{win}) = \sum_{m=1}^{N} \text{Prob}(\text{win with best at } m)\,\text{Prob}(\text{best at } m)$$

$$= \sum_{m=k+1}^{N} \frac{k}{m-1}\,\frac{1}{N}$$

$$\text{Prob(win)} = \sum_{n=k+1}^{N} \frac{k}{n-1}\frac{1}{N}$$

$$\simeq \frac{k}{N}\sum_{n=k}^{N-1}\frac{1}{n}$$

Harmonic Series: $\sum_{n=1}^{M}\frac{1}{n} = \log M + \text{const} + \text{smaller}$

(integral test: sum $\approx \int_1^M \frac{1}{t}\,dt = \log M$)

$$\sum_{n=k}^{N-1}\frac{1}{n} = \sum_{n=1}^{N-1}\frac{1}{n} - \sum_{n=1}^{k-1}\frac{1}{n} \approx \log(N-1) - \log(k-1)$$

$$= \log\frac{N-1}{k-1} \approx \log\frac{N}{k}$$

$$\text{Prob(win)} = \frac{k}{N}\log\frac{N}{k}$$

$\text{Prob(win)} = \frac{k}{N} \log\left(\frac{N}{k}\right)$ $\quad k : 1 \to N$

$x = k/N$ so $x : 0 \to 1$

$f(x) = x \log(1/x) = -x \log x$

deriv $= 0$ and endpoints : $x = 1/N$ or $x = N$ "bad"

$f'(x) = -\log x - x \cdot \frac{1}{x}$

$= -\log x - 1$

so $f'(x) = 0 \Rightarrow \log x = -1$

so $x = 1/e$

$x = \frac{k}{N}$ so $\frac{k}{N} = 1/e$

$\to k = N/e$

$f\left(\frac{N}{e}\right) = \frac{1}{e} \log e = \frac{1}{e}$

about 37%

I think

Simple

Student paints house in 2 days
Prof paints house in 3 days

The two paint a house in

upper	lower
2	
1.5	1

(a) (b) (c) (d) (e)

often only one in range

d days

$\frac{1}{2}d + \frac{1}{3}d = 1$

$\frac{5}{6}d = 1$

$d = \frac{6}{5} = 1.2$

Study $1^k + 2^k + \cdots + n^k$

Know $1 + x + x^2 + \cdots + x^k = \dfrac{1 - x^{k+1}}{1-x}$

$x\frac{d}{dx}$ $\quad x + 2x^2 + \cdots + kx^k$

$1^2 x + 2^2 x^2 + \cdots + k^2 x^k$

$1^3 x + 2^3 x^2 + \cdots + k^3 x^k$

$x = 1$

$1-x$

at $x=1$

denom $=0$

Bad

L'Hopital