Math/Stat 341 and Math 433
Probability and Mathematical Modeling II: Continuous Systems

Steven J Miller
Williams College

sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/341

Lawrence 231
Williams College, February 23, 2015

Goal

- Understand continuous models.
- Solve continuous deterministic systems.
- Introduce stochastic processes.
- Discuss General Solutions.
- Zeckendorf Decompositions.

Continuous Systems

Differential Equations: I: First Order

Lots of differential equations can study.

Consider $f'(x) = af(x)$ with initial condition $f(0) = C$.

Differential Equations: I: First Order

Lots of differential equations can study.

Consider $f'(x) = af(x)$ with initial condition $f(0) = C$.

Special case: $a = 1$ solution $f(x) = Ce^x$....

Differential Equations: I: First Order

Lots of differential equations can study.

Consider $f'(x) = af(x)$ with initial condition $f(0) = C$.

Special case: $a = 1$ solution $f(x) = Ce^x$....

Solution: $f(x) = Ce^{ax}$ ($f(0) = C$ yields unique soln).

Differential Equations: I: First Order

Lots of differential equations can study.

Consider $f'(x) = af(x)$ with initial condition $f(0) = C$.

Special case: $a = 1$ solution $f(x) = Ce^x$....

Solution: $f(x) = Ce^{ax}$ ($f(0) = C$ yields unique soln).

Check: $f(x) = Ce^{ax}$ then $f'(x) = aCe^{ax} = af(x)$.

Differential Equations: II: Second Order

What about $f''(x) \;=\; af'(x) + bf(x)$?

Differential Equations: II: Second Order

What about $f''(x) \;=\; af'(x) + bf(x)$?

Similar to our difference equations! Try exponential!

Differential Equations: II: Second Order

What about $f''(x) \;=\; af'(x) + bf(x)$?

Similar to our difference equations! Try exponential!

$f(x) = e^{\rho x}$ ($e^{\rho x} = (e^{\rho})^x$ like r^n from before) yields

$$\rho^2 e^{\rho x} \;=\; a\rho e^{\rho x} + be^{\rho x}.$$

Differential Equations: II: Second Order

What about $f''(x) \;=\; af'(x) + bf(x)$?

Similar to our difference equations! Try exponential!

$f(x) = e^{\rho x}$ ($e^{\rho x} = (e^{\rho})^x$ like r^n from before) yields

$$\rho^2 e^{\rho x} \;=\; a\rho e^{\rho x} + b e^{\rho x}.$$

Yields characteristic equation

$$\rho^2 - a\rho - b \;=\; 0 \quad \text{with roots } \rho_1, \rho_2,$$

general solution (if $\rho_1 \neq \rho_2$)

$$f(x) \;=\; \alpha e^{\rho_1 x} + \beta e^{\rho_2 x}.$$

Differential Equations: III: System

In general have several variables and/or related quantities.

Consider a system involving $f(x)$ and $g(x)$:

$$\begin{aligned} f'(x) &= af(x) + bg(x) \\ g'(x) &= cf(x) + dg(x). \end{aligned}$$

How do we solve?

Differential Equations: III: System

In general have several variables and/or related quantities.

Consider a system involving $f(x)$ and $g(x)$:

$$\begin{aligned} f'(x) &= af(x) + bg(x) \\ g'(x) &= cf(x) + dg(x). \end{aligned}$$

How do we solve? Think back to similar examples.

Differential Equations: III: System: Solution

$$\begin{aligned} f'(x) &= af(x) + bg(x) \\ g'(x) &= cf(x) + dg(x). \end{aligned}$$

In linear algebra solved for one variable in terms of others.

Differential Equations: III: System: Solution

$$\begin{aligned} f'(x) &= af(x) + bg(x) \\ g'(x) &= cf(x) + dg(x). \end{aligned}$$

In linear algebra solved for one variable in terms of others.

$$g(x) = \tfrac{1}{b}f'(x) - \tfrac{a}{b}f(x),$$

Differential Equations: III: System: Solution

$$\begin{aligned} f'(x) &= af(x) + bg(x) \\ g'(x) &= cf(x) + dg(x). \end{aligned}$$

In linear algebra solved for one variable in terms of others.

$g(x) \;=\; \frac{1}{b}f'(x) - \frac{a}{b}f(x)$, substitute:

$$\left[\frac{1}{b}f'(x) - \frac{a}{b}f(x)\right]' \;=\; cf(x) + d\left[\frac{1}{b}f'(x) - \frac{a}{b}f(x)\right]$$

Differential Equations: III: System: Solution

$$\begin{aligned} f'(x) &= af(x) + bg(x) \\ g'(x) &= cf(x) + dg(x). \end{aligned}$$

In linear algebra solved for one variable in terms of others.

$g(x) \ = \ \frac{1}{b}f'(x) - \frac{a}{b}f(x)$, substitute:

$$\begin{aligned} \left[\frac{1}{b}f'(x) - \frac{a}{b}f(x)\right]' &= cf(x) + d\left[\frac{1}{b}f'(x) - \frac{a}{b}f(x)\right] \\ f''(x) &= (a+d)f'(x) + (cb - ad)f(x), \end{aligned}$$

reducing to previously solved problem!

Differential Equations: III: Matrix Formulation for System

$$V'(x) = AV(x), \quad V(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Differential Equations: III: Matrix Formulation for System

$$V'(x) \;=\; AV(x), \quad V(x) \;=\; \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \quad A \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Formally looks like $f'(x) = af(x)$, guess solution is $V(x) = e^{Ax}V(0)$, where

$$e^{Ax} \;=\; I + Ax + \frac{1}{2!}A^2x^2 + \frac{1}{3!}A^3x^3 + \cdots \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}A^k x^k.$$

Differential Equations: III: Matrix Formulation for System

$$V'(x) = AV(x), \quad V(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Formally looks like $f'(x) = af(x)$, guess solution is $V(x) = e^{Ax}V(0)$, where

$$e^{Ax} = I + Ax + \frac{1}{2!}A^2x^2 + \frac{1}{3!}A^3x^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k x^k.$$

Can justify term-by-term differentiation of series for e^{Ax}, see importance of matrix exponential.

Differential Equations: III: Matrix Formulation for System

$$V'(x) = AV(x), \quad V(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Formally looks like $f'(x) = af(x)$, guess solution is $V(x) = e^{Ax}V(0)$, where

$$e^{Ax} = I + Ax + \frac{1}{2!}A^2x^2 + \frac{1}{3!}A^3x^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^kx^k.$$

Can justify term-by-term differentiation of series for e^{Ax}, see importance of matrix exponential.

Mentioned Baker-Campbell-Hausdorf formula; in general product of matrices is hard but $\left(e^{Ax}\right)' = Ae^{Ax} = e^{Ax}A$.

Application: Battle of Trafalgar

Modified from *Mathematics in Warfare* by F. W. Lancaseter.

Battle of Trafalgar

Wikipedia: “The battle was the most decisive naval victory of the war. Twenty-seven British ships of the line led by Admiral Lord Nelson aboard HMS Victory defeated thirty-three French and Spanish ships of the line under French Admiral Pierre-Charles Villeneuve off the southwest coast of Spain, just west of Cape Trafalgar, in Caños de Meca.

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -Nr(t) \\ r'(t) &= -Mb(t). \end{aligned}$$

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -N r(t) \\ r'(t) &= -M b(t). \end{aligned}$$

Can solve using techniques from before: what do you expect solution to look like?

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -Nr(t) \\ r'(t) &= -Mb(t). \end{aligned}$$

Can solve using techniques from before: what do you expect solution to look like?

If take derivatives again find

$$b''(t) = -Nr'(t)$$

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -Nr(t) \\ r'(t) &= -Mb(t). \end{aligned}$$

Can solve using techniques from before: what do you expect solution to look like?

If take derivatives again find

$$b''(t) = -Nr'(t) = NMb(t),$$

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -Nr(t) \\ r'(t) &= -Mb(t). \end{aligned}$$

Can solve using techniques from before: what do you expect solution to look like?

If take derivatives again find

$$b''(t) = -Nr'(t) = NMb(t), \quad \text{yields}$$

$$b(t) = \beta_1 e^{\sqrt{NM}t} + \beta_2 e^{-\sqrt{NM}t}, \quad r(t) = \alpha_1 e^{\sqrt{NM}t} + \alpha_2 e^{-\sqrt{NM}t}.$$

The Square Law: I

Forces $r(t)$ and $b(t)$, effective fighting values N and M:

$$\begin{aligned} b'(t) &= -Nr(t) \\ r'(t) &= -Mb(t). \end{aligned}$$

Can solve using techniques from before: what do you expect solution to look like?

If take derivatives again find

$$b''(t) = -Nr'(t) = NMb(t), \quad \text{yields}$$

$$b(t) = \beta_1 e^{\sqrt{NM}t} + \beta_2 e^{-\sqrt{NM}t}, \quad r(t) = \alpha_1 e^{\sqrt{NM}t} + \alpha_2 e^{-\sqrt{NM}t}.$$

$b'(t)/b(t) = r'(t)/r(t)$ yields $Nr(t)^2 = Mb(t)^2$ (square law).

Trafalgar

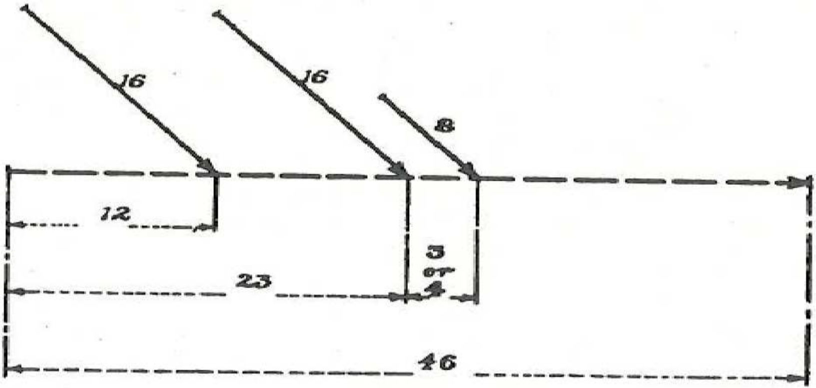
Nelson outnumbered – how could he win?

Trafalgar

Nelson outnumbered – how could he win?

Analysis of Nelson's Plan: I

Nelson assumed for the purpose of framing his plan of attack that his own force would consist of forty sail of the line, against forty-six of the com-



Analysis of Nelson's Plan: II

If for the purpose of comparison we suppose the total forces had engaged under the conditions described by Villeneuve as "the usage of former days," we have:—

Strength of combined fleet, 46^2	....	$= 2116$
" British " 40^2	....	$= 1600$
Balance in favour of enemy	....	516

Analysis of Nelson's Plan: III

Dealing with the position arithmetically, we have:—

Strength of British (in arbitrary n^2 units),

$$32^2 + 8^2 = 1088$$

And combined fleet,

$$23^2 + 23^2 = 1058$$

British advantage 30

Battle of Trafalgar

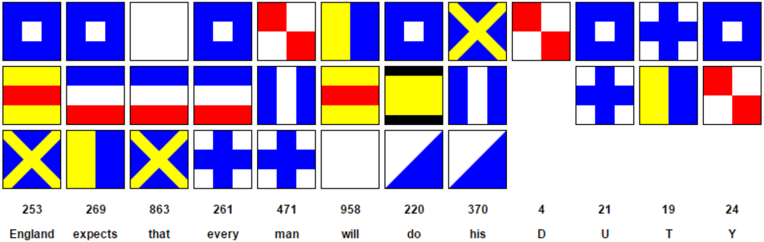
Wikipedia: “The battle was the most decisive naval victory of the war. Twenty-seven British ships of the line led by Admiral Lord Nelson aboard HMS Victory defeated thirty-three French and Spanish ships of the line under French Admiral Pierre-Charles Villeneuve off the southwest coast of Spain, just west of Cape Trafalgar, in Caños de Meca.”

Battle of Trafalgar

Wikipedia: “The battle was the most decisive naval victory of the war. Twenty-seven British ships of the line led by Admiral Lord Nelson aboard HMS Victory defeated thirty-three French and Spanish ships of the line under French Admiral Pierre-Charles Villeneuve off the southwest coast of Spain, just west of Cape Trafalgar, in Caños de Meca. The Franco-Spanish fleet lost twenty-two ships, without a single British vessel being lost.”

AfterMATH of Battle of Trafalgar


AfterMATH of Battle of Trafalgar: English expectation



British: 0 of 27 ships, 1,666 dead or wounded.
Franco-Spanish: 22 of 33 ships, 13,781 captured, dead or wounded.

AfterMATH of Battle of Trafalgar: Issues & Remedies with Model

Biggest issue is deterministic.

Make fighting effectiveness random variables!

Leads to stochastic differential equations.

http://en.wikipedia.org/wiki/Stochastic_differential_equation.

Introduction to
Zeckendorf Decompositions

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = ?$

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 17 = F_8 + 17$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + 1$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.
Example: $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
Observe: 51 miles $\approx$ 82.1 kilometers.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.
Example: $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
Observe: 51 miles $\approx$ 82.1 kilometers.
Reason: $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and 1 mile $\approx$ 1.609 km.

Old Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of number of summands in Zeckendorf decomposition for $m \in [F_n, F_{n+1})$ is Gaussian.

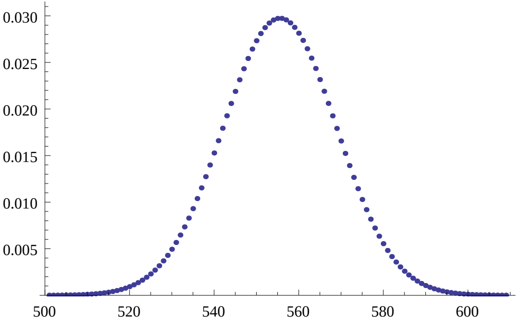


Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1,$$

Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2,$$

Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3,$$

Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3, \quad 5,$$

Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3, \quad 5, \quad 8,$$

Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad \ldots .$$

Equivalent Definition of the Fibonaccis

Fibonaccis are the only sequence such that each integer can be written uniquely as a sum of non-adjacent terms.

$$1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad \dots.$$

- Key to entire analysis: $F_{n+1} = F_n + F_{n-1}$.
- View as bins of size 1, cannot use two adjacent bins:

$$[1] \quad [2] \quad [3] \quad [5] \quad [8] \quad [13] \quad \cdots.$$

- SMALL '15, '16, ...: How does the notion of legal decomposition affect the sequence and results?

Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \; n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \; n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the a_i's (e.g. cannot use the recurrence relation to remove any summand).

- Central Limit Type Theorem

Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \dots, 9\}$ $(1 \leq i < m)$, $a_m \in \{1, \dots, 9\}$.

Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \ldots, 9\}$ $(1 \le i < m)$, $a_m \in \{1, \ldots, 9\}$.
- For $N \in [H_n, H_{n+1})$, first term is $a_n H_n = a_n 10^{n-1}$.

Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \dots, 9\}$ $(1 \leq i < m)$, $a_m \in \{1, \dots, 9\}$.
- For $N \in [H_n, H_{n+1})$, first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i. The A_i's are independent.

Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \dots, 9\}$ $(1 \leq i < m)$, $a_m \in \{1, \dots, 9\}$.
- For $N \in [H_n, H_{n+1})$, first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i. The A_i's are independent.
- For large n, the contribution of A_n is immaterial. A_i $(1 \leq i < n)$ are identically distributed random variables with mean 4.5 and variance 8.25.

Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \dots, 9\}$ $(1 \le i < m)$, $a_m \in \{1, \dots, 9\}$.
- For $N \in [H_n, H_{n+1})$, first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i. The A_i's are independent.
- For large n, the contribution of A_n is immaterial. A_i $(1 \le i < n)$ are identically distributed random variables with mean 4.5 and variance 8.25.
- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \to$ Gaussian with mean $4.5n + O(1)$ and variance $8.25n + O(1)$.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(g)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length g.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(g)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length g.

Bulk: What is $P(g) = \lim_{n\to\infty} P_n(g)$?

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(g)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length g.

Bulk: What is $P(g) = \lim_{n\to\infty} P_n(g)$?

Individual: Similar questions about gaps for a fixed $m \in [F_n, F_{n+1})$: distribution of gaps, longest gap.

New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m = \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

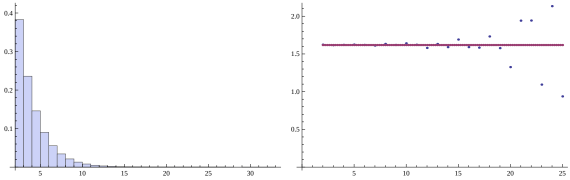


Figure: Distribution of gaps in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results: Longest Gap

Fair coin: largest gap tightly concentrated around $\log n / \log 2$.

Theorem (Longest Gap)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\mathrm{Prob}\left(L_n(m) \leq f(n)\right) \;\approx\; e^{-e^{\log n - f(n) \cdot \log \phi}}$$

- $\mu_n = \dfrac{\log\left(\frac{\phi^2}{\phi^2+1}n\right)}{\log \phi} + \dfrac{\gamma}{\log \phi} - \dfrac{1}{2} +$ Small Error.

- If $f(n)$ grows ***slower*** (resp. ***faster***) than $\log n / \log \phi$, then $\mathrm{Prob}(L_n(m) \leq f(n))$ goes to **0** (resp. ***1***).

Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C+P-1$ cookies in a line.
Cookie Monster eats $P-1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.
Divides the cookies into P sets.

Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line.

Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.

Divides the cookies into P sets.

Example: 8 cookies and 5 people ($C = 8$, $P = 5$):

Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line.
Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.
Divides the cookies into P sets.
Example: 8 cookies and 5 people ($C = 8$, $P = 5$):

Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line.
Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.
Divides the cookies into P sets.
Example: 8 cookies and 5 people ($C = 8$, $P = 5$):

Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

Number of sols to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.
If $x_i \geq c_i$ same as $y_1 + \cdots + y_P = C - (c_1 + \cdots + c_P)$.

Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

Number of sols to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.
If $x_i \geq c_i$ same as $y_1 + \cdots + y_P = C - (c_1 + \cdots + c_P)$.

Let $p_{n,k} = \# \{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

Number of sols to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.
If $x_i \geq c_i$ same as $y_1 + \cdots + y_P = C - (c_1 + \cdots + c_P)$.

Let $p_{n,k} = \# \{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n,\ i_j - i_{j-1} \geq 2.$$

Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

Number of sols to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.
If $x_i \geq c_i$ same as $y_1 + \cdots + y_P = C - (c_1 + \cdots + c_P)$.

Let $p_{n,k} = \#\left\{N \in [F_n, F_{n+1})\text{: the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\right\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n,\ i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

Number of sols to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.
If $x_i \geq c_i$ same as $y_1 + \cdots + y_P = C - (c_1 + \cdots + c_P)$.

Let $p_{n,k} = \#\left\{N \in [F_n, F_{n+1})\text{: the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\right\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \; i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1, \; d_j := i_j - i_{j-1} - 2 \; (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \; d_j \geq 0.$$

Cookie counting $\Rightarrow$ $p_{n,k} = \binom{n-2k+1 \, + \, k-1}{k-1} = \binom{n-k}{k-1}$.

Generalizing Lekkerkerker: Erdos-Kac type result

Theorem (KKMW 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

Sketch of proof: Use Stirling's formula,

$$n! \;\approx\; n^n e^{-n}\sqrt{2\pi n}$$

to approximates binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.

Continuous to assist discrete: $n! = \Gamma(n+1)$, where

$$\Gamma(s) \;=\; \int_0^\infty e^{-x}x^{s-1}dx, \quad \mathrm{Re}(s) > 0.$$

(Sketch of the) Proof of Gaussianity

The probability density for the number of Fibonacci numbers that add up to an integer in $[F_n, F_{n+1})$ is $f_n(k) = \binom{n-1-k}{k}/F_{n-1}$. Consider the density for the $n+1$ case. Then we have, by Stirling

$$\begin{aligned} f_{n+1}(k) &= \binom{n-k}{k}\frac{1}{F_n} \\ &= \frac{(n-k)!}{(n-2k)!k!}\frac{1}{F_n} = \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{n-k+\frac{1}{2}}}{k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}}\frac{1}{F_n} \end{aligned}$$

plus a lower order correction term.

Also we can write $F_n = \frac{1}{\sqrt{5}}\phi^{n+1} = \frac{\phi}{\sqrt{5}}\phi^n$ for large n, where ϕ is the golden ratio (we are using relabeled Fibonacci numbers where $1 = F_1$ occurs once to help dealing with uniqueness and $F_2 = 2$). We can now split the terms that exponentially depend on n.

$$f_{n+1}(k) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}\right)\left(\phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}\right).$$

Define

$$N_n = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}, \quad S_n = \phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}.$$

Thus, write the density function as

$$f_{n+1}(k) = N_n S_n$$

where N_n is the first term that is of order $n^{-1/2}$ and S_n is the second term with exponential dependence on n.

(Sketch of the) Proof of Gaussianity

Model the distribution as centered around the mean by the change of variable $k = \mu + x\sigma$ where μ and σ are the mean and the standard deviation, and depend on n. The discrete weights of $f_n(k)$ will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

$$f_n(k)dk = f_n(\mu + \sigma x)\sigma dx.$$

Using the change of variable, we can write N_n as

$$\begin{aligned}
N_n &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n-k}{k(n-2k)}}\frac{\phi}{\sqrt{5}} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}}\frac{\sqrt{5}}{\phi} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-(\mu+\sigma x)/n}{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}}\frac{\sqrt{5}}{\phi} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C-y}{(C+y)(1-2C-2y)}}\frac{\sqrt{5}}{\phi}
\end{aligned}$$

where $C = \mu/n \approx 1/(\phi+2)$ (note that $\phi^2 = \phi + 1$) and $y = \sigma x/n$. But for large n, the y term vanishes since $\sigma \sim \sqrt{n}$ and thus $y \sim n^{-1/2}$. Thus

$$N_n \approx \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C}{C(1-2C)}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{(\phi+1)(\phi+2)}{\phi}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{5(\phi+2)}{\phi}} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

since $\sigma^2 = n\frac{\phi}{5(\phi+2)}$.

(Sketch of the) Proof of Gaussianity

For the second term S_n, take the logarithm and once again change variables by $k = \mu + x\sigma$,

$$\begin{aligned}
\log(S_n) &= \log\left(\phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}\right)\\
&= -n\log(\phi) + (n-k)\log(n-k) - (k)\log(k)\\
&\quad - (n-2k)\log(n-2k)\\
&= -n\log(\phi) + (n-(\mu+x\sigma))\log(n-(\mu+x\sigma))\\
&\quad - (\mu+x\sigma)\log(\mu+x\sigma)\\
&\quad - (n-2(\mu+x\sigma))\log(n-2(\mu+x\sigma))\\
&= -n\log(\phi)\\
&\quad + (n-(\mu+x\sigma))\left(\log(n-\mu)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right)\\
&\quad - (\mu+x\sigma)\left(\log(\mu)+\log\left(1+\frac{x\sigma}{\mu}\right)\right)\\
&\quad - (n-2(\mu+x\sigma))\left(\log(n-2\mu)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right)\\
&= -n\log(\phi)\\
&\quad + (n-(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-1\right)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right)\\
&\quad - (\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right)\\
&\quad - (n-2(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-2\right)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

Note that, since $n/\mu = \phi + 2$ for large n, the constant terms vanish. We have $\log(S_n)$

$$\begin{aligned}
= \quad & -n\log(\phi) + (n-k)\log\left(\frac{n}{\mu} - 1\right) - (n-2k)\log\left(\frac{n}{\mu} - 2\right) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu+x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n-2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & -n\log(\phi) + (n-k)\log(\phi+1) - (n-2k)\log(\phi) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu+x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n-2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & n(-\log(\phi) + \log\left(\phi^2\right) - \log(\phi)) + k(\log(\phi^2) + 2\log(\phi)) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu+x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n-2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right) \\
= \quad & (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) - (\mu+x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) \\
& - (n-2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

Finally, we expand the logarithms and collect powers of $x\sigma/n$.

$$\begin{aligned}
\log(S_n) &= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n-\mu}-\frac{1}{2}\left(\frac{x\sigma}{n-\mu}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\mu}-\frac{1}{2}\left(\frac{x\sigma}{\mu}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-2\frac{x\sigma}{n-2\mu}-\frac{1}{2}\left(2\frac{x\sigma}{n-2\mu}\right)^2+\dots\right)\\
&= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}-\frac{1}{2}\left(\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\frac{n}{\phi+2}}-\frac{1}{2}\left(\frac{x\sigma}{\frac{n}{\phi+2}}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-\frac{2x\sigma}{n\frac{\phi}{\phi+2}}-\frac{1}{2}\left(\frac{2x\sigma}{n\frac{\phi}{\phi+2}}\right)^2+\dots\right)\\
&= \frac{x\sigma}{n}n\left(-\left(1-\frac{1}{\phi+2}\right)\frac{(\phi+2)}{(\phi+1)}-1+2\left(1-\frac{2}{\phi+2}\right)\frac{\phi+2}{\phi}\right)\\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n\left(-2\frac{\phi+2}{\phi+1}+\frac{\phi+2}{\phi+1}+2(\phi+2)-(\phi+2)+4\frac{\phi+2}{\phi}\right)\\
&\quad +O\left(n\left(x\sigma/n\right)^3\right)
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

$$\begin{aligned}
\log(S_n) \quad &= \quad \frac{x\sigma}{n} n \left(-\frac{\phi+1}{\phi+2}\frac{\phi+2}{\phi+1} - 1 + 2\frac{\phi}{\phi+2}\frac{\phi+2}{\phi}\right) \\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n(\phi+2)\left(-\frac{1}{\phi+1}+1+\frac{4}{\phi}\right) \\
&\quad +O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= \quad -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4}{\phi(\phi+1)}+1\right)+O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= \quad -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4+2\phi+1}{\phi(\phi+1)}\right)+O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= \quad -\frac{1}{2}x^2\sigma^2\left(\frac{5(\phi+2)}{\phi n}\right)+O\left(n\,(x\sigma/n)^3\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

But recall that

$$\sigma^2 = \frac{\phi n}{5(\phi+2)}.$$

Also, since $\sigma \sim n^{-1/2}$, $n\left(\frac{x\sigma}{n}\right)^3 \sim n^{-1/2}$. So for large n, the $O\left(n\left(\frac{x\sigma}{n}\right)^3\right)$ term vanishes. Thus we are left with

$$\begin{aligned} \log S_n &= -\frac{1}{2}x^2 \\ S_n &= e^{-\frac{1}{2}x^2}. \end{aligned}$$

Hence, as n gets large, the density converges to the normal distribution:

$$\begin{aligned} f_n(k)dk &= N_n S_n dk \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2} \sigma dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \end{aligned}$$

$\square$

Code: Problem and Basic Functions

Problem: Compute Zeckendorf decompositions and look at leading (i.e., first) digits to compare to Benford's law.

Here are some basic functions that we will need.

```
fd[x_] := Floor[10^Mod[Log[10, x], 1]]
fib[n_] := Fibonacci[n + 1];
lenfib[n_] := Floor[Log[1.0 fib[n]] / Log[E 1.0]]
```

Code: Main Program

```
zeckdecomp[m_, printcheck_] := Module[{},
   listn = {};
   For[d = 1, d ≤ 9, d++, digits[d] = 0];
   current = m;
   goldenmean = (1 + Sqrt[5]) / 2;
   While[current ≥ 1,
    { (* 1 2 3 5 8 13 21 34 55 89 144 233*)
     If[current ≤ 232,
      {
       If[current ≤ 232, newterm = 11];
       If[current ≤ 143, newterm = 10];
       If[current ≤ 88, newterm = 9];
       If[current ≤ 54, newterm = 8];
       If[current ≤ 33, newterm = 7];
       If[current ≤ 20, newterm = 6];
       If[current ≤ 12, newterm = 5];
       If[current ≤ 7, newterm = 4];
       If[current ≤ 4, newterm = 3];
       If[current ≤ 2, newterm = 2];
       If[current ≤ 1, newterm = 1];
      },
```

Code: Main Program

```
{
 x = Floor[(Log[current * Sqrt[5]] / Log[goldenmean]) - 1];
 If[fib[x+1] ≤ current, newterm = x+1,
  If[fib[x] ≤ current, newterm = x,
   If[fib[x-1] ≤ current, newterm = x-1]
  ]];
}]; (* end of if *)
listn = AppendTo[listn, newterm];
d = fd[fib[newterm]];
digits[d] = digits[d]+1;
current = current - fib[newterm];
}];
```

Code: Main Program

```
If[printcheck == 1,
  {
   Print[listn];
   listfib = fib[listn];
   Print[listfib];
   Print["m = ", m, " and sum of terms is ", Sum[listfib[[i]], {i, 1, Length[listfib]}]];
   Print["Difference is ", m - Sum[listfib[[i]], {i, 1, Length[listfib]}]];
   Print["Digits are "];
    For[d = 1, d ≤ 9, d++, Print[d, " " , digits[d]]];
  }];
 Return[listn];
];
```

Code: Main Program

```
zeckdecomp[14 531 997, 1];
{34, 32, 30, 27, 24, 22, 20, 15, 12, 7, 4}
{9 227 465, 3 524 578, 1 346 269, 317 811, 75 025, 28 657, 10 946, 987, 233, 21, 5}
m = 14 531 997 and sum of terms is 14 531 997
Difference is 0
Digits are
1 2
2 3
3 2
4 0
5 1
6 0
7 1
8 0
9 2
```

Summary

Summary of Two Lectures

- Difference/Differential Equations model world.
- Deterministic vs Stochastic.
- Prevalence of Central Limit Theorem.
- Approximate Continuous with Discrete.
- Convert Discrete to Continuous!

Homework Problems

Problems to Think About: I: Trafalgar

- In the naval battle model with $r(t)$ and $b(t)$, assume $M = N = 1$ (though it doesn't matter). If the inial force concentrations are $B_0 > R_0$, how long will the battle rage before Blue defeats Red?

- If Red divides its forces into two components $R_{0,1} + R_{0,2} = R_{0,1}$, which splits Blue into two components $B_{0,1} + B_{0,2} = B_0$, how should this be done to maximize Red's fighting strength, using the square law? If you want, assume $B_0 = 46$ and $R_0 = 40$ (or use 33 and 27, the actual battle numbers).

- Redo the last problem, but allow Red to split its forces into k parts, which split Blue into k parts as well. What is the optimal k and the optimal splitting for red? Again, if you want choose specific numbers.

Problems to Think About: II: Zeckendorf

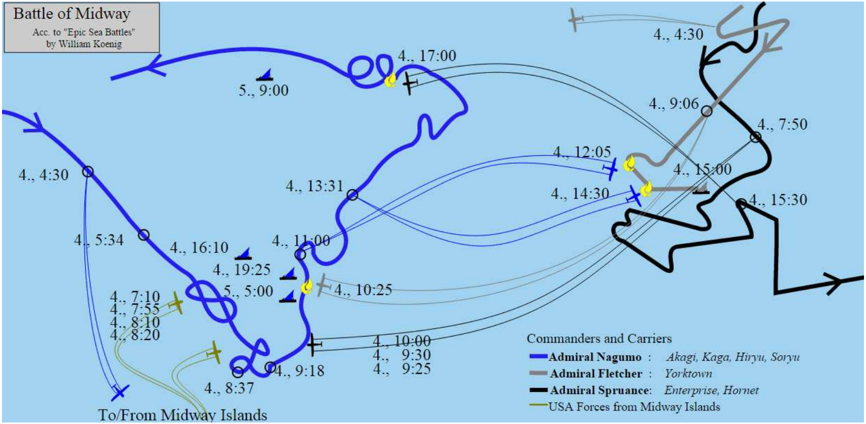
- Construct a sequence of positive integers such that every number can be written uniquely as a sum of these integers without ever using three consecutive numbers. Is there a nice recurrence relation describing this sequence?

- Consider the Gamma function $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$. Where is the integrand largest when $s = n + 1$ (so we are looking at $\Gamma(n+1) = n!$)? Can you use this to approximate $n!$?

- How many ways are there to divide C cookies among P people, but now we do not require each cookie to be given to a person? *Hint: there is a simple, clean answer.*

Bonus

Battle of Midway: I



Battle of Midway: II

United States	Empire of Japan
Commanders and leaders	
Chester W. Nimitz Frank Jack Fletcher Raymond Spruance Marc A. Mitscher Thomas C. Kinkaid	Isoroku Yamamoto Nobutake Kondō Chūichi Nagumo Tamon Yamaguchi † Ryusaku Yanagimoto †
Strength	
3 carriers 7 heavy cruisers 1 light cruiser 15 destroyers 233 carrier-based aircraft 127 land-based aircraft 16 submarines[1]	4 carriers 2 battleships 2 heavy cruisers 1 light cruiser 12 destroyers 248 carrier-based aircraft[2] 16 floatplanes Did not participate in battle: 2 light carriers 5 battleships 4 heavy cruisers 2 light cruisers ~35 support ships
Casualties and losses	
1 carrier sunk 1 destroyer sunk ~150 aircraft destroyed 307 killed[3]	4 carriers sunk 1 heavy cruiser sunk 1 heavy cruiser damaged 248 aircraft destroyed[4] 3,057 killed[5]

Codebreakers (Passage from Wikipedia entry 'Battle of Midway')

Cryptanalysts had broken the Japanese Navy's JN-25b code. Since the early spring of 1942, the US had been decoding messages stating that there would soon be an operation at objective "AF". It was not known where "AF" was, but Commander Joseph J. Rochefort and his team at Station HYPO were able to confirm that it was Midway by telling the base there by secure undersea cable to radio an uncoded false message stating that the water purification system it depended upon had broken down and that the base needed fresh water. The code breakers then picked up a Japanese message that "AF was short on water." HYPO was also able to determine the date of the attack [deleted], and to provide Nimitz with a complete IJN order of battle, [deleted] with a very good picture of where, when, and in what strength the Japanese would appear. Nimitz knew that the Japanese had negated their numerical advantage by dividing their ships into four separate task groups, all too widely separated to be able to support each other. Nimitz calculated that the aircraft on his three carriers, plus those on Midway Island, gave the U.S. rough parity with Yamamoto's four carriers, mainly because American carrier air groups were larger than Japanese ones. The Japanese, by contrast, remained almost totally unaware of their opponent's true strength and dispositions even after the battle began.