Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 19: 10-29-21: https://youtu.be/oYRoyqV2jAI

Lecture 19: 10/23/19: Differentiating Identiteis (Gaussian, Exponential, Geometric, Negative Binomial):
https://youtu.be/T3PwM91_YaY

Plan for the day: Lecture 19: October 29, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Differentiating Identities (Gaussian, Exponential, Geometric, Negative Binomial)

General items.

- Care in interchanging order….

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Gaussian!

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Exponential

$$\Pr(X = k) = (1-p)^{k-1} p$$

Geometric

$$f(k;r,p) \equiv \Pr(X = k) = \binom{k+r-1}{r-1} (1-p)^k p^r$$

Negative Binomial

Prove $I=\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$$

Polar

$x = r\cos\theta$ $\quad dx\,dy$

$y = r\sin\theta$ $\quad = r\,dr\,d\theta$

$x^2+y^2=r^2$ $\quad = dr \cdot r\,d\theta$

$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\,dy\,dx$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \frac{1}{2\pi} e^{-r^2/2}\,r\,dr\,d\theta$$

$u = r^2/2 \quad du = r\,dr$

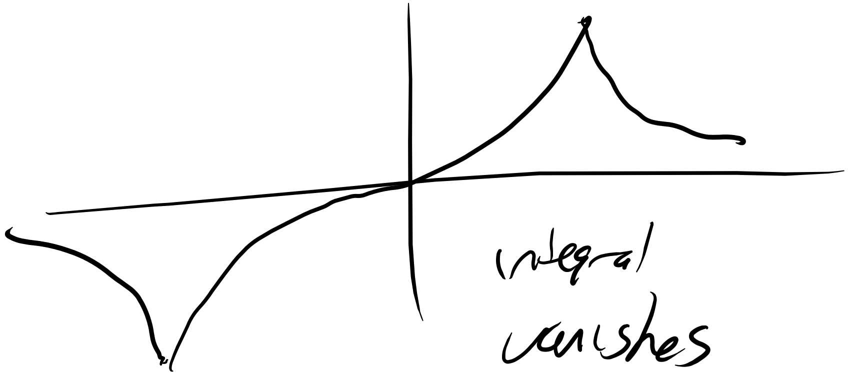
$r: 0 \to \infty \quad u: 0 \to \infty$

$$= \frac{2\pi}{2\pi} \int_{r=0}^{\infty} e^{-r^2/2}\,r\,dr = \int_{u=0}^{\infty} e^{-u}\,du = e^{-u}\Big|_{\infty}^{0} = 1$$

Rescale: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\, dx = 1$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\, dx = 0$$

odd



$f(x) = f(-x)$ even

$f(x) = -f(-x)$ odd

$$\mathbb{E}[X] = 0$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\, dx = uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\, du$$

$$u = x \qquad dv = e^{-x^2/2\sigma^2} \sigma^2 x\, dx/\sigma^2$$

$$du = dx \qquad v = -\sigma^2 e^{-x^2/2\sigma^2}$$

$$E[X^2] = \cancelto{0}{uv}\Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi\sigma^2}} \sigma^2 \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx$$

1

$$= \sigma^2$$

$E[X^{2k}]$ by induction ..... $= (2k-1)!!$

$$= (2k-1)(2k-3)\cdots(3)(1)$$

Differentiating Identities : $\mathbb{E}[X^{2k}]$ $X \sim N(0,1)$

Want $\displaystyle\int_{-\infty}^{\infty} x^{2k} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

Start with $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2}\, dx = \sigma$

Apply $\sigma \frac{d}{d\sigma}$ to both sides, $\frac{d}{d\sigma}\int = \int \frac{d}{d\sigma}$ (Real Analysis)

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, \sigma \frac{d}{d\sigma}\left(e^{-x^2/2\sigma^2}\right) dx = \sigma \frac{d}{d\sigma}(\sigma) = \sigma$$

$$\sigma = \int_{-\infty}^{\infty} -\frac{x^2}{2} \cdot \frac{-2}{\sigma^3} \cdot \frac{\sigma}{\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}\, dx \Rightarrow \sigma^2 = \int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2} \frac{dx}{\sqrt{2\pi\sigma^2}}$$

Exponential: $X \sim Exp(\lambda)$ means $1/\lambda * e^{-x/\lambda}$ if $x \geq 0$

Prob is zero if x is negative, and

$$E[X] = \int_0^\infty x \cdot \frac{1}{\lambda} e^{-x/\lambda}\, dx \qquad \int \text{ by parts}$$

Identity: $\int_0^\infty \frac{1}{\lambda} e^{-x/\lambda}\, dx = 1 \Rightarrow \boxed{\int_0^\infty e^{-x/\lambda}\, dx = \lambda}$

Apply $\boxed{\lambda \frac{d}{d\lambda}}$ yield $\lambda \frac{d}{d\lambda}(\lambda) = \lambda \frac{d}{d\lambda} \int_0^\infty e^{-x/\lambda}\, dx \qquad \frac{d}{d\lambda}\int = \int \frac{d}{d\lambda}$

$$\lambda = \lambda \int_0^\infty \frac{d}{d\lambda}\left(e^{-x/\lambda}\right) dx = \lambda \int_0^\infty (-x) \frac{-1}{\lambda^2} e^{-x/\lambda}\, dx$$

$$\lambda = \int_0^\infty x \frac{1}{\lambda} e^{-x/\lambda}\, dx = E[X]$$

Know $\mathbb{E}[X] = \lambda$ $\quad Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$\lambda = \int_0^\infty x \, \frac{1}{\lambda} e^{-x/\lambda} \, dx$$

$$\lambda^2 = \int_0^\infty x \, e^{-x/\lambda} \, dx$$ apply $\lambda \frac{d}{d\lambda}$

$$\lambda \frac{d}{d\lambda} (\lambda^2) = \int_0^\infty x \lambda \frac{d}{d\lambda} \left[e^{-x/\lambda} \right] dx$$ as $\frac{d}{dx} \int = \int \frac{d}{d\lambda}$

$$2\lambda^2 = \int_0^\infty \lambda x \cdot \frac{x}{\lambda^2} e^{-x/\lambda} \, dx = \int_0^\infty x^2 \frac{1}{\lambda} e^{-x/\lambda} \, dx$$

$$Var(X) = 2\lambda^2 - \lambda^2 = \lambda^2$$

$$St\ Dev(X) = \sigma = \lambda = \mu$$

Geometric! $X \in \{0, 1, 2, \dots\}$

$\text{Prob}(X = n) = (1-p)\,p^n$

p = prob failure
$1-p$ = prob success

Start with $\sum_{n=0}^{\infty} (1-p)\,p^n = 1$ or $\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}$

X = # failures before the first success

$E[X] = \sum_{n=0}^{\infty} n(1-p)\,p^n$ apply $\boxed{p\frac{d}{dp}}$ to $\boxed{\sum_{n=0}^{\infty} p^n = (1-p)^{-1}}$

Get $p(-1)(1-p)^{-2}(-1) = \sum_{n=0}^{\infty} p \cdot n\,p^{n-1}$

$$\frac{p}{(1-p)^2} = \sum_{n=0}^{\infty} n \cdot p^n \Longrightarrow \frac{p}{1-p} = \sum_{n=0}^{\infty} n(1-p)\,p^n = E[X]$$

$$\cancel{f(k;r,p) \equiv \Pr(X=k) = \binom{k+r-1}{r-1}(1-p)^k p^r}$$

Negative Binomial

Geometric density $(1-p)\,p^n$

$r = k$

$k = n$

(k successes)

Geometric is just the case $k=1$

$$\text{Prob}(X=n) = \binom{n+k-1}{k-1}(1-p)^k p^n$$

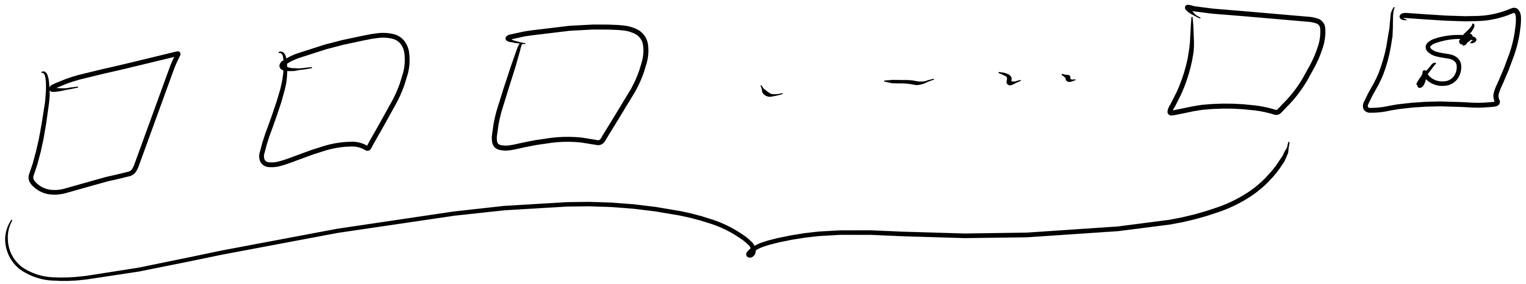
Let $X_1, \dots, X_k$ be iid rv Geom(p)

$$X = X_1 + \dots + X_k$$

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_k] = k\,\frac{p}{1-p}$$

$$Prob(X = n) = \binom{n+k-1}{k-1}(1-p)^k p^n$$

k successes n failures



$k-1$ successes
n failures

$n+k-1$ boxes
choose $k-1$ to be success

$$\binom{n+k-1}{k-1}(1-p)^k p^n$$

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} (1-p)^k p^n = 1$$

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} p^n = (1-p)^{-k} \quad \text{apply } p\frac{d}{dp}$$

$$p\frac{d}{dp}$$

$$\sum_{n=0}^{\infty} n \binom{n+k-1}{k-1} p^n = p(-k)(1-p)^{-k-1}(-1)$$

$$E[X] = \underbrace{\sum_{n=0}^{\infty} n \binom{n+k-1}{k-1} (1-p)^k p^n}_{E[X]} = \frac{kp}{1-p} = k\frac{p}{1-p}$$