Math/Stat 341: Probability: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21

Lecture 24: 11-12-21: https://youtu.be/i0NX8vb9rWU (slides)

11/06/19: CLT for random walk of fair coin tosses, intro to generating fns via sums Poisson rvs: https://youtu.be/kVBIVl9uDTU

Plan for the day: Lecture 2: November , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/handouts/341Notes_Chap1.pdf

- Central Limit Theorem for fair coin
- Random Walks....
- Generating Functions
- Poisson Random Variables

General items.

- Power of Stirling's Formula
- Intuition from Special Cases, but dangers.... (prime counting?)
- Finding good approach through algebra: Generating Functions

Definition 20.2.1 (Normal distribution) *A random variable X is normally distributed (or has the normal distribution, or is a Gaussian random variable) with mean μ and variance σ^2 if the density of X is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We often write $X \sim N(\mu, \sigma^2)$ to denote this. If $\mu = 0$ and $\sigma^2 = 1$, we say X has the standard normal distribution.

Theorem 20.2.2 (Central Limit Theorem (CLT)) *Let $X_1, \ldots, X_N$ be independent, identically distributed random variables whose moment generating functions converge for $|t| < \delta$ for some $\delta > 0$ (this implies all the moments exist and are finite). Denote the mean by μ and the variance by σ^2, let*

$$\overline{X}_N = \frac{X_1 + \cdots + X_N}{N}$$

and set

$$Z_N = \frac{\overline{X}_N - \mu}{\sigma/\sqrt{N}}.$$

Then as $N \to \infty$, the distribution of Z_N converges to the standard normal (see Definition 20.2.1 for a statement).

The Gamma function. The Gamma function $\Gamma(s)$ is

$$\Gamma(s) \; = \; \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0.$$

$\Gamma(n+1) = n!$, $n = 0, 1, 2, \ldots$ “$x^s \frac{dx}{x}$”

Stirling's formula: As $n \to \infty$, we have

$$n! \; \approx \; n^n e^{-n} \sqrt{2\pi n};$$

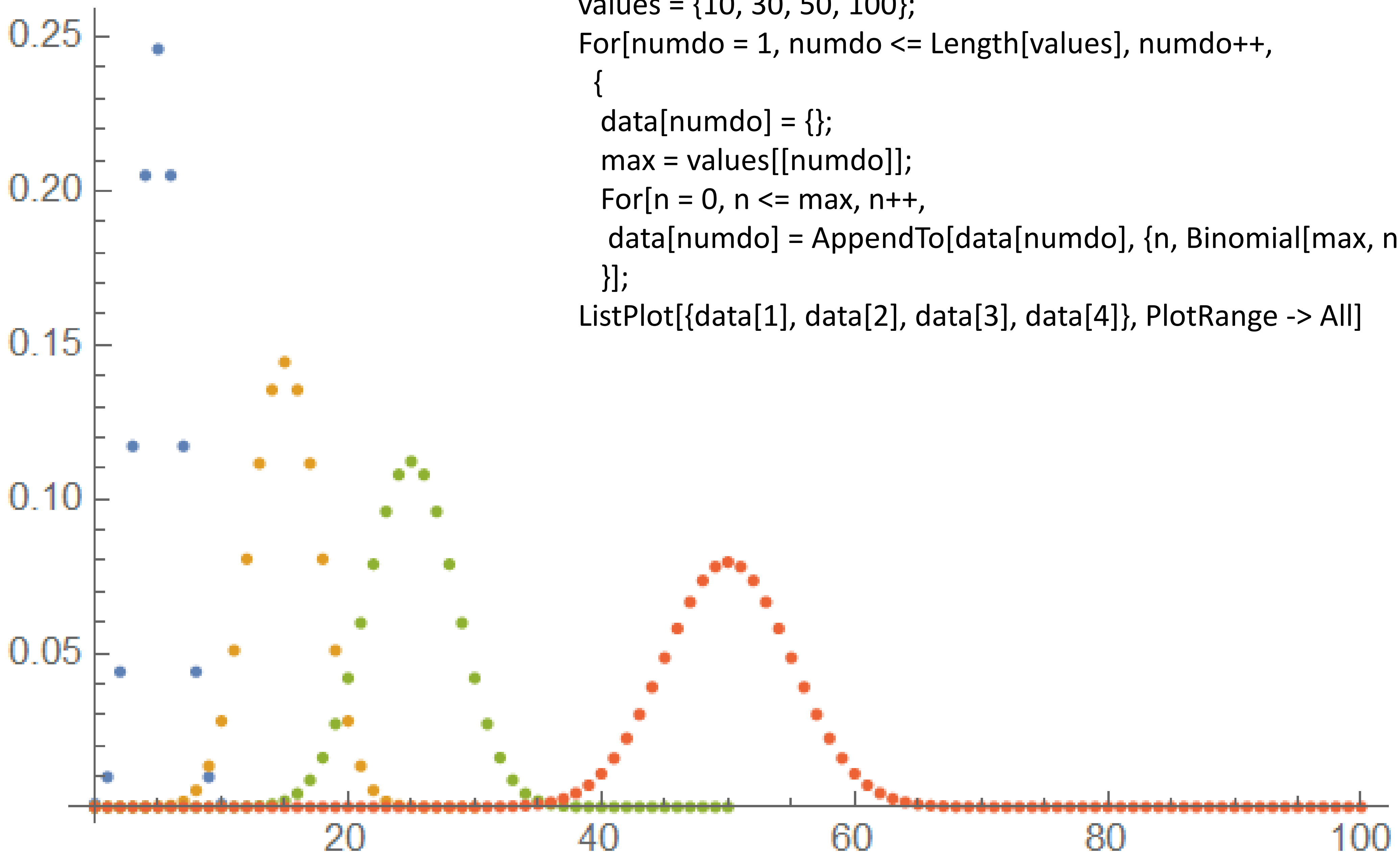
by this we mean

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \; = \; 1.$$

More precisely, we have the following series expansion:

$$n! \; = \; n^n e^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots \right).$$

```
values = {10, 30, 50, 100};
For[numdo = 1, numdo <= Length[values], numdo++,
  {
   data[numdo] = {};
   max = values[[numdo]];
   For[n = 0, n <= max, n++,
    data[numdo] = AppendTo[data[numdo], {n, Binomial[max, n]/2^max}]];
   }];
ListPlot[{data[1], data[2], data[3], data[4]}, PlotRange -> All]
```



$$\text{Prob}(X_i = n) = \begin{cases} 1/2 & \text{if } n = 1 \\ 1/2 & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases} \qquad S_{2N} = X_1 + \cdots + X_{2N}$$

$\mathbb{E}[S_{2N}] = 0$

$\text{Var}(S_{2N}) = 2N$

$S_{2N} \sim \text{Bin}(\frac{1}{2}, 2N)$

$$\text{Var}(X_i) = \frac{1}{2}(1-0)^2 + \frac{1}{2}(-1-0)^2 = 1$$

$\text{Prob}(S_{2N} = 2k)$ means have $N+k$ heads, $N-k$ tails

$$= \binom{2N}{N+k}\left(\frac{1}{2}\right)^{N+k}\left(\frac{1}{2}\right)^{N-k} = \binom{2N}{N+k}\frac{1}{2^{2N}}$$

as coin fair!

$$\text{Prob}(S_{2N} = 2k) = \frac{2N!}{(N+k)!\,(N-k)!}\,\frac{1}{2^{2N}}$$

St Dev is of size $\sqrt{2N}$

Prob we are $\log N$ st dev from mean is at most $\left(\frac{1}{\log N}\right)^2 \to 0$

with prob ≈ 1, we have $0 - (\log N)\sqrt{2N} \le 2k \le 0 + (\log N)\sqrt{2N}$

means $N+k,\ N-k \approx N$ so factorials are large!

$$\text{Prob}(S_{2N} = 2k) = \frac{(2N)!}{(N+k)!\,(N-k)!}\,\frac{1}{2^{2N}} \qquad n! \approx n^n e^{-n}\sqrt{2\pi n}$$

$$= \frac{2^{2N} N^{2N} e^{-2N}\sqrt{2\pi\cdot 2N}}{(N+k)^{N+k}(N-k)^{N-k}e^{-2N}\sqrt{2\pi(N+k)}\sqrt{2\pi(N-k)}}\,\frac{1}{2^{2N}}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{\sqrt{2N}}{\sqrt{(N+k)(N-k)}}\left(1+\frac{k}{N}\right)^{-(N+k)}\left(1-\frac{k}{N}\right)^{-(N-k)}$$

$$\approx \frac{1}{\sqrt{\pi N}}\left(1+\frac{k}{N}\right)^{-(N+k)}\left(1-\frac{k}{N}\right)^{-(N-k)}$$

Try: 1st $\rightarrow \left(1+\frac{k}{N}\right)^{-N} = \frac{1}{(1+k/N)^N} = \frac{1}{e^k}$

Note: $\left(1+\frac{x}{N}\right)^N \longrightarrow e^x$

$$\left(1-\frac{k}{N}\right)^{-N} = \frac{1}{(1-k/N)^N} = \frac{1}{e^{-k}}$$

Product ≈ 1 BAD!

Lemma 18.3.1 *For any $\epsilon \le 1/9$, for $N \to \infty$ with $|k| \le (2N)^{1/2+\epsilon}$, we have*

$$\left(1+\frac{k}{N}\right)^{N+\frac{1}{2}+k}\left(1-\frac{k}{N}\right)^{N+\frac{1}{2}-k} \longrightarrow e^{k^2/N}e^{O(N^{-1/6})}.$$

$|k| \ll N$

$|k| \sim \sqrt{N}$

$\log(1-x) \approx -x - \frac{x^2}{2}$

$\log(1+x) \approx x - \frac{x^2}{2}$

$$P = \left(1+\frac{k}{N}\right)^{N+k}\left(1-\frac{k}{N}\right)^{N-k}$$

$$\log P = (N+k)\log\left(1+\frac{k}{N}\right) + (N-k)\log\left(1-\frac{k}{N}\right)$$

$$= (N+k)\left[\frac{k}{N} - \frac{1}{2}\frac{k^2}{N^2} + O\left(\frac{k^3}{N^3}\right)\right] + (N-k)\left[-\frac{k}{N} - \frac{1}{2}\frac{k^2}{N^2} + O\left(\frac{k^3}{N^3}\right)\right]$$

$$= 2\frac{k^2}{N} - \frac{k^2}{N} + \text{DAMN SMALL} = \frac{k^2}{N} + \text{much smaller.}$$

$$P = e^{k^2/N}\, e^{\text{much smaller}}$$

$$\text{Prob}(S_{2N} = 2k) \approx \frac{1}{\sqrt{\pi N}} e^{-k^2/N} \underbrace{e^{\text{much smaller}}}_{\text{Goes to 1 as } N \to \infty}$$

$$E[S_{2N}] = 0 \qquad \text{Var}(S_{2N}) = 2N$$

$$\hookrightarrow N(0, 2N) \quad \text{or} \quad \frac{1}{\sqrt{2\pi \cdot 2N}} e^{-x^2/2\cdot 2N}$$

$$e^{-k^2/N} = e^{-(2k)^2/4N} = e^{-(2k)^2/2\cdot 2N}$$

$$\frac{1}{\sqrt{\pi N}} = \frac{1}{\sqrt{2\pi \cdot 2N/4}} = \frac{2}{\sqrt{2\pi \cdot 2N}}$$

$$\text{Prob}(S_{2N} = 2k) = \frac{2}{\sqrt{2\pi \cdot 2N}} e^{-(2k)^2/2\cdot 2N} \quad \checkmark$$

$$\text{Prob}(S_{2N} = 2k) = \binom{2N}{N+k}\frac{1}{2^{2N}} \approx \frac{2}{\sqrt{2\pi\cdot(2N)}}e^{-(2k)^2/2(2N)}.$$

Difficulty is seeing the path thru the algebra!

Input: $E[S_{2N}] = 0$

$Var(S_{2N}) = 2N$

$$e^{-k^2/N} = e^{-(2k)^2/2\cdot 2N}$$

Definition 19.2.1 (Generating Function) *Given a sequence* $\{a_n\}_{n=0}^{\infty}$*, we define its generating function by*

$$G_a(s) = \sum_{n=0}^{\infty} a_n s^n$$

for all s *where the sum converges.*

$a_n = 1$

$G_1(s) = \sum_{n=0}^{\infty} 1 \cdot s^n = \frac{1}{1-s}$ if $|s|<1$

Generating Function (Example: Binet's Formula)

Binet's Formula

$F_0 = 0$

$$F_1 = F_2 = 1; \; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$\begin{aligned}
(1) \quad & \Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
& \Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
& \Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2 \sum_{n\geq 1} F_n x^n \\
& \Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x) \\
& \Rightarrow g(x) = x/(1 - x - x^2).
\end{aligned}$$

Partial Fraction Expansion (Example: Binet's Formula)

- Generating function: $g(x) = \sum_{n>0} \boldsymbol{F}_n x^n = \frac{x}{1-x-x^2}$.

- Partial fraction expansion:

$$\Rightarrow \; g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x}\right).$$

Coefficient of x^n (power series expansion):

$$\boldsymbol{F}_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

X has a Poisson distribution with parameter λ means

$$a_n = \text{Prob}(X = n) = \begin{cases} \frac{\lambda^n e^{-\lambda}}{n!} & \text{if } n \geq 0 \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases} \qquad G_a(s) = \sum_{n=0}^{\infty} a_n s^n$$

$$G_a(s) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} s^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} = e^{-\lambda} e^{\lambda s}$$

$$= e^{\lambda(s-1)} \quad \text{or} \quad e^{-\lambda(1-s)} \qquad \text{valid for all } s!$$

$$X_k \sim \text{Poiss}(\lambda_k) \qquad X = X_1 + X_2 \sim \text{Poiss}(\lambda_1 + \lambda_2)$$

$$G_{X_1}(s) = e^{\lambda_1(s-1)} \qquad G_{X_2}(s) = e^{\lambda_2(s-1)}$$

$$G_X(s) = e^{(\lambda_1+\lambda_2)(s-1)} = e^{\lambda_1(s-1) + \lambda_2(s-1)} = G_{X_1}(s)\, G_{X_2}(s)$$

The Generating Fn of a sum of indep RV is the prod of the Gen fns ...

Lemma 19.4.2 *Let $G_a(s)$ be the generating function for $\{a_m\}_{m=0}^{\infty}$ and $G_b(s)$ the generating function for $\{b_n\}_{n=0}^{\infty}$. Then the generating function of $c = a * b$ is $G_c(s) = G_a(s)G_b(s)$.*

We can see why this is useful for the Central Limit Theorem….

A sum of independent random variables is a convolution.

Thus when we are studying the sum that arises in the CLT we see the generating function of the sum is the product of the generating functions.

As the random variables are identically distributed it is just one generating function raised to a large power, which we have a Pavlovian response and take logarithms….

Sadly generating functions do not always exist and are not always the most accessible object to study, so we study cousins: the moment generating function (wanna guess what that gives!) and the characteristic function (this involves complex analysis, but unlike the other two will always exist in a neighborhood of the origin).