Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/

Lecture 21: 11-3-21: https://youtu.be/9QQCnIrC53c (slides)

Lecture: 10/30/19: Pythagoras, Gamma Function, Chi-Square Distribution, Surface Area: https://youtu.be/X9Ujt8oicAI

Plan for the day: Lecture 21: November 3, 2021:

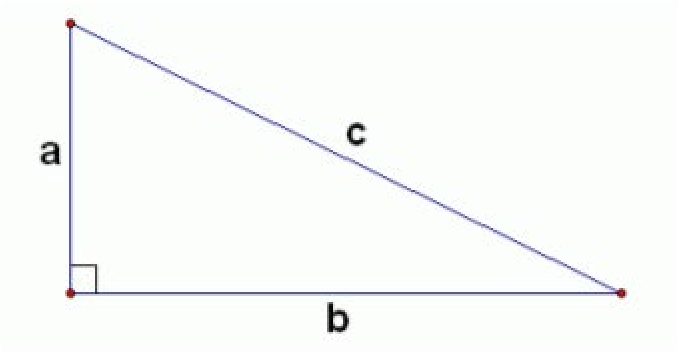
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Pythagorean Formula: Extending the Pythagorean Formula: http://youtu.be/idIHcgapMG4 (slides here)
- Gamma Function
- Chi-Square Distribution
- Surface Area

General items.

- Integrating without integrating…

Geometry Gem: Pythagorean Theorem



Theorem (Pythagorean Theorem)

Right triangle with sides a, b and hypotenuse c, then $a^2 + b^2 = c^2$.

Most students know the statement, but the proof?

Why are proofs important? Can help see big picture.

Geometric Proofs of Pythagoras

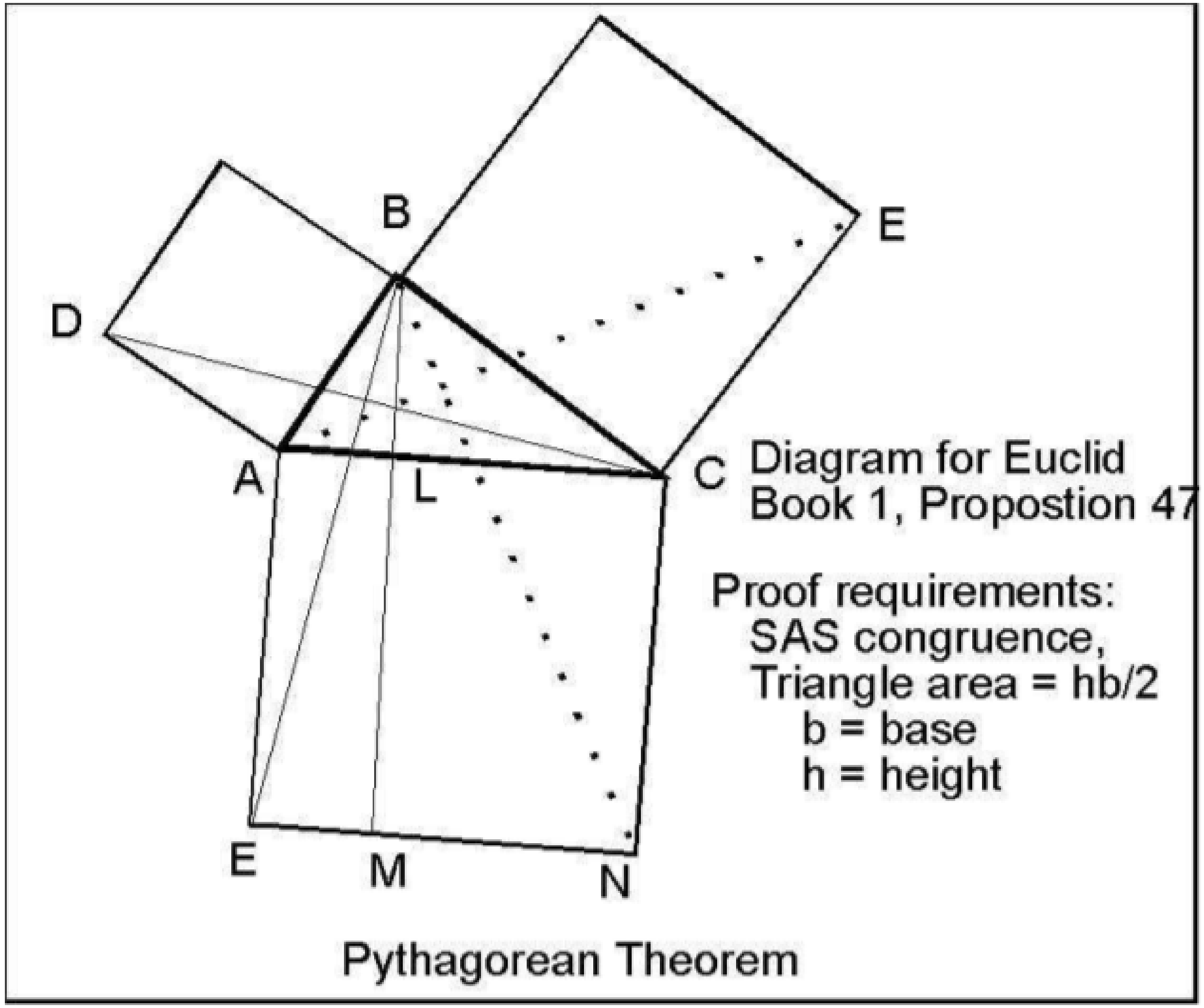


Figure: Euclid's Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?

Geometric Proofs of Pythagoras

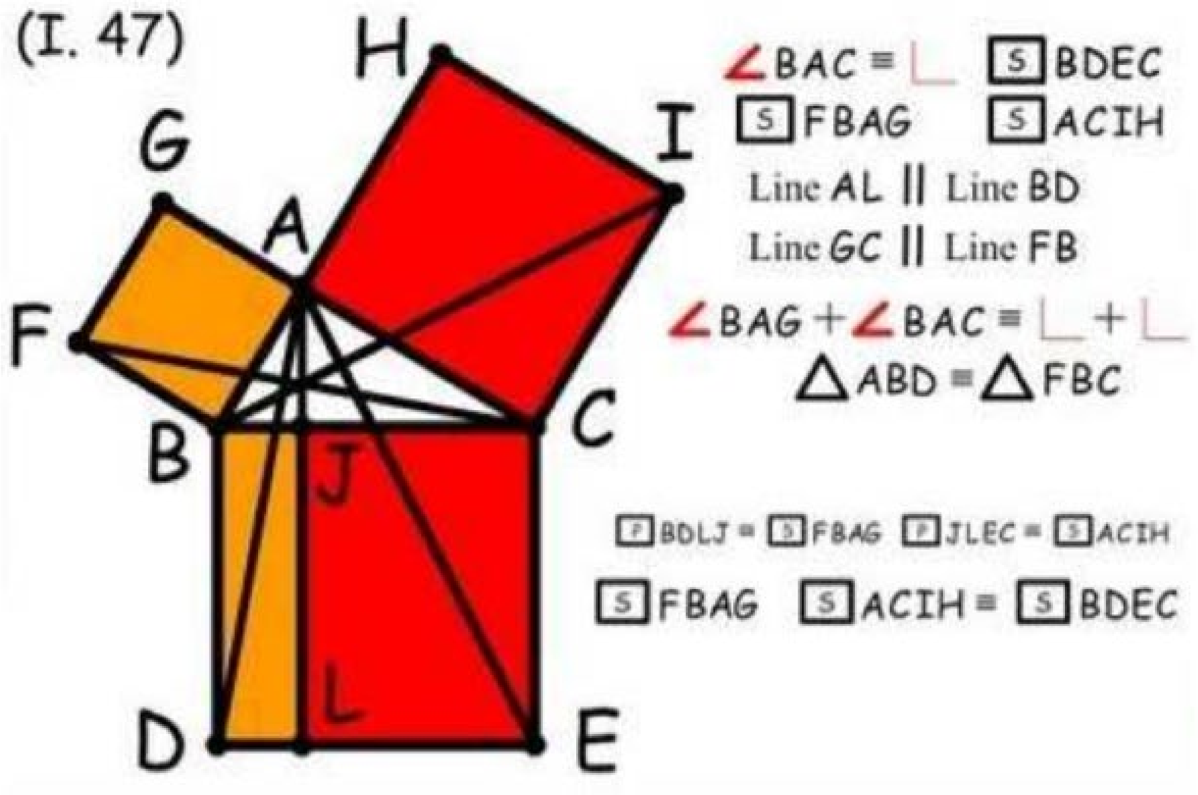


Figure: Euclid's Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?

Geometric Proofs of Pythagoras

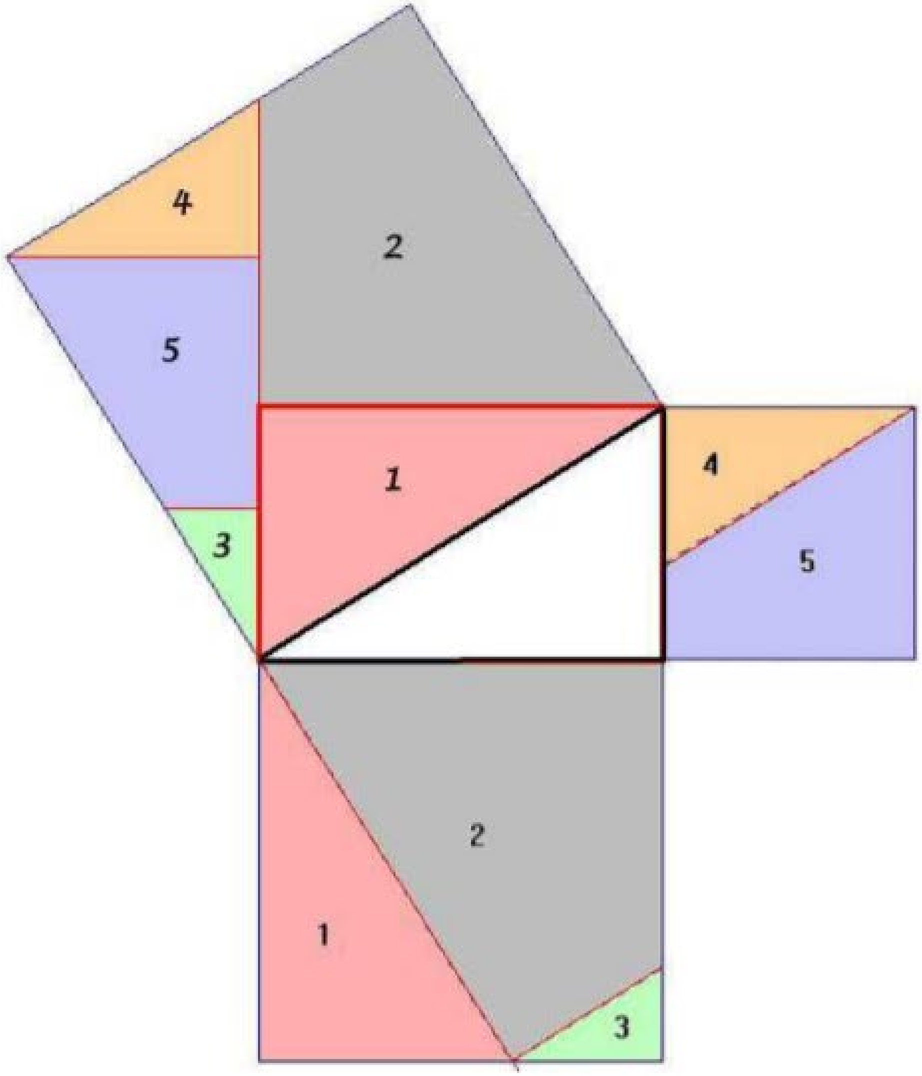
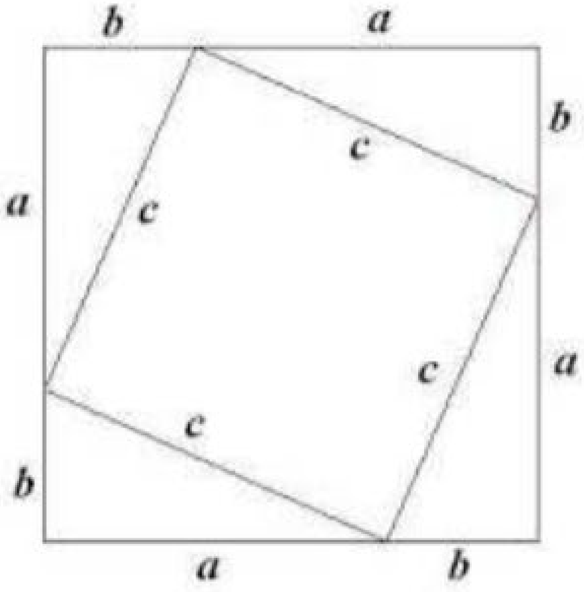


Figure: A nice matching proof, but how to find these slicings!

Geometric Proofs of Pythagoras



Big square: $(a+b)^2$

$$= a^2 + 2ab + b^2$$

Four triangles $= 2ab$

Little square $= c^2$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

$$a^2 + b^2 = c^2$$

Figure: Four triangles proof: I

Geometric Proofs of Pythagoras

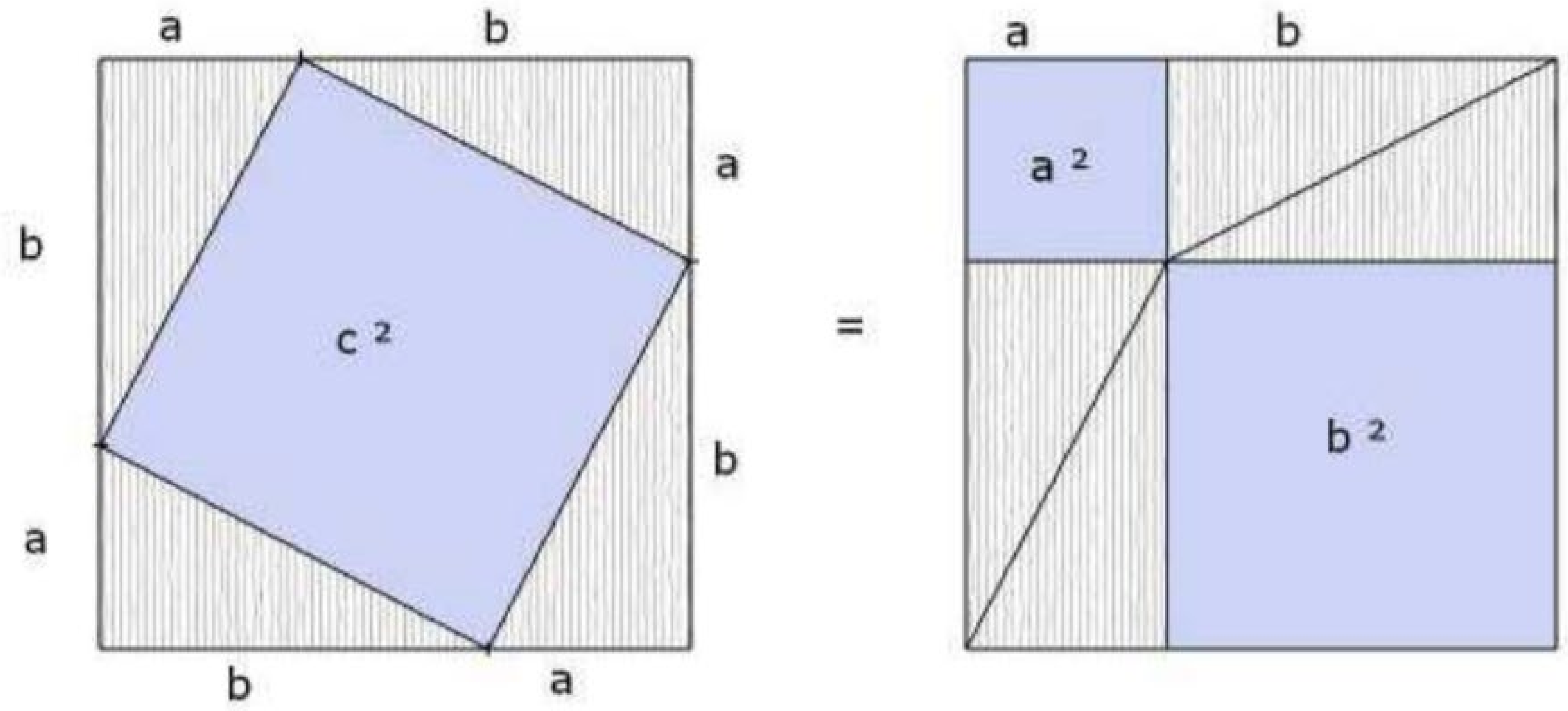


Figure: Four triangles proof: II

Geometric Proofs of Pythagoras

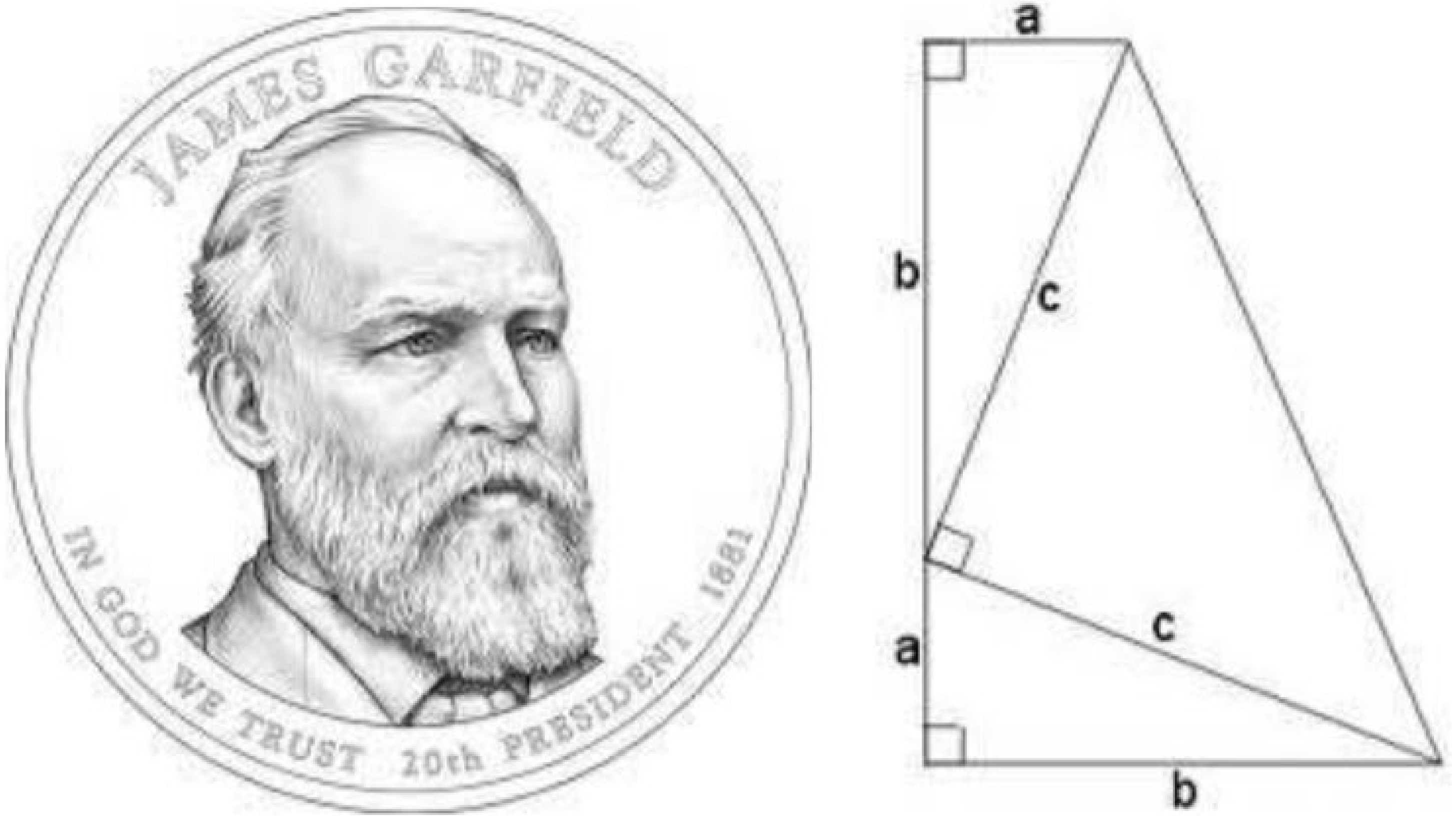


Figure: President James Garfield's (Williams 1856) Proof.

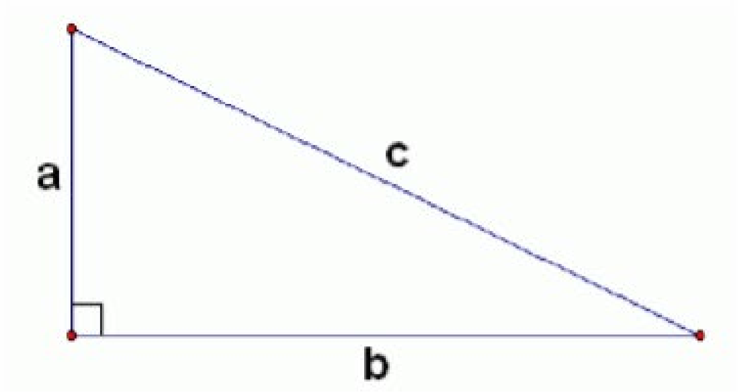
Geometric Proofs of Pythagoras

Lots of different proofs.

Difficulty: how to find these combinations?

At the end of the day, do you know *why* it's true?

Possible Pythagorean Theorems....

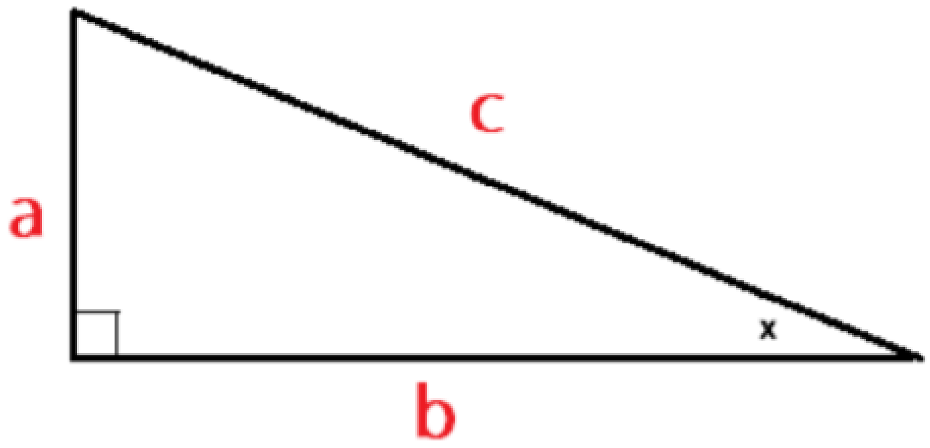


- $c^2 = a^3 + b^3$.
- $c^2 = a^2 + 2b^2$.
- $c^2 = a^2 - b^2$.
- $c^2 = a^2 + ab + b^2$.
- $c^2 = a^2 + 110ab + b^2$.

Possible Pythagorean Theorems....

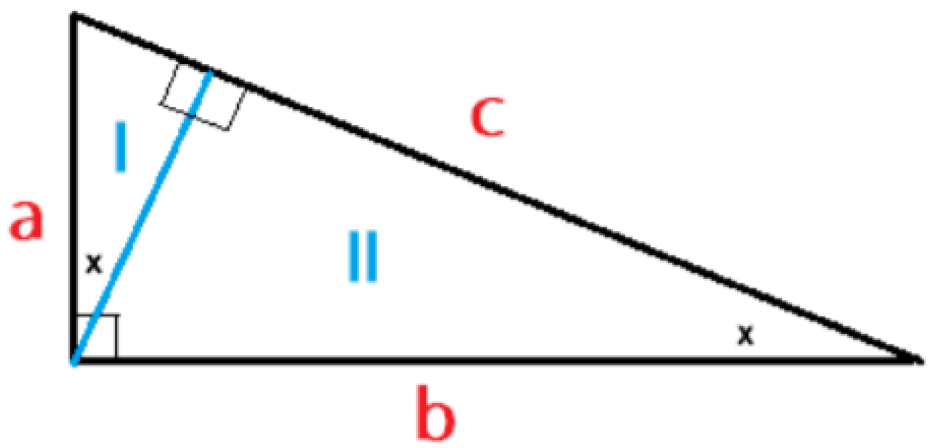
- ◇ $c^2 = a^3 + b^3$. No: wrong dimensions.
- ◇ $c^2 = a^2 + 2b^2$. No: asymmetric in a, b.
- ◇ $c^2 = a^2 - b^2$. No: can be negative.
- ◇ $c^2 = a^2 + ab + b^2$. Maybe: passes all tests.
- ◇ $c^2 = a^2 + 110ab + b^2$. No: violates $a + b > c$.

Dimensional Analysis Proof of the Pythagorean Theorem



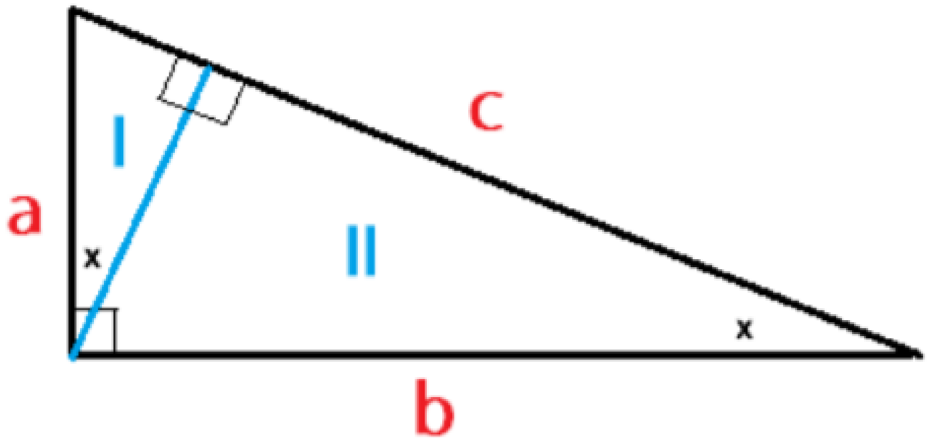
- Area is a function of hypotenuse c and angle x.
- $\mathrm{Area}(c, x) = f(x)c^2$ for some function f (similar triangles).
- Must draw an auxiliary line, but where? Need right angles!

Dimensional Analysis Proof of the Pythagorean Theorem



- ◇ Area is a function of hypotenuse c and angle x.
- ◇ $\mathrm{Area}(c, x) = f(x)c^2$ for some function f (CPCTC).
- ◇ Must draw an auxiliary line, but where? Need right angles!

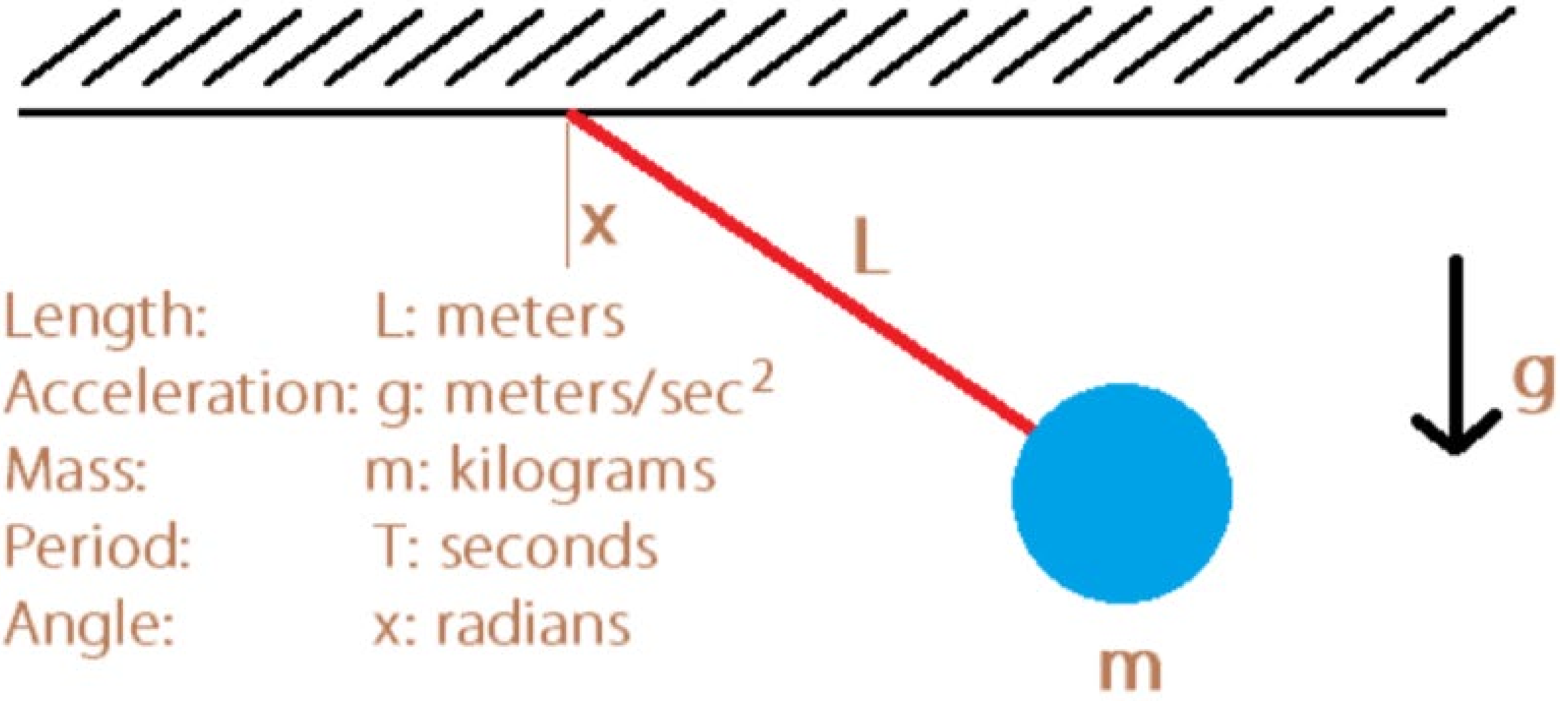
Dimensional Analysis Proof of the Pythagorean Theorem



- Area is a function of hypotenuse c and angle x.
- $\text{Area}(c, x) = f(x)c^2$ for some function f (CPCTC).
- Must draw an auxiliary line, but where? Need right angles!
- $f(x)a^2 + f(x)b^2 = f(x)c^2 \Rightarrow a^2 + b^2 = c^2$.

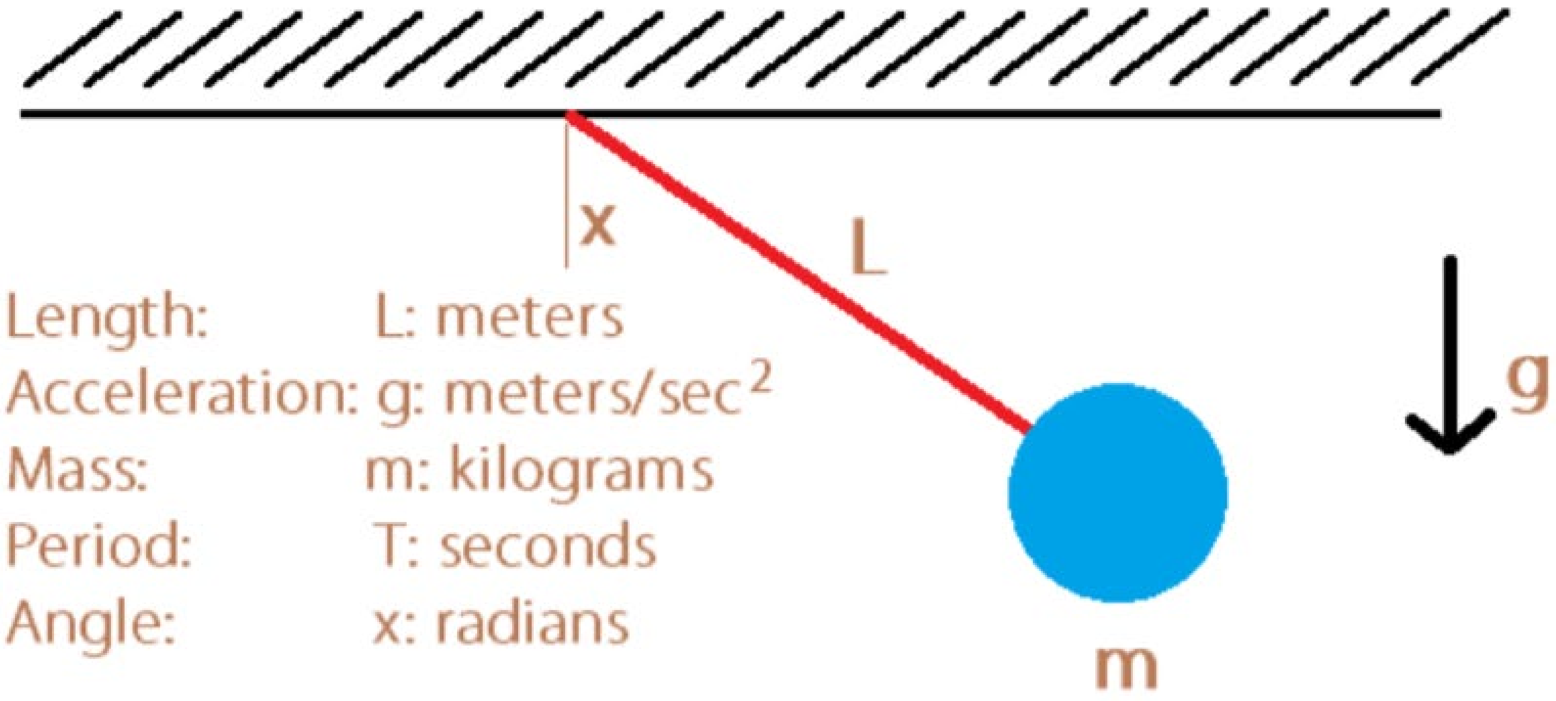
assume

Dimensional Analysis and the Pendulum



Length:	L: meters
Acceleration:	g: meters/sec^2
Mass:	m: kilograms
Period:	T: seconds
Angle:	x: radians

Dimensional Analysis and the Pendulum



Period: Need combination of quantities to get seconds.

$$T = f(x)\sqrt{L/g}.$$

For $s > 0$ (or actually $\Re(s) > 0$), the **Gamma function** $\Gamma(s)$ is

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx = \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

Existence of $\Gamma(s)$

e^{-x} great decay at ∞, x^{s-1} only poly growth

near 0, $e^{-x} \approx 1$, looks like $\int_0^\epsilon x^{s-1} dx = \frac{x^s}{s}\Big|_0^\epsilon$ = ok if $Re(s) > 0$

Say $s = -1/2$ get $\frac{x^{-1/2}}{-1/2}\Big|_0^\epsilon$ blows up

$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$

$\Gamma(3) = \int_0^\infty e^{-x} x^2 dx = 2$ $= 2!$

$\Gamma(2) = \int_0^\infty e^{-x} x\, dx$ = integrate by parts = 1 $= 0!$ $\Gamma(4) = 6$ $= 3!$

$\int_0^\infty x e^{-x} dx$ = Mean of exp dist with $\lambda = 1$ $= 1$ $= 1!$

Functional equation of $\Gamma(s)$: The Gamma function satisfies

$$\Gamma(s+1) \;=\; s\Gamma(s).$$

This allows us to extend the Gamma function to all s. We call the extension the Gamma function as well, and it's well-defined and finite for all s save the negative integers and zero.

Proved by $\int$ by parts

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^{s+1-1}\,dx = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$u = x^s \qquad dv = e^{-x}dx$

$du = s\,x^{s-1} \qquad v = -e^{-x}$

$$= \int_0^\infty e^{-x} s\,x^{s-1}\,dx = s\Gamma(s)$$

$$\Gamma(n+1) = n!$$

$\Gamma(s)$ **and the Factorial Function.** If n is a non-negative integer, then $\Gamma(n+1) = n!$. Thus the Gamma function is an extension of the factorial function.

The cosecant identity. If s is not an integer, then

$$\Gamma(s)\Gamma(1-s) \;=\; \pi\csc(\pi s) \;=\; \frac{\pi}{\sin(\pi s)}.$$

$$\Gamma(1/2) = \sqrt{\pi}.$$

view as $(-1/2)!$

Fundamental Relation of the Beta Function: For $a, b > 0$ we have

$$B(a,b) \;:=\; \int_0^1 t^{a-1}(1-t)^{b-1}dt \;=\; \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Beta distribution: Let $a, b > 0$. If X is a random variable with the **Beta distribution** with parameters a and b, then its density is

$$f_{a,b} \;=\; \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1}(1-t)^{b-1}dt & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

We write $X \sim B(a,b)$.

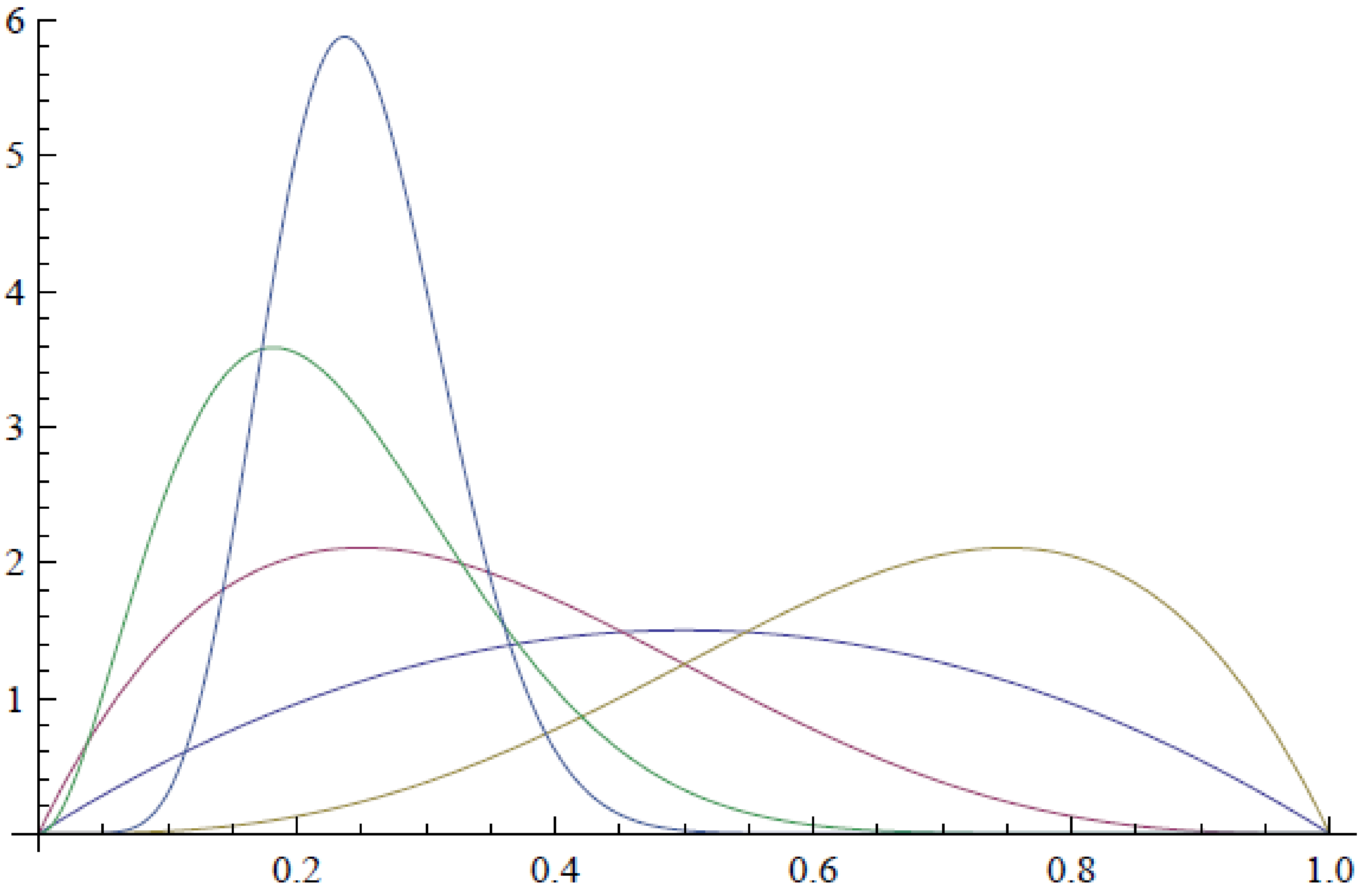


Figure 15.1: Plots of Beta densities for (a, b) equal to (2, 2), (2, 4), (4, 2), (3, 10), and (10, 30).

The Normal Distribution and the Gamma Function

$$\mu_{2m} = \int_{-\infty}^{\infty} x^{2m} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = (2m-1)!!$$

$$\mu_{2m} = \frac{2^m}{\sqrt{\pi}} \Gamma\left(m + \frac{1}{2}\right)$$

Build intuition: take $m = 0$

$$\mu_0 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$u = x^2/2$ $\quad du = x\,dx$

$x = (2u)^{1/2}$ $\quad dx = x^{-1} du$

$dx = (2u)^{-1/2} du$

$$\mu_0 = \frac{2}{\sqrt{2\pi}} \int_{x=0}^{\infty} e^{-x^2/2} dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{u=0}^{\infty} e^{-u}\, 2^{-1/2} u^{-1/2} du$$

$$= \frac{1}{\sqrt{\pi}} \int_{u=0}^{\infty} e^{-u} u^{\frac{1}{2}-1} du$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

$$\Gamma(s) := \int_0^{\infty} e^{-x} x^{s-1} dx$$

Generalize: have x^{2m} incorporated

$x^{2m} = (2u)^m = 2^m u^m$

yield $\mu_{2m} = \frac{2^m}{\sqrt{\pi}} \Gamma\left(m + \frac{1}{2}\right)$

The Gamma and Weibull Distributions. A random variable X has the **Gamma distribution** with (positive) parameters k and σ if its density is

$$f_{k,\sigma}(x) = \begin{cases} \frac{1}{\Gamma(k)\sigma^k} x^{k-1} e^{-x/\sigma} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We call k the **shape** parameter and σ the **scale** parameter, and write $X \sim \Gamma(k, \sigma)$ or $X \sim \text{Gamma}(k, \sigma)$.

A random variable X has the **Weibull** distribution with (positive) parameters k and σ if its density is

$$f_{k,\sigma}(x) = \begin{cases} (k/\sigma)(x/\sigma)^{k-1} e^{-(x/\sigma)^k} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We call k the **shape** parameter and σ the **scale** parameter, and write $X \sim W(k, \sigma)$.

Chi-square distribution: If X is a chi-square distribution with $\nu \geq 0$ degrees of freedom, then X has density

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2-1)} e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We write $X \sim \chi^2(\nu)$ to denote this.

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx$$

$\int_0^\infty x^{\frac{\nu}{2}-1} e^{-x/2} dx$ $\quad \frac{x}{2} = u$ so $dx = 2du$

after alg get claimed normalization constant

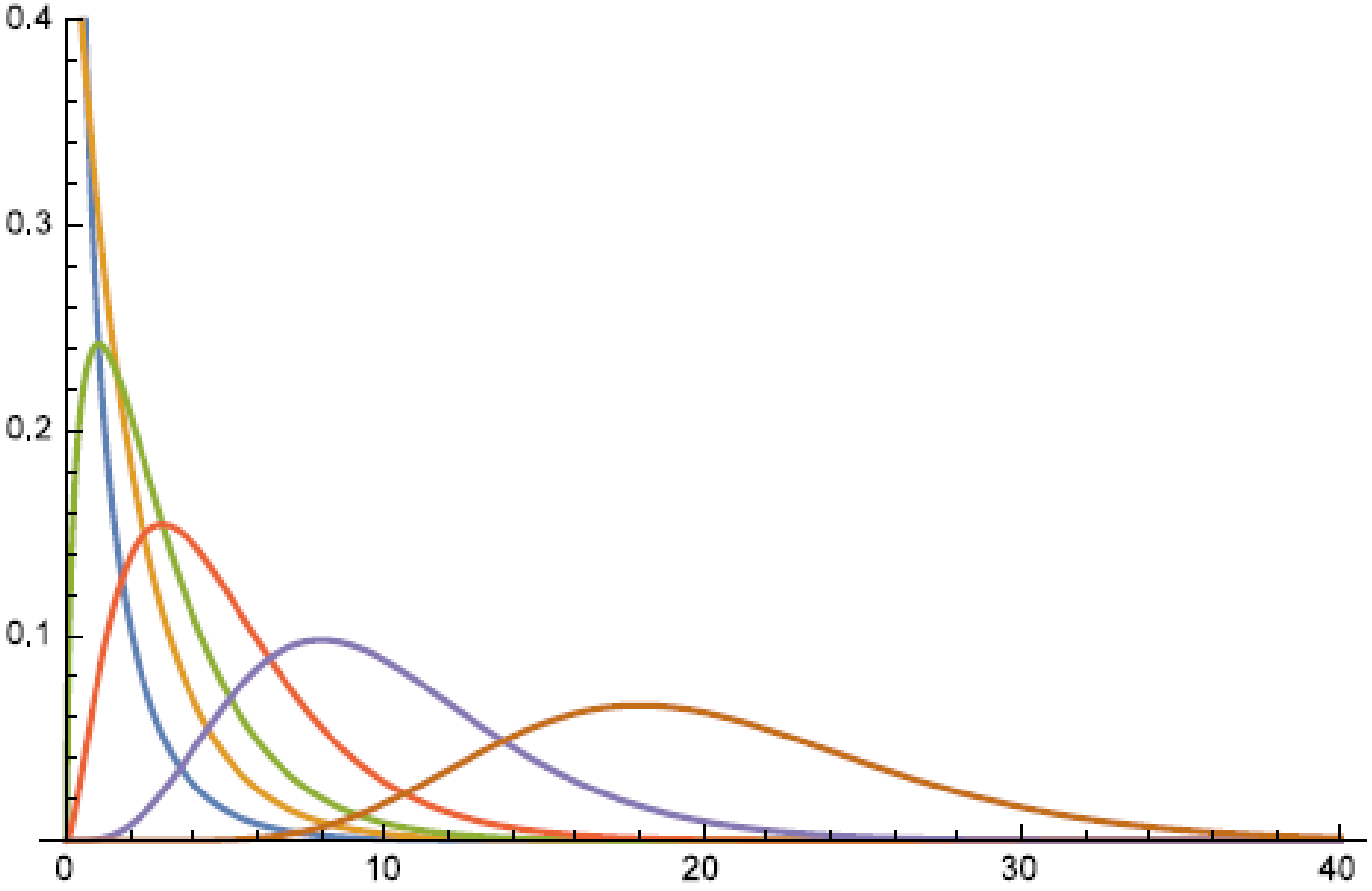


Figure 16.1: Plot of chi-square distributions with $\nu \in \{1, 2, 3, 5, 10, 20\}$; as the degree of freedom increases, the location of the bump moves rightward.

Relation between Chi-square and Normal Random Variables. If $X \sim N(0,1)$ then $X^2 \sim \chi^2(1)$.

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2-1)} e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

CDF technique : $Y = X^2$, $X \sim N(0,1)$

$y \geq 0$

$$\text{Prob}(Y \leq y) = \text{Prob}(X^2 \leq y)$$

$$= \text{Prob}(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = F_X'(\sqrt{y})(\sqrt{y})' - F_X'(-\sqrt{y})(-\sqrt{y})'$$

$$= f_X(\sqrt{y})\tfrac{1}{2}y^{-\frac{1}{2}} + f_X(-\sqrt{y})\tfrac{1}{2}y^{-1/2} = f_X(\sqrt{y})\,y^{-\frac{1}{2}}$$

(use f_X is even)

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} y^{-1/2} = \frac{y^{\frac{1}{2}-1} e^{-y/2}}{2^{1/2}\Gamma(1/2)}$$

Mean and Variance of $X \sim \chi^2(1)$

$$\mathbb{E}[X] = \int_0^\infty x f(x) dx = \int_0^\infty \frac{x}{2^{(1/2)}\Gamma(1/2)} x^{\frac{1}{2}-1} e^{-x/2} dx.$$

Chi-square distribution and sums of normal random variables: Let k be a positive integer, and $X_1, \ldots, X_k$ independent standard normal random variables; this means each $X_i \sim N(0,1)$. Then if $Y_k = X_1^2 + \cdots + X_k^2$, $Y_k \sim \chi^2(k)$. More generally, let $Y_{\nu_1}, \ldots, Y_{\nu_m}$ be m independent chi-square random variables, where $Y_{\nu_i} \sim \chi^2(\nu_i)$. Then $Y = Y_{\nu_1} + \cdots + Y_{\nu_m}$ is a chi-square random variable with $\nu_1 + \cdots + \nu_m$ degrees of freedom.

If $Y_{\nu_1} \sim \chi^2(\nu_1)$ and $Y_{\nu_2} \sim \chi^2(\nu_2)$ are two independent, chi-square random variables, then $Y_{\nu_1} + Y_{\nu_2} \sim \chi^2(\nu_1 + \nu_2)$.

$$((Y_1 + Y_2) + Y_3) + Y_4$$

$$\chi^2(\nu_1+\nu_2) + \chi^2(\nu_3)$$

$$\chi^2(\nu_1+\nu_2+\nu_3)$$

(Induction)

Grouping ( )

Change of Variables Theorem: Let V and W be bounded open sets in $\mathbb{R}^k$. Let $h : V \to W$ be a 1-1 and onto map, given by

$$h(u_1, \ldots, u_k) = (h_1(u_1, \ldots, u_k), \ldots, h_k(u_1, \ldots, u_k)).$$

Let $f : W \to \mathbb{R}$ be a continuous, bounded function. Then

$$\int \cdots \int_W f(x_1, \ldots, x_k) dx_1 \cdots dx_k$$
$$= \int \cdots \int_V f(h(u_1, \ldots, u_k)) J(u_1, \ldots, u_v) du_1 \cdots du_k,$$

where J is the Jacobian

$$J = \begin{vmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_k}{\partial u_1} & \cdots & \frac{\partial h_k}{\partial u_k} \end{vmatrix}.$$

Letting the densities be f_{ν_1} and f_{ν_2}, the density of $Y = Y_{\nu_1} + Y_{\nu_2}$ is

$$f_Y(y) = (f_{\nu_1} * \cdots * f_{\nu_2})(y) = \int_{-\infty}^{\infty} f_{\nu_1}(t) f_{\nu_2}(y-t)\,dt$$

$$= \int_0^y c_{\nu_1} t^{\frac{\nu_1}{2}-1} e^{-t/2} \cdot c_{\nu_2} (y-t)^{\frac{\nu_2}{2}-1} e^{-(y-t)/2}\,dt.$$

Chi-square distribution: If X is a chi-square distribution with $\nu \geq 0$ degrees of freedom, then X has density

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2-1)} e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We write $X \sim \chi^2(\nu)$ to denote this.

$$= c_{\nu_1} c_{\nu_2} e^{-y/2} \int_{t=0}^{y} t^{\frac{\nu_1}{2}-1} (y-t)^{\frac{\nu_2}{2}-1}\,dt$$

$t = yu$

$dt = y\,du$

remove y from integration and bounds

$$= c_{\nu_1} c_{\nu_2} e^{-y/2} \int_{u=0}^{1} (yu)^{\frac{\nu_1}{2}-1} (y-yu)^{\frac{\nu_2}{2}-1}\, y\,du$$

$$= c_{\nu_1} c_{\nu_2} e^{-y/2} y^{\frac{\nu_1}{2}-1+\frac{\nu_2}{2}-1+1} \underbrace{\int_{u=0}^{1} u^{\frac{\nu_1}{2}-1} (1-u)^{\frac{\nu_2}{2}-1}\,du}_{\text{Something indep of } y!}$$

$$= C c_{\nu_1} c_{\nu_2} e^{-y/2} y^{\frac{\nu_1+\nu_2}{2}-1}$$

Must be $\chi^2(\nu_1+\nu_2)$ ∎

Sums of squares by the Change of Variables Theorem

We now return to our problem. Let $Y = X_1^2 + \cdots + X_k^2$. We again use the cumulative distribution function technique and find

$$\begin{aligned} F_Y(y) &= \text{Prob}(X_1^2 + \cdots + X_k^2 \leq y) \\ &= \int \cdots \int_{x_1^2+\cdots+x_k^2 \leq y} \frac{1}{\sqrt{2\pi}}\, e^{-x_1^2/2} \cdots \frac{1}{\sqrt{2\pi}}\, e^{-x_k^2/2} dx_1 \cdots dx_k \\ &= \int \cdots \int_{x_1^2+\cdots+x_k^2 \leq y} \frac{1}{(2\pi)^{k/2}}\, e^{-(x_1^2+\cdots+x_k^2)/2} dx_1 \cdots dx_k. \end{aligned}$$