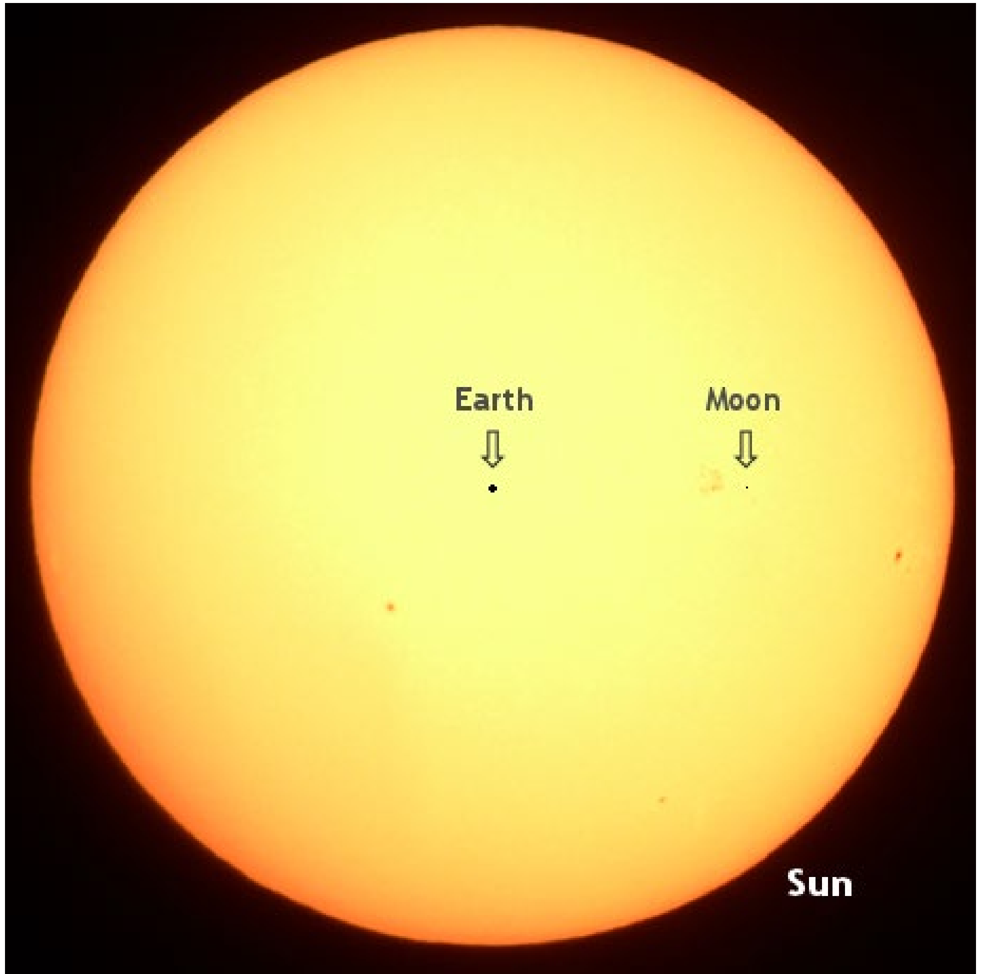
Introduction to Logarithms

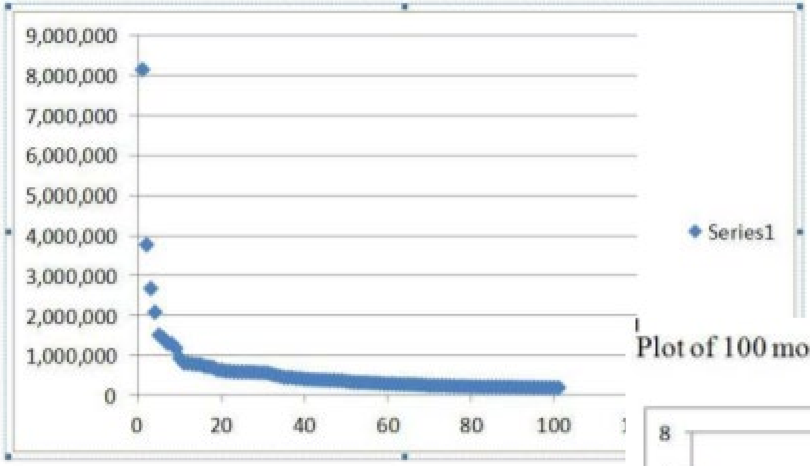
Steven Miller, Williams College
sjm1@Williams.edu

Why do we care about Logarithms

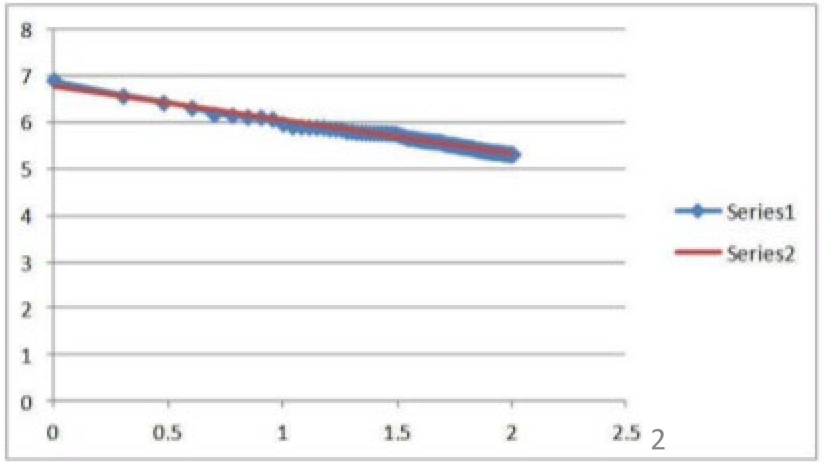
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities



Plot of 100 most populous cities: log-log plot



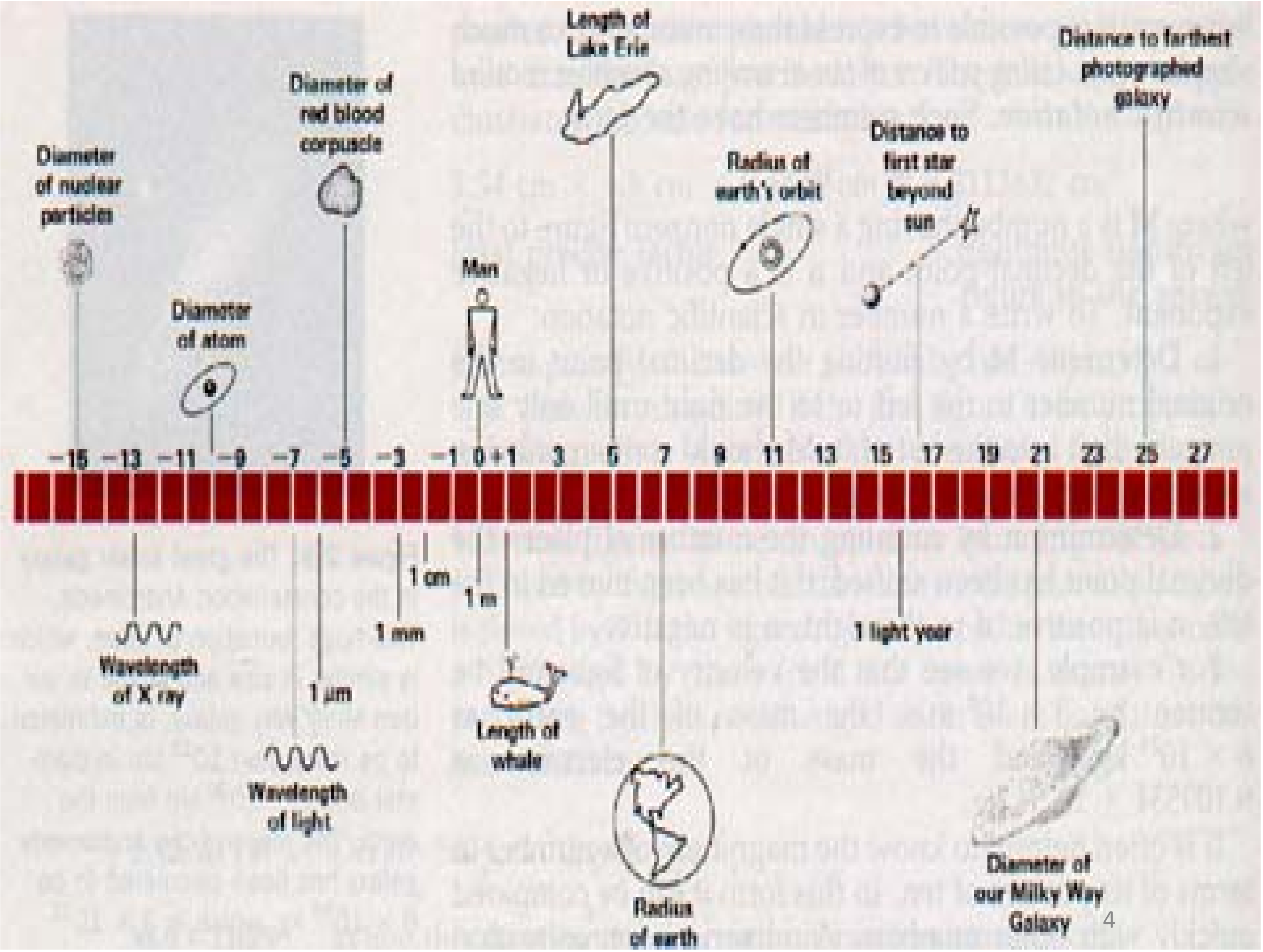
Definition of Logarithms

- **If $x = b^y$ then $\log_b x = y$.**
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x, which is its logarithm base e (e is approximately 2.71828).

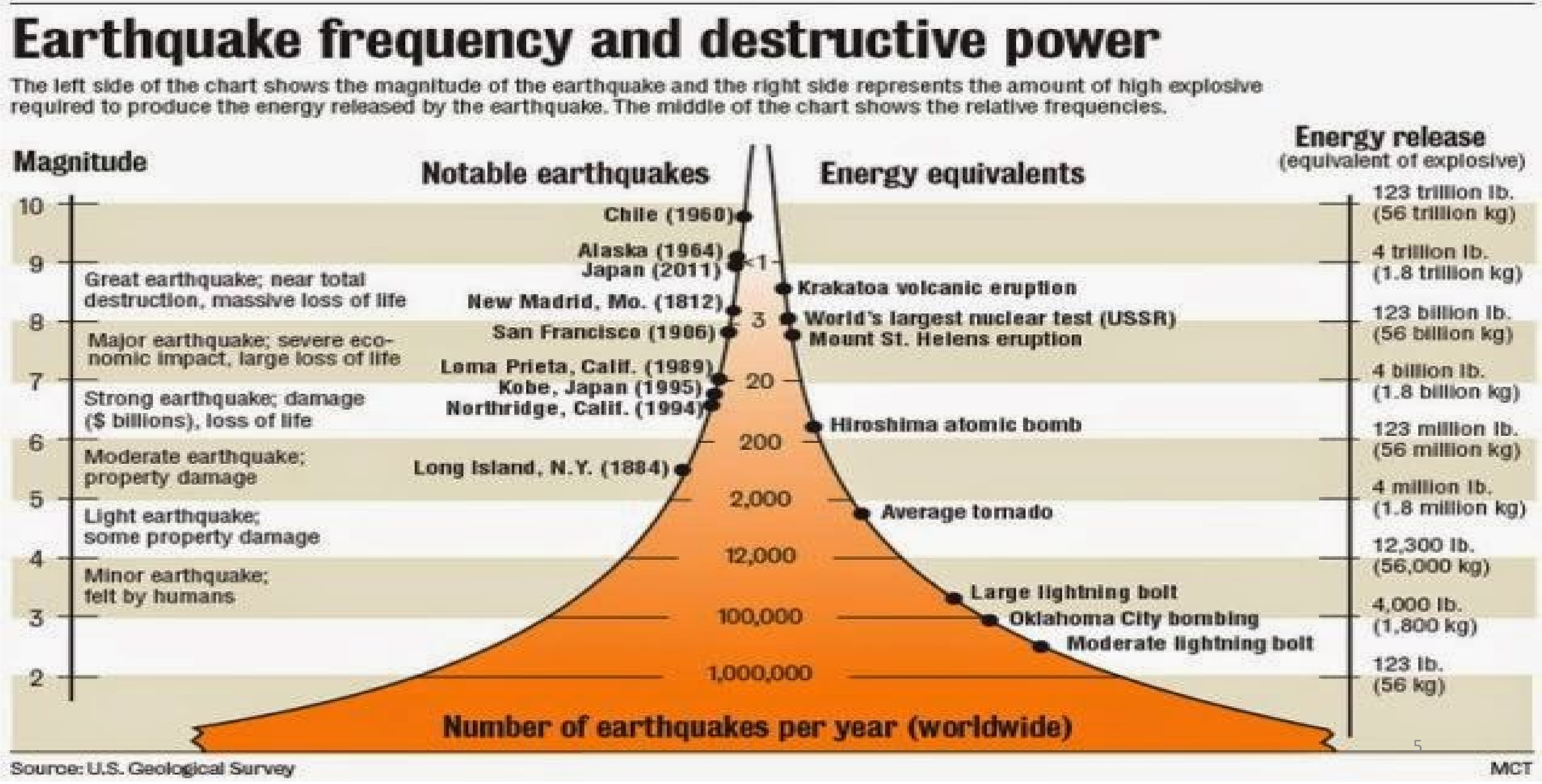
- **Examples: $\log_b x = y$ means we need y powers of b to get x.**
  - $100 = 10^2$ becomes $\log_{10} 100 = 2$. In base e it is about 4.6.
  - $1 = 10^0$ becomes $\log_{10} 1 = 0$. In base e it is still 0.
  - $.001 = 10^{-3}$ becomes $\log_{10} .001 = -3$. In base e it is about -6.9.

Examples of Logarithms

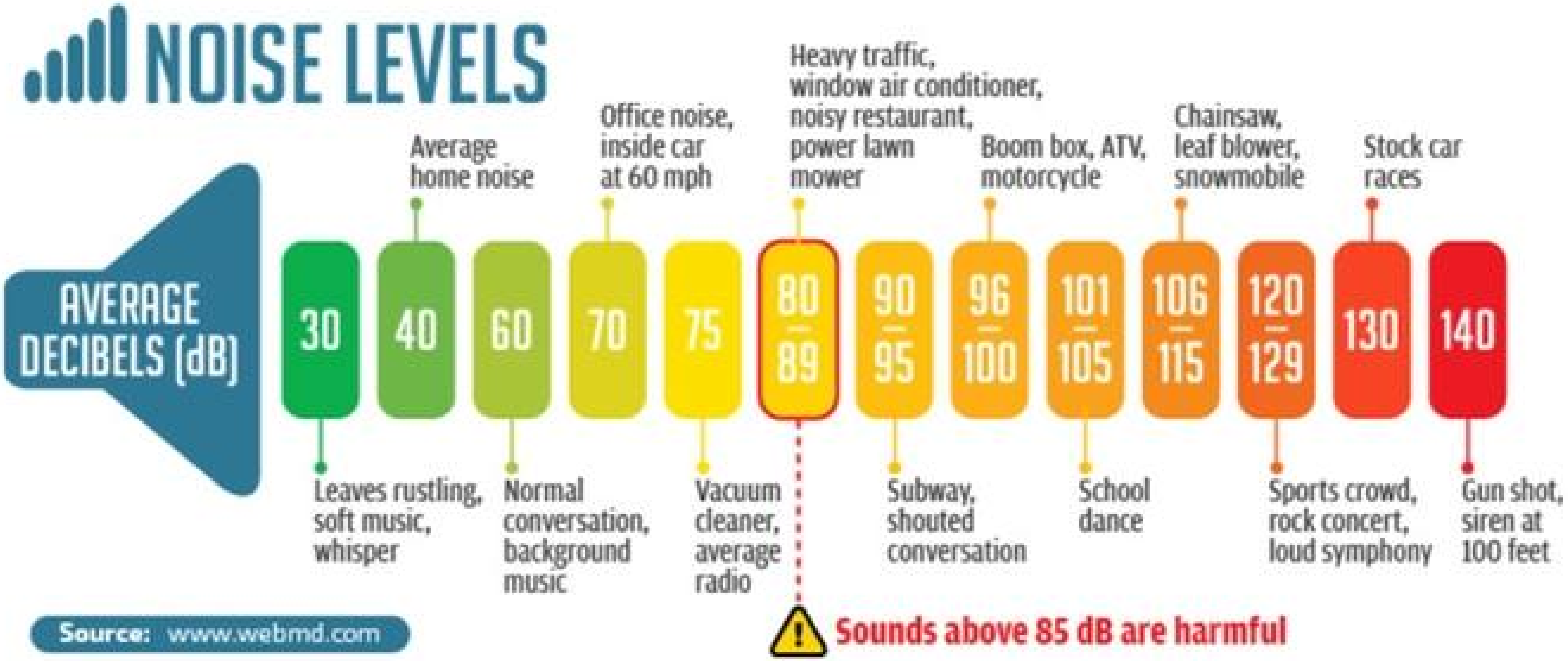
Order of Magnitude of some Lengths	
LENGTH	**meters**
radius of proton	10^{-15}
radius of atom	10^{-10}
radius of virus	10^{-7}
radius of amoeba	10^{-4}
height of human being	10^{0}
radius of earth	10^{7}
radius of sun	10^{9}
earth-sun distance	10^{11}
radius of solar system	10^{13}
distance of sun to nearest star	10^{16}
radius of milky way galaxy	10^{21}
radius of visible Universe	10^{26}



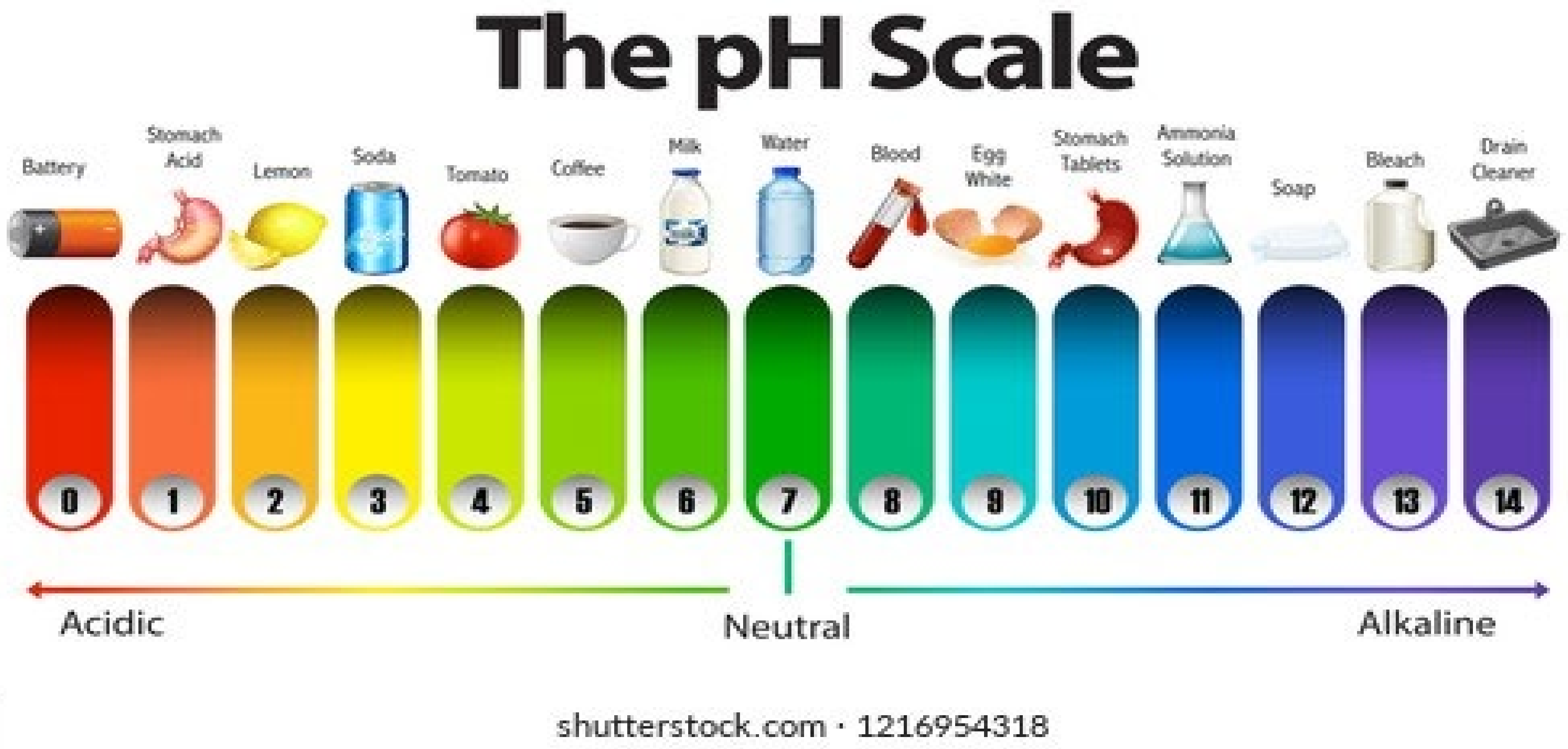
Examples of Logarithms



Examples of Logarithms



Examples of Logarithms



Recall: Definition of Logarithms

- **If $x = b^y$ then $\log_b x = y$.**
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
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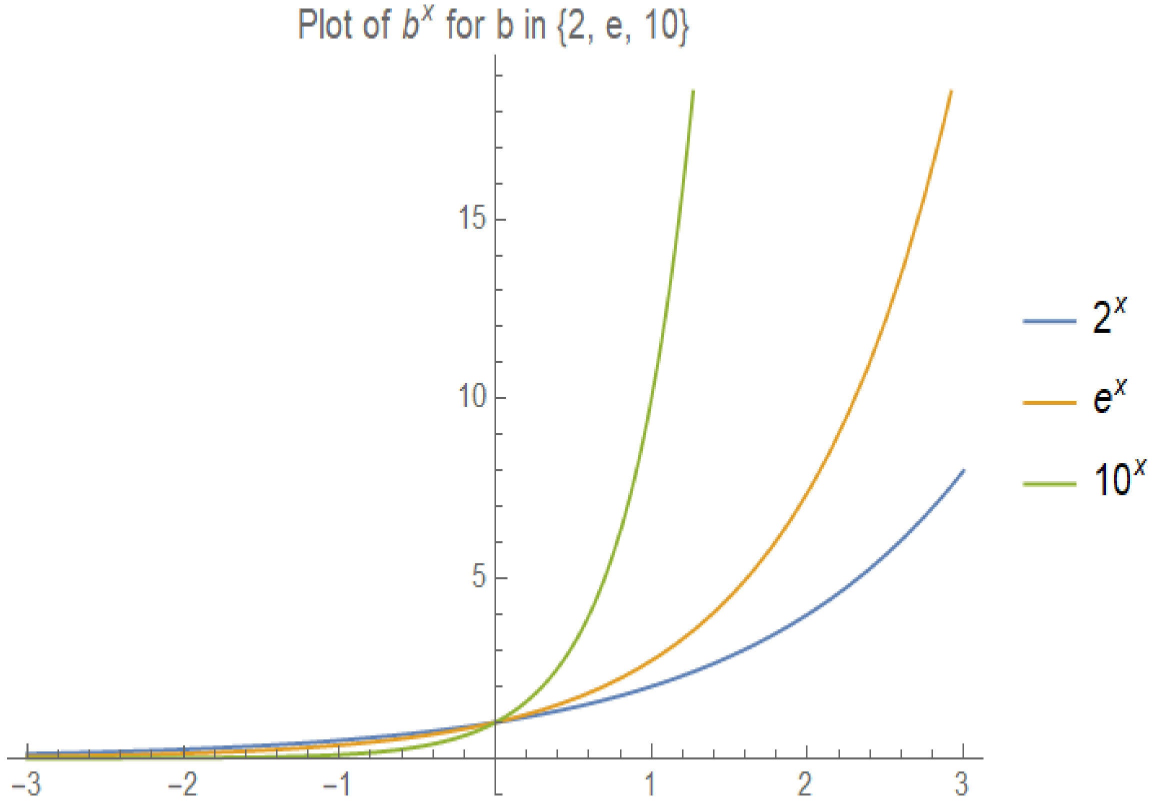
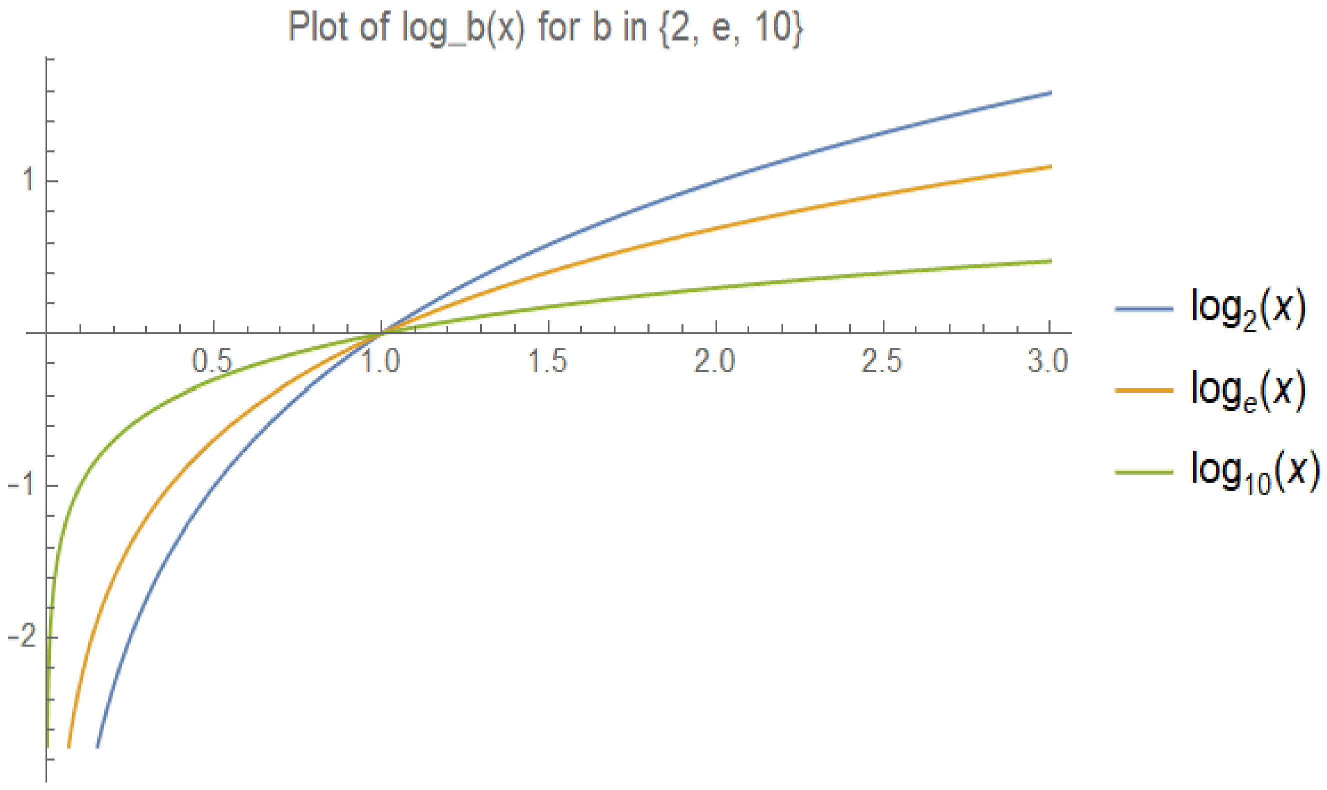
Plots of Exponentiation and Logarithms

- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y.

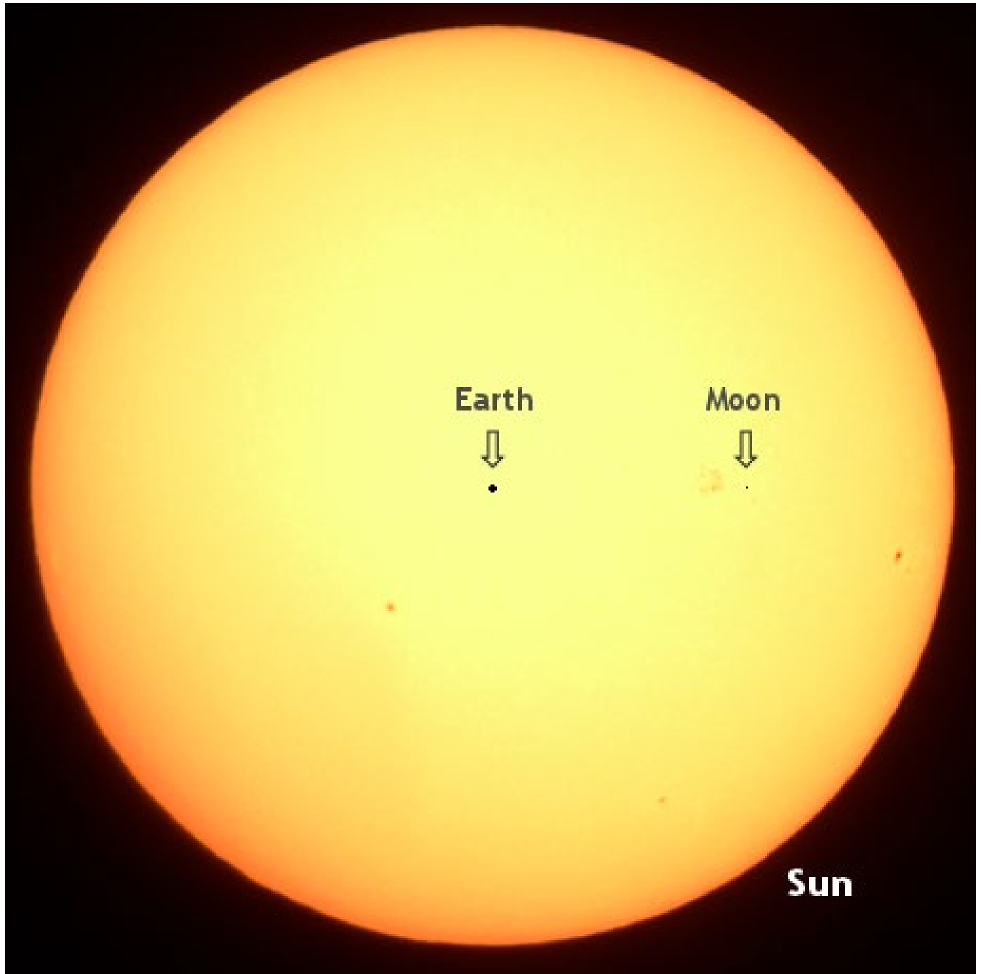




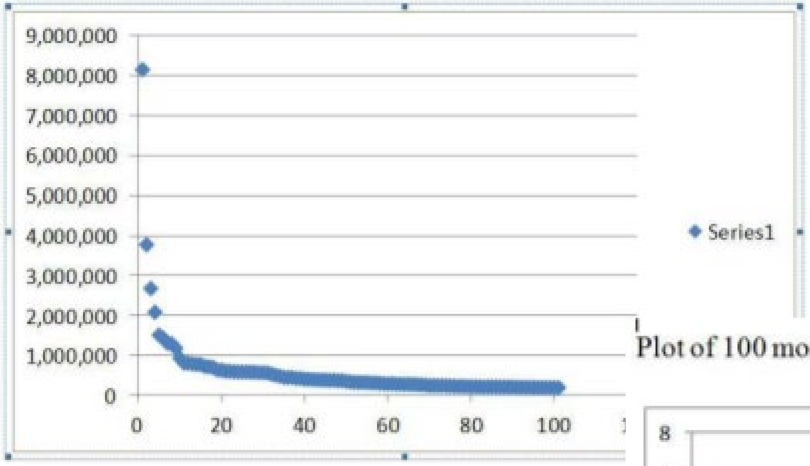
$$10^n \le x \le 10^{n+1} \quad \text{implies} \quad n \le \log_{10} x \le n + 1.$$

Why do we care about Logarithms

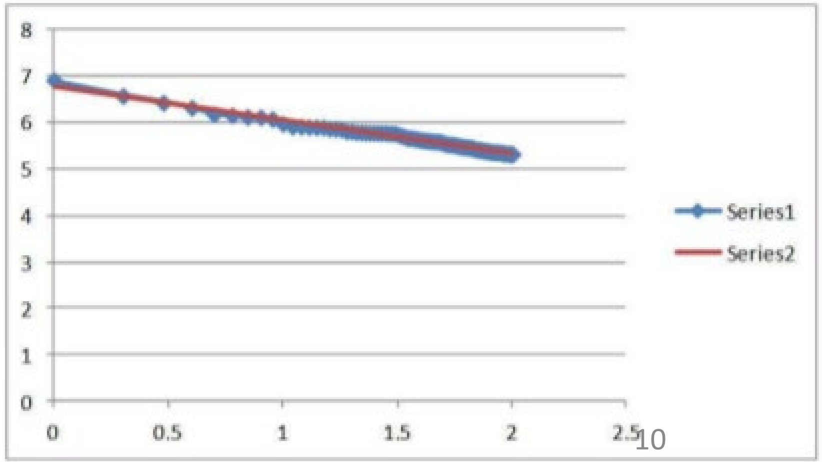
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities

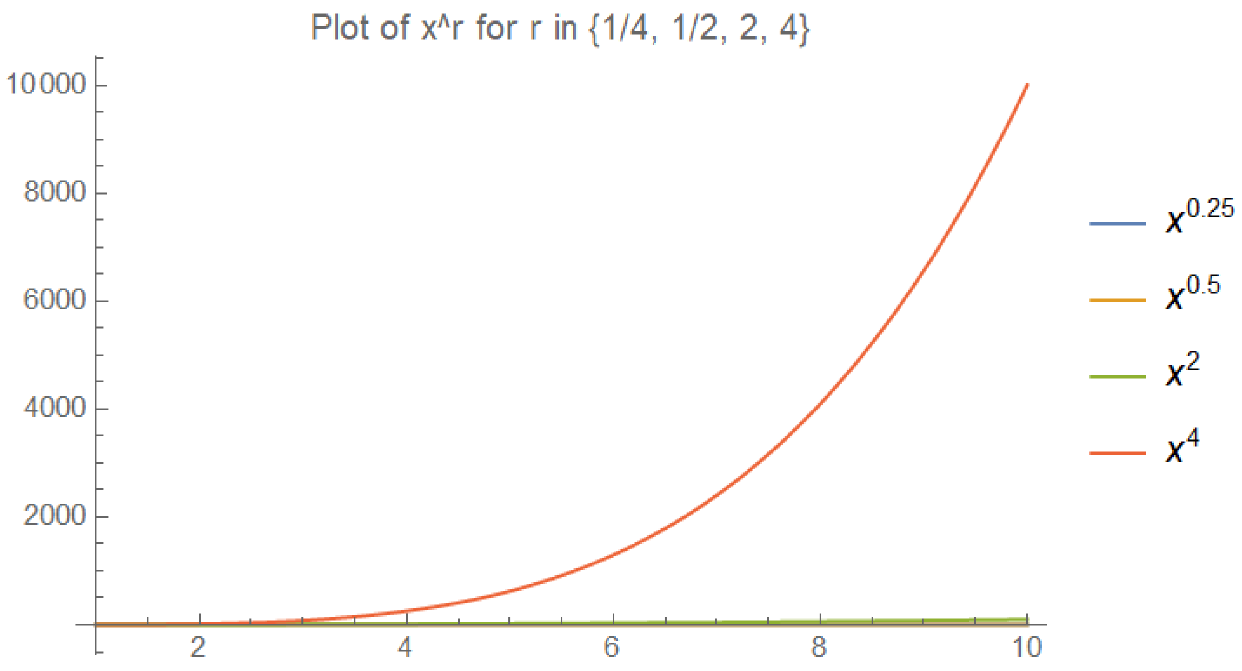


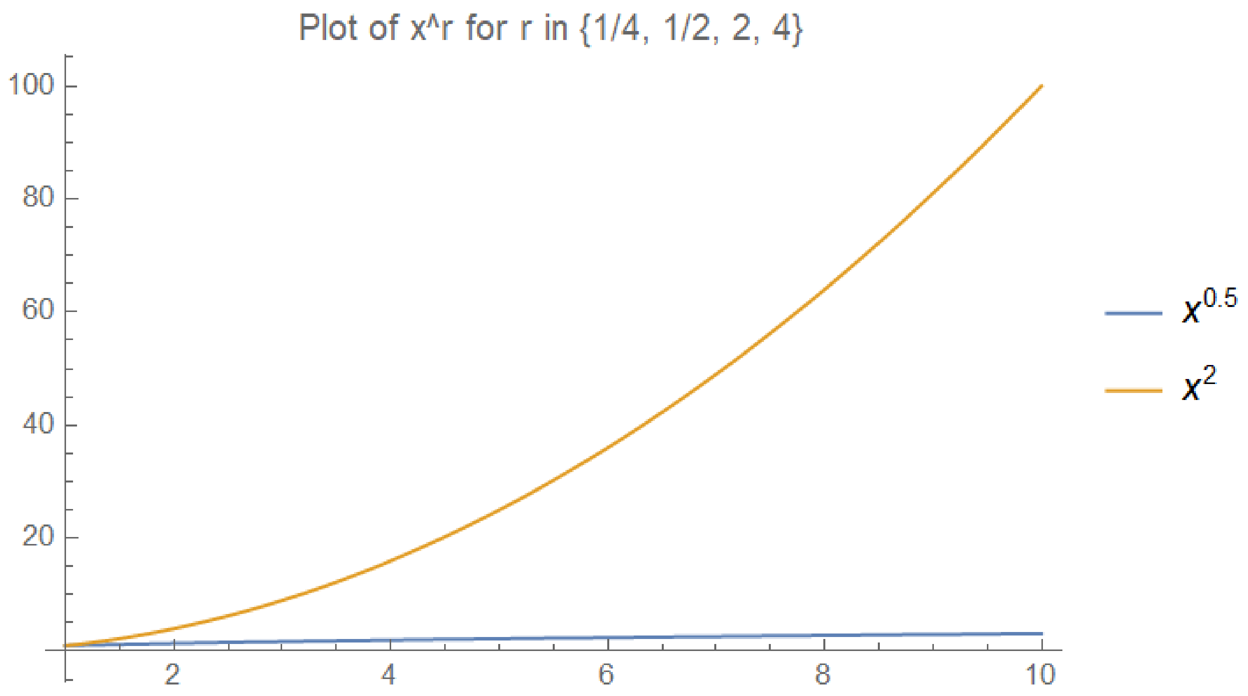
Plot of 100 most populous cities: log-log plot



Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).

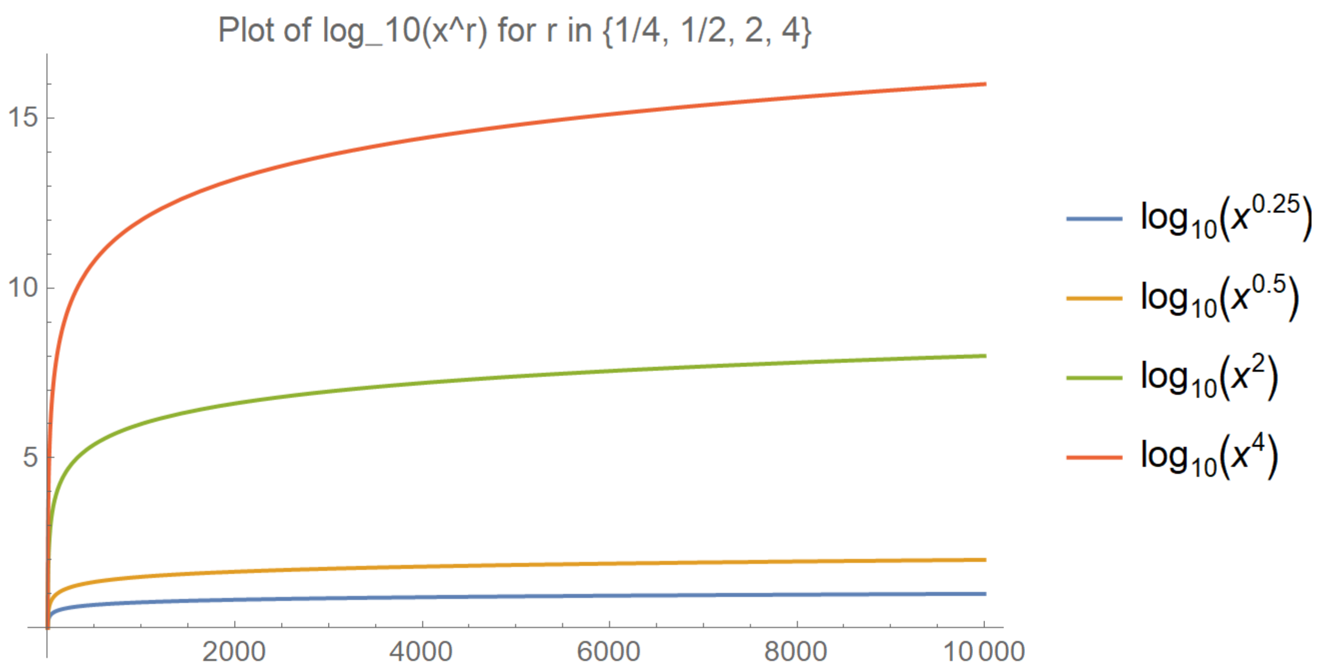


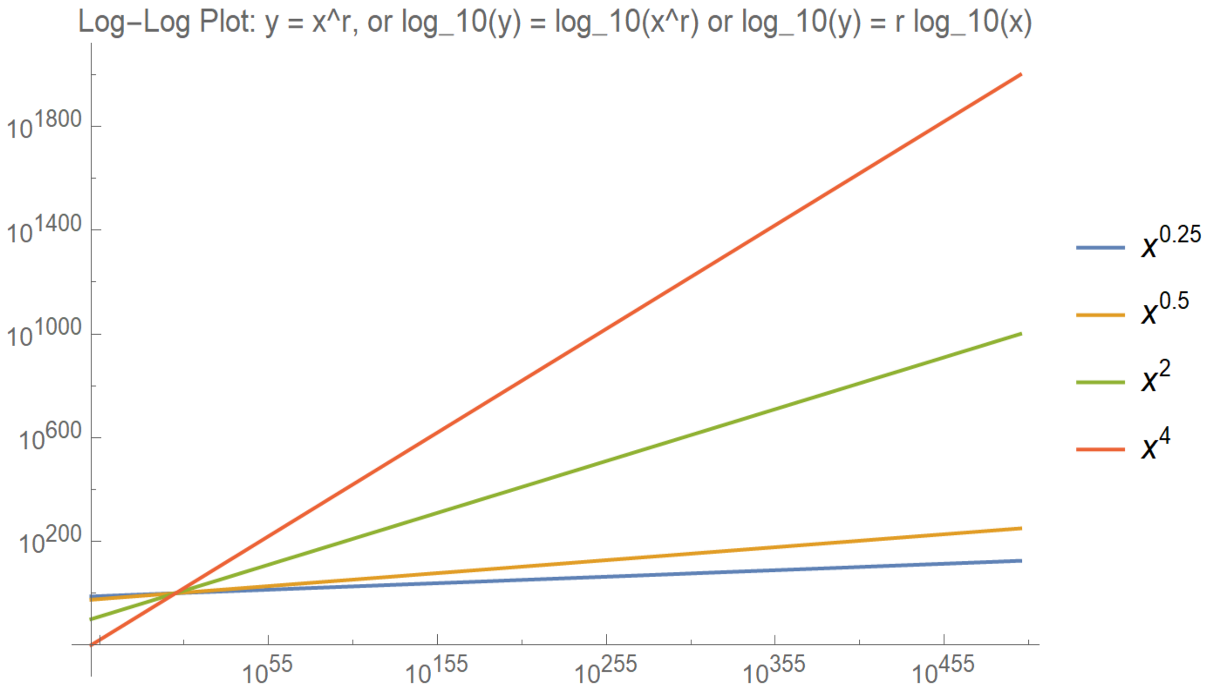


Notice that even on a small range, from 1 to 10, the polynomial of highest degree drowns out the others and can barely see.

Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).





Left: Semi-log plot: $y = \log x^r$. Right: log-log plot: $\log y = \log x^r$.
Note that we can now see the four functions on one plot, and the log-log plot now has linear relations.

Review: Exponent Laws

Laws

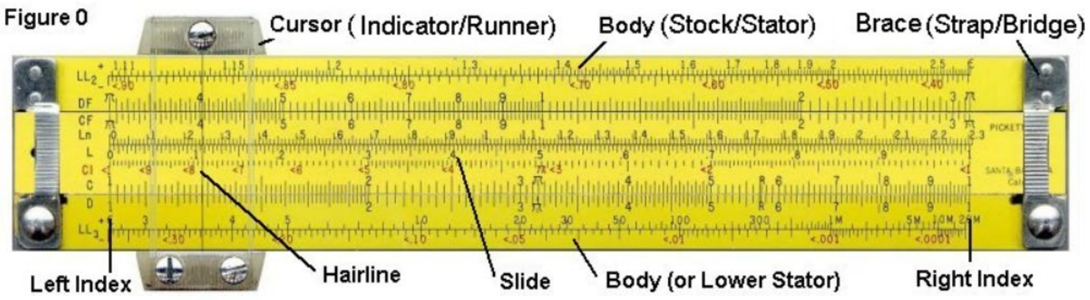
- $b^m\ b^n = b^{m+n}$
- $b^m\ /\ b^n = b^{m-n}$
- $(b^m)^n = b^{mn}$

Examples

- $10^3 10^2 = (10 * 10 * 10) * (10 * 10) = 10^5$
- $10^3/10^2 = (10 * 10 * 10)/(10 * 10) = 10^1$
- $(10^3)^2 = 10^3 * 10^3 = (10 * 10 * 10) * (10 * 10 * 10) = 10^6$

Logarithm Laws

Parts of a Slide Rule



Remember if $x = b^y$ then $\log_b x = y$.

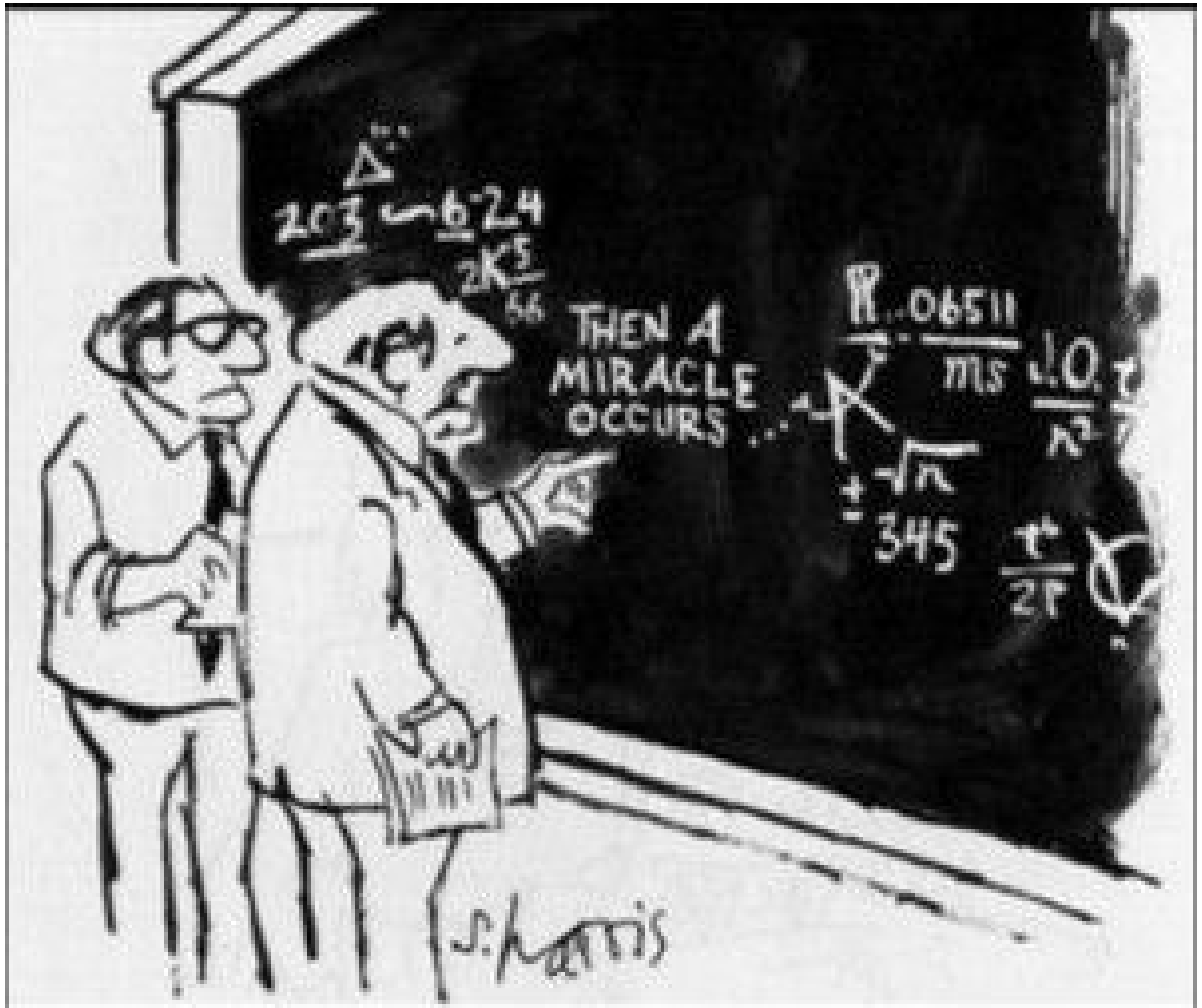
Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

These allow us to simplify computations with logarithms.

THEOREM

- $\log_b(x^n) = \mathrm{n} \log_b x$. Log of a power is that power times the log.
- $\log_b(x_1\ x_2) = \log_b(x_1) + \log_b(x_2)$. Log of a product is the sum of the logs.
- $\log_b(x_1 / x_2) = \log_b(x_1) - \log_b(x_2)$. Log of a quotient is the difference of the logs.
- $\log_b x = \log_c x / \log_c b$. If know logs in one base, know in all.

OPTIONAL – PROOFS OF THE LOG LAWS



"I think you should be more explicit here in step two."

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\mathbf{\log_b(x^n) = n\log_b x}$**.** **Log of a power is that power times the log.**

Proof:

- $\log_b x = y$ means $x = b^y$.

Logarithm Laws: Proofs

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- $\boldsymbol{\log_b(x^n) = n \log_b x.}$ **Log of a power is that power times the log.**

Proof:

- $\log_b x = y$ means $x = b^y$.
- Thus $x^n = (b^y)^n = b^{ny}$.

Logarithm Laws: Proofs

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Proof:

- $\log_b x = y$ means $x = b^y$.
- Thus $x^n = (b^y)^n = b^{ny}$.
- Taking logarithms: $\log_b(xn) = ny = n \log_b x$. ∎

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\mathbf{log}_{\boldsymbol{b}}(\boldsymbol{x}_1\ \boldsymbol{x}_2) = \mathbf{log}_{\boldsymbol{b}}(\boldsymbol{x}_1) + \mathbf{log}_{\boldsymbol{b}}(\boldsymbol{x}_2)$. **Log of a product is the sum of the logs.**

Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

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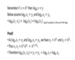
- Thus $x_1\ x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.



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- Thus $x_1\ x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.
- Therefore $\log_b(x_1\ x_2) = y_1 + y_2 = \log_b x_1 + \log_b x_2$. ■

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

- $\boldsymbol{\log_b x = \log_c x / \log_c b}$. **Know logs in one base, know in all.**

Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.

Logarithm Laws: Proofs

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Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^y = (c^v)^y = cv^y$.

Logarithm Laws: Proofs

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- As also have $x = c^u$ we have $u = vy$ or $y = u/v$.

Logarithm Laws: Proofs

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- Thus $x = b^y = (c^v)^y = cv^y$.
- As also have $x = c^u$ we have $u = vy$ or $y = u/v$.
- Substituting gives $\log_b x = \log_c x / \log_c b$. ■

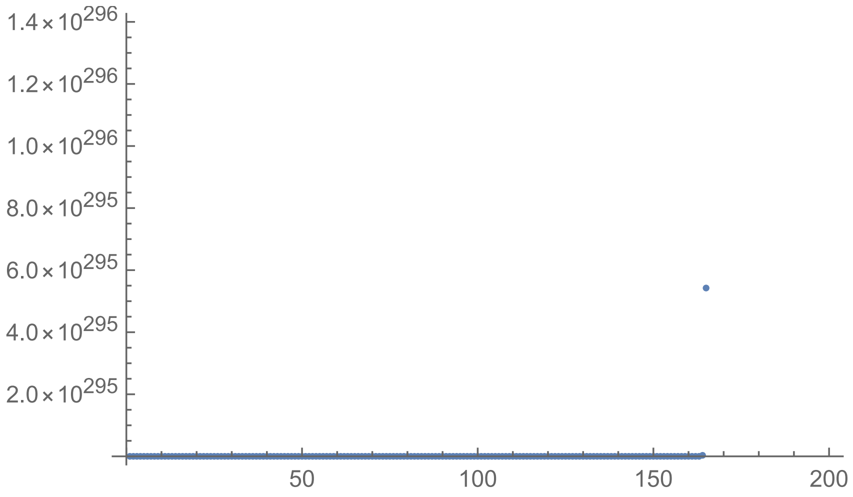
Example: Factorial Function:
Number ways to order $\boldsymbol{n}$ *objects when order matters:*

$$\boldsymbol{n}! = \boldsymbol{n} * (\boldsymbol{n} - \mathbf{1}) * \bullet\bullet\bullet * \mathbf{3} * \mathbf{2} * \mathbf{1}.$$

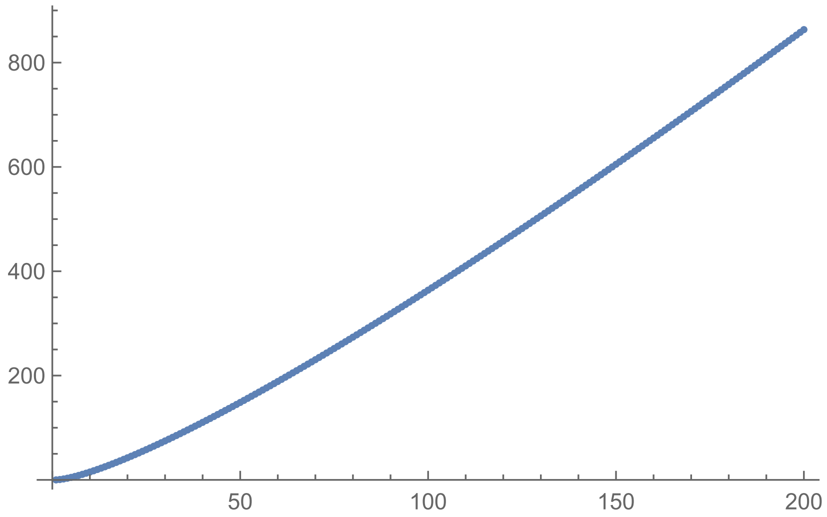
```
list = {}; semiloglist = {}; logloglist = {};
For[n = 1, n <= 200, n++,
  {
   list = AppendTo[list, {n, n!}];
   semiloglist = AppendTo[semiloglist, {n, Log[n!]}];
   logloglist = AppendTo[logloglist, {Log[n], Log[n!]}];
   }];
Print[ListPlot[list]]; Print[ListPlot[semiloglist]]; Print[ListPlot[logloglist]];
```

Example: Factorial Function:
Number ways to order n *objects when order matters:*

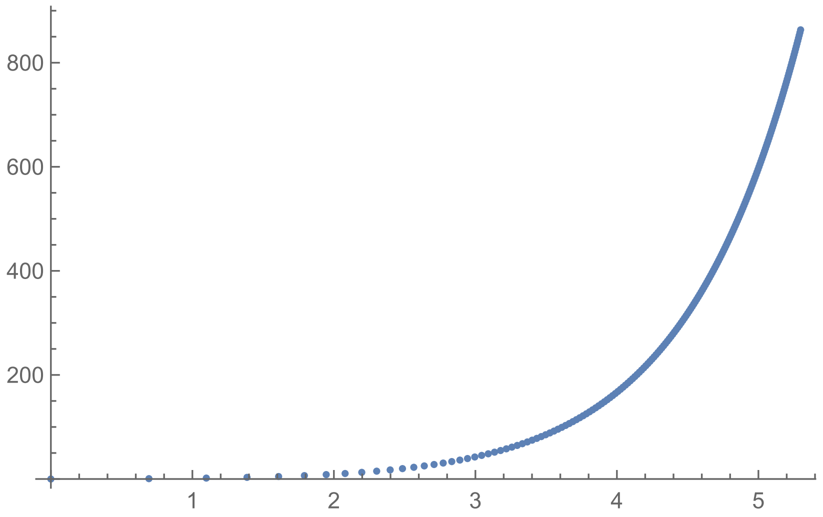
$$n! = n * (n-1) * \bullet\bullet\bullet * 3 * 2 * 1.$$



Normal Plot



Semi-log Plot



Log-Log Plot

For large n, have $n! \approx n^n e^{-n}\sqrt{2\pi n}$, so $\log n \approx \frac{n}{e} \log n$ (plus a much smaller term).

From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

Steven J. Miller, Williams College
Steven.J.Miller@williams.edu
http://web.williams.edu/Mathematics/sjmiller/public_html/

ATMIM Spring Conference
Assabet Valley Regional Technical High School, 3/23/13

Using in the Classroom

These slides are from the keynote address at the 2013 Spring Conference of ATMIM. If you are interested in using any of these topics (or anything from the math riddles page) in your class, please email me at sjm1@williams.edu, and I am happy to talk with you about implementation.

Thanks

Wanted to thank many people who encouraged me and provided opportunities.

- Parents and brother.
- Math teachers from preschool to graduate school.
- Colleagues and students for many discussions.
- Henry Bolton (henry.bolter@gmail.com) from Teachers As Scholars: http://www.teachersasscholars.org/
- Mr. Anthony for stepping up so many times.

Some Issues for the Future

- World is rapidly changing – powerful computing cheaply and readily available.
- What skills are we teaching? What skills should we be teaching?
- One of hardest skills: how to think / attack a new problem, how to see connections, what data to gather.

Goals of the Talk: Opportunities Everywhere!

- Ask Questions! Often simple questions lead to good math.
- Gather data: observe, program and simulate.
- Use games to get to mathematics.
- Discuss implementation: `Please interrupt!`

Joint work with Cameron (age 6) and Kayla (age $4 - 2\epsilon$) Miller

My math riddles page:
http://mathriddles.williams.edu/

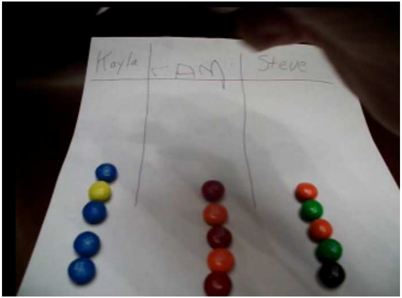
The M&M Game

Motivating Question

Cam (4 years): If you're born on the same day, do you die on the same day?

M&M Game Rules

Cam (4 years): `If you're born on the same day, do you die on the same day?`



(1) Everyone starts off with k M&Ms (we did 5).
(2) All toss fair coins, eat an M&M if and only if head.

Be active – ask questions!

What are natural questions to ask?

Be active – ask questions!

What are natural questions to ask?

Question 1: How likely is a tie (as a function of k)?

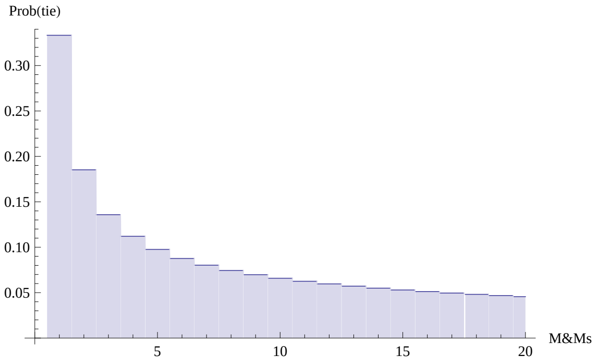
Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiousity is good and to be encouraged! Value to the journey and not knowing the answer.

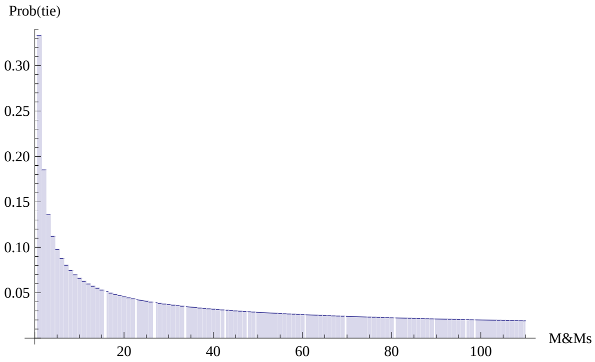
Let's gather some data!

Probability of a tie in the M&M game (2 players)



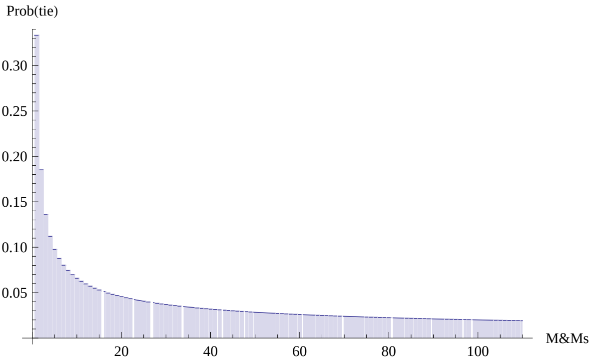
Prob(tie) $\approx$ 33% (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

Probability of a tie in the M&M game (2 players)



But we’re celebrating 110 years of service, so....

Probability of a tie in the M&M game (2 players)



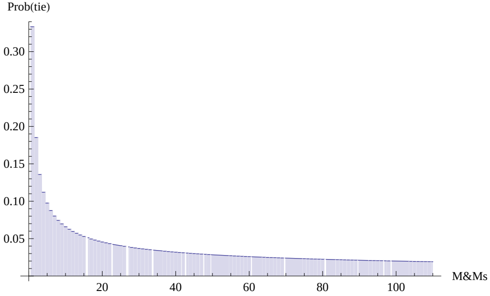
... where will the next 110 bring us?

Never too early to lay foundations for future classes.

Welcome to Statistics and Inference!

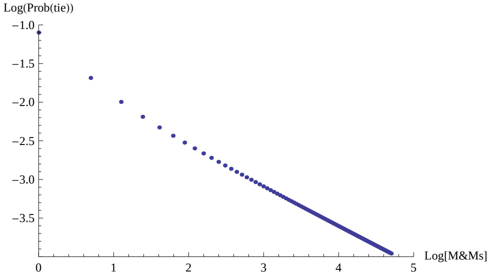
- Goal: Gather data, see pattern, extrapolate.
- Methods: Simulation, analysis of special cases.
- Presentation: It matters how we show data, and which data we show.

Viewing M&M Plots



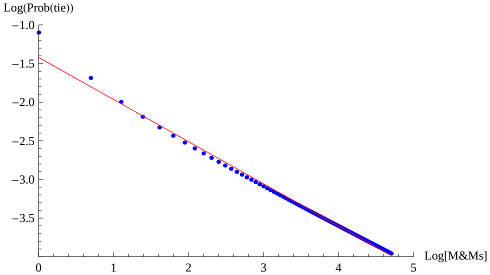
Hard to predict what comes next.

Viewing M&M Plots: Log-Log Plot



Not *just* sadistic teachers: logarithms useful!

Viewing M&M Plots: Log-Log Plot

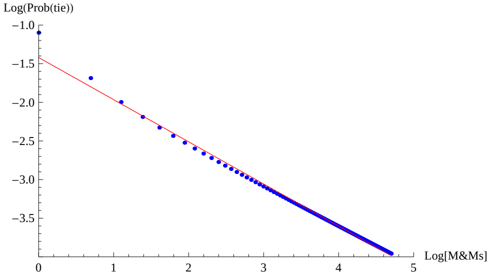


Best fit line:

$\log(\text{Prob}(\text{tie})) = -1.42022 - 0.545568 \log(\#\text{M\&Ms})$ or

$\text{Prob}(k) \approx 0.2412/k^{.5456}$.

Viewing M&M Plots: Log-Log Plot



Best fit line:
$\log(\text{Prob}(\text{tie})) = -1.42022 - 0.545568 \log(\#\text{M\&Ms})$ or $\text{Prob}(k) \approx 0.2412/k^{.5456}$.

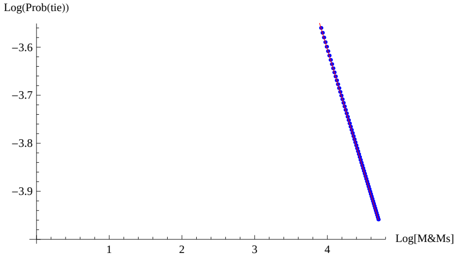
Predicts probability of a tie when $k = 220$ is 0.01274, but answer is 0.0137. **What gives?**

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.



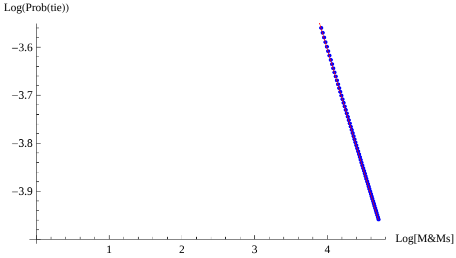
Best fit line:

$\log(\text{Prob}(\text{tie})) = -1.58261 - 0.50553 \log(\#\text{M\&Ms})$ or

$\text{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.



Best fit line:

$\log(\text{Prob(tie)}) = -1.58261 - 0.50553 \log(\#\text{M\&Ms})$ or $\text{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Get 0.01344 for $k = 220$ (answer 0.01347); **much better!**

From Shooting Hoops
to the Geometric Series Formula

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p.
- **Magic** always gets basket with probability q.

Let x be the probability **Bird** wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

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Break into cases:

- **Bird** wins on 1^{st} shot: p.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1 - p)(1 - q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

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- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p+rp+r^2p+r^3p+\cdots \\ &= p\left(1+r+r^2+r^3+\cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Solving the Hoop Game: The Power of Perspective

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Have

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1 - p)(1 - q)$$

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1 - p)(1 - q)x$$

Solving the Hoop Game: The Power of Perspective

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will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

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$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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Have

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \;=\; p \quad \text{or} \quad x \;=\; \frac{p}{1-r}.$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \; = \; p \quad \text{or} \quad x \; = \; \frac{p}{1-r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots \; = \; \frac{1}{1-r}.$$

Lessons from Hoop Problem

- Power of Perspective: Memoryless process.
- Can circumvent algebra with deeper understanding! (Hard)
- Depth of a problem not always what expect.
- Importance of knowing more than the minimum: connections.
- Math is fun!

The M&M Game

Solving the M&M Game

Overpower with algebra: Assume k M&Ms, two people, fair coins:

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

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is a binomial coefficient.

“Simplifies” to $4^{-k}\ {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function! (Look up / write report.)

Obviously way beyond the classroom – is there a better way?

Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.

Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.
Each person has exactly $k-1$ heads in first $n-1$ tosses, then ends with a head.

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$

Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

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If neither eat, as if toss didn't happen. Now game is finite.

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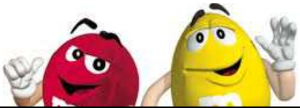
If neither eat, as if toss didn't happen. Now game is finite.

Much better perspective: each "turn" one of three equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is 1/3 or about 33%

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^{n} \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



Solving the M&M Game (cont)

Interpretation: Let Cam have c M&Ms and Kayla have k; write as (c, k).

Then each of the following happens 1/3 of the time after a 'turn':

- $(c, k) \longrightarrow (c - 1, k - 1)$.
- $(c, k) \longrightarrow (c - 1, k)$.
- $(c, k) \longrightarrow (c, k - 1)$.

Solving the M&M Game (cont): Assume $k = 4$

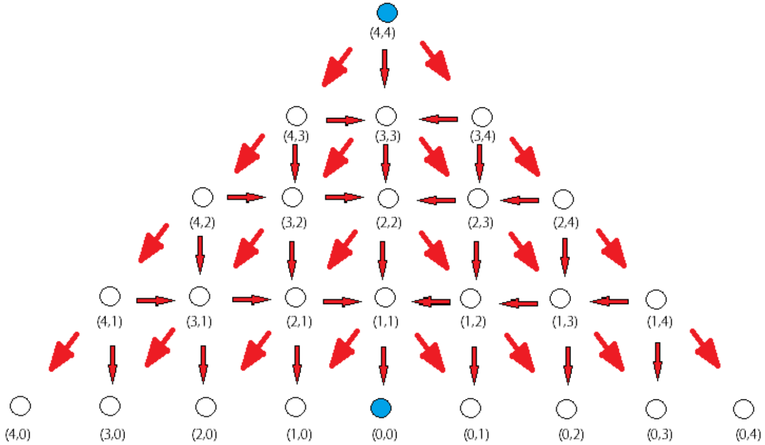


Figure: The M&M game when $k = 4$. Count the paths! Answer 1/3 of probability hit (1,1).

Solving the M&M Game (cont): Assume $k = 4$

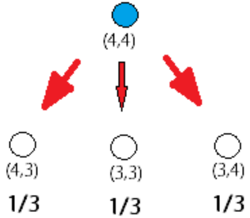


Figure: The M&M game when $k = 4$, going down one level.

Solving the M&M Game (cont): Assume $k = 4$

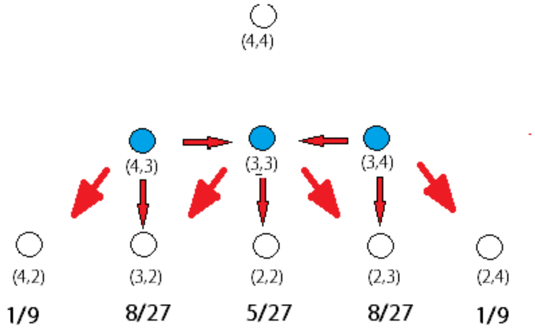


Figure: The M&M game when $k = 4$, removing probability from the second level.

Solving the M&M Game (cont): Assume $k = 4$

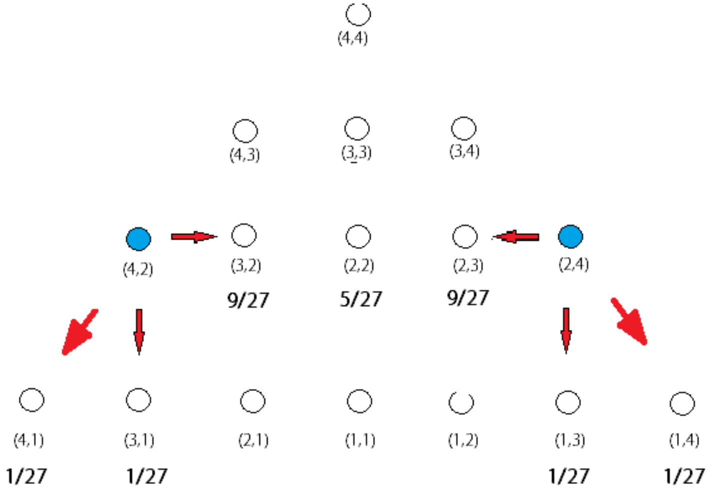


Figure: Removing probability from two outer on third level.

Solving the M&M Game (cont): Assume $k = 4$

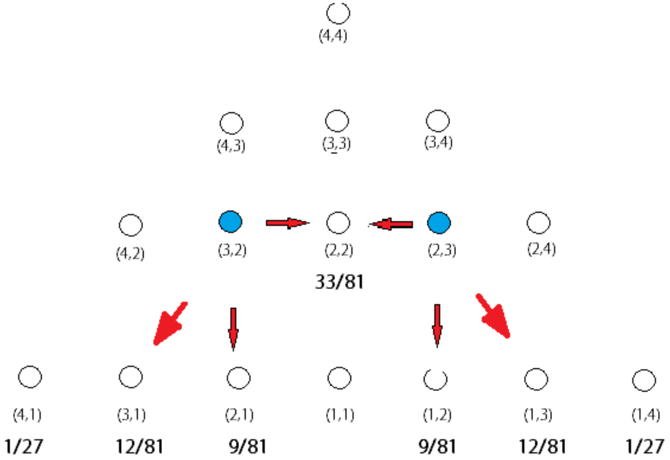


Figure: Removing probability from the (3,2) and (2,3) vertices.

Solving the M&M Game (cont): Assume $k = 4$

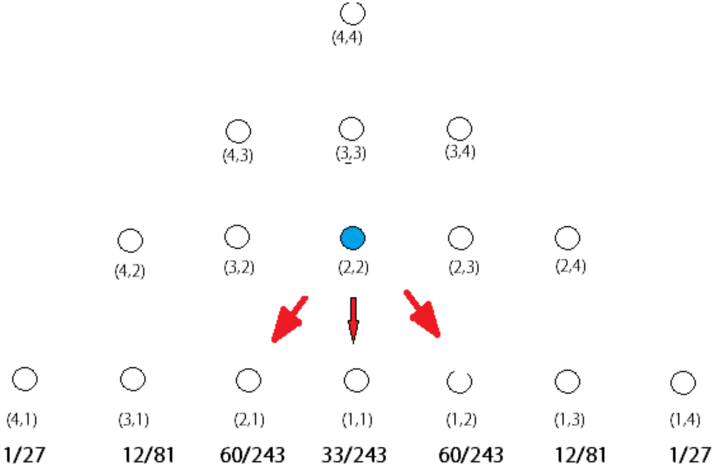


Figure: Removing probability from the (2,2) vertex.

Solving the M&M Game (cont): Assume $k = 4$

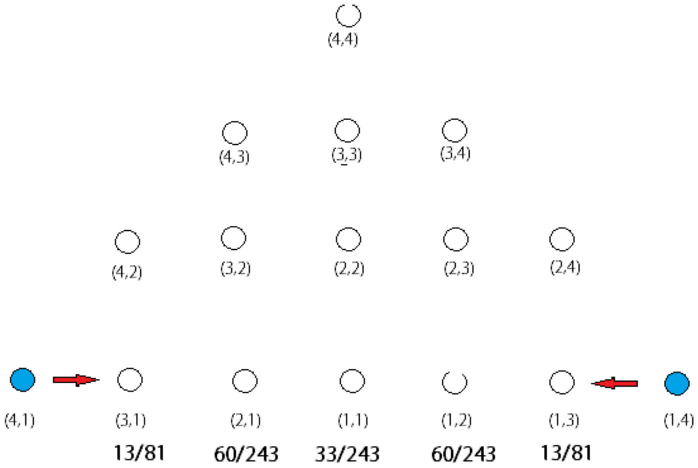


Figure: Removing probability from the (4,1) and (1,4) vertices.

Solving the M&M Game (cont): Assume $k = 4$

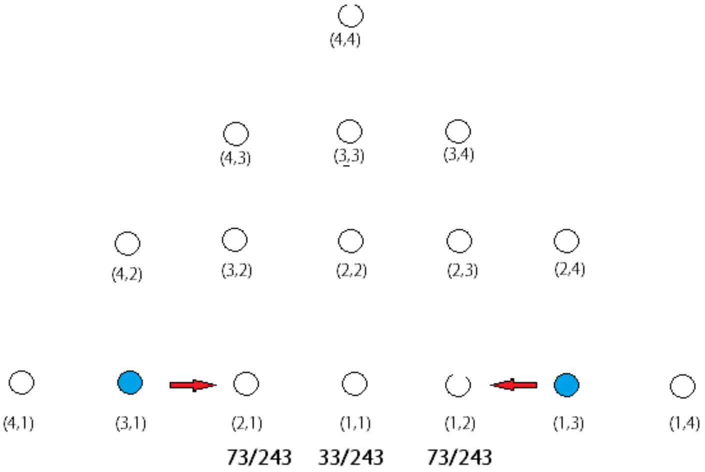


Figure: Removing probability from the (3,1) and (1,3) vertices.

Solving the M&M Game (cont): Assume $k = 4$

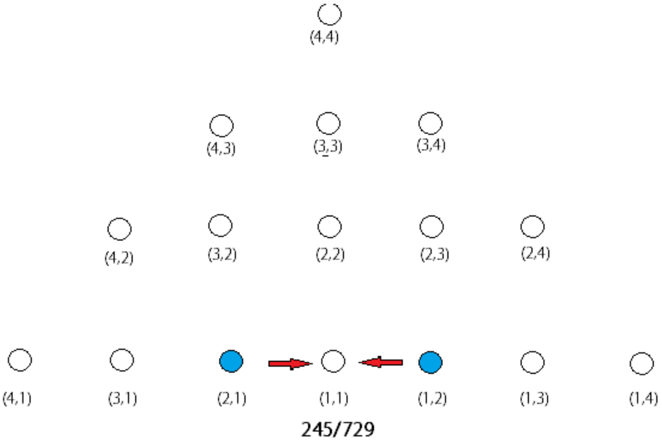


Figure: Removing probability from (2,1) and (1,2) vertices. Answer is 1/3 of (1,1) vertex, or 245/2187 (about 11%).

Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ...
http://www.youtube.com/watch?v=kkGeOWYOFoA.

Binet's Formula (can prove via 'generating functions'):

$$F_n \;=\; \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

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M&Ms: For $c, k \geq 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \geq 1$:

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

Interpreting Proof: Finding the Recurrence

What if we didn't see the 'simple' recurrence?

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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The following recurrence is 'natural':

$$x_{c,k} \;=\; \frac{1}{4}x_{c,k} + \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}.$$

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Obtain 'simple' recurrence by algebra: subtract $\frac{1}{4}x_{c,k}$:

$$\begin{aligned}\frac{3}{4}x_{c,k} &= \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}\\ \text{therefore } x_{c,k} &= \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.\end{aligned}$$

Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1.$

Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1$.

- $x_{1,0} = x_{0,1} = 0$.
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%$.

Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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- $x_{2,0} = x_{0,2} = 0$.
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} \;=\; x_{1,2}$.
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} \;=\; \frac{5}{27} \approx 18.5\%$.

Try Simpler Cases!!!

Try and find an easier problem and build intuition.

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Try and find an easier problem and build intuition.

Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.

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Generalization of the Catalan problem. There don't have (1,1) and stay on or below the main diagonal.

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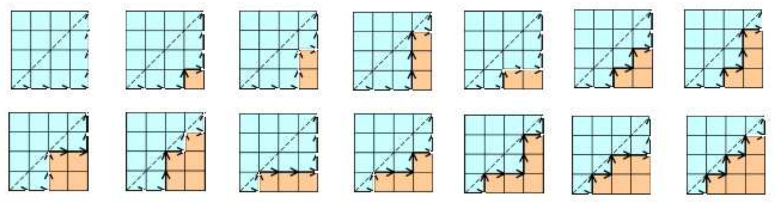
Generalization of the Catalan problem. There don't have (1,1) and stay on or below the main diagonal.



Interpretation: Catalan numbers are valid placings of (and).

Aside: Fun Riddle Related to Catalan Numbers

Young Saul, a budding mathematician and printer, is making himself a fake ID. He needs it to say he's 21. The problem is he's not using a computer, but rather he has some symbols he's bought from the store, and that's it. He has one 1, one 5, one 6, one 7, and an unlimited supply of + - * / (the operations addition, subtraction, multiplication and division). Using each number exactly once (but you can use any number of +, any number of -, ...) how, oh how, can he get 21 from 1,5, 6,7? Note: you can't do things like 15+6 = 21. You have to use the four operations as 'binary' operations: ((1+5)*6) + 7. Problem submitted by ohadbp@infolink.net.il, phrasing by yours truly.

Solution involves valid sentences: $((w+x)+y)+z$, $w+((x+y)+z)$, $\dots$.

For more riddles see my riddles page:
`http://mathriddles.williams.edu/`.

Examining Probabilities of a Tie

When $k = 1$, Prob(tie) = 1/3.

When $k = 2$, Prob(tie) = 5/27.

When $k = 3$, Prob(tie) = 11/81.

When $k = 4$, Prob(tie) = 245/2187.

When $k = 5$, Prob(tie) = 1921/19683.

When $k = 6$, Prob(tie) = 575/6561.

When $k = 7$, Prob(tie) = 42635/531441.

When $k = 8$, Prob(tie) = 355975/4782969.

Examining Ties: Multiply by 3^{2k-1} to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS: `http://oeis.org/`.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS: http://oeis.org/.

Our sequence: http://oeis.org/A084771.

The web exists! Use it to build conjectures, suggest proofs....

OEIS (continued)

A084771 Coefficients of 1/sqrt(1-10*x+9*x^2); also, a(n) is the central coefficient of (1+5*x+4*x^2)^n. 5

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - N-E. Fahssi, Mar 30 2008

Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]

Sums of squares of coefficients of (1+2*x)^n. [Joerg Arndt, Jul 06 2011]

The Hankel transform of this sequence gives A103488 . - Philippe DELEHAM, Dec 02 2007

REFERENCES Paul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Paths, Journal of Integer Sequences, Vol. 15, 2012, #12.4.8.- From N. J. A. Sloane, Oct 08 2012

Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 (2006), Article 06.1.1.

LINKS Table of n, a(n) for n=0..19.

Tony D. Noe, On the Divisibility of Generalized Central Trinomial Coefficients, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.

FORMULA G.f.: 1/sqrt(1-10*x+9*x^2).

Binomial transform of A059304. G.f.: Sum_{k>=0} binomial(2*k, k)*(2*x)^k/(1-x)^(k+1). E.g.f.: exp(5*x)*BesselI(0, 4*x). - Vladeta Jovovic (vladeta(AT)eunet.rs), Aug 20 2003

a(n) = sum(k=0..n, sum(j=0..n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j))). - Paul Barry, May 19 2006

a(n) = sum(k=0..n, 4^k*(C(n,k))^2) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]

Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [Vaclav Kotesovec, Sep 11 2012]

Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0. - R. J. Mathar,

Takeaways

Lessons

- Always ask questions.
- Many ways to solve a problem.
- Experience is useful and a great guide.
- Need to look at the data the right way.
- Often don't know where the math will take you.
- Value of continuing education: more math is better.
- Connections: My favorite quote: `If all you have is a hammer, pretty soon every problem looks like a nail.`

Generating Functions

Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n \\
&\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x) \\
&\Rightarrow g(x) = x/(1 - x - x^2).
\end{aligned}$$

Partial Fraction Expansion (Example: Binet's Formula)

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x}\right).$$

Coefficient of x^n (power series expansion):

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).