Math 341: Probability
First Lecture

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Williams College

Introduction and Objectives

Introduction / Objectives

Probability theory: model the real world, predict likelihood of events.

One of the three most important quantitative classes (statistics, programming).

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Objectives

- Obviously learn probability.
- Emphasize techniques / asking the right questions.
- Model problems and analyze model.
- Elegant solutions vs brute force (parameters in closed form versus numerical solutions).
- Looking at equations and getting a sense: $\log -5$ Method: $\frac{p \pm pq}{p+q \pm 2pq}$.

Types of Problems

- Biology: will a species survive?
- Physics / Chemistry / Number Theory: Random Matrix Theory.
- Gambling: Double-plus-one.
- Economics: Stock market / economy.
- Finance: Monte Carlo integration.
- Marketing: Movie schedules.
- Cryptography: Markov Chain Monte Carlo.
- 8 ever 9 never (bridge).

My (applied) experiences

- Marketing: parameters for linear programming (SilverScreener).
- Data integrity: detecting fraud with Benford's Law (IRS, Iranian elections).
- Sabermetrics: Pythagorean Won-Loss Theorem.

Course Mechanics

Grading / Administrative

- Fast Pace: Class Participation: 5%. HW: 15%. Writing Assigbment: 10% (Solutions to HW Problems from Chapters, Project in Probability / Statistics / Data Science). Midterm: 30% (if there are two exams only best counts). 'Final' exam: 40%. May do a project for 10% of your grade (reduces other categories proportionally).
- Pre-reqs: Calc III, basic combinatorics / set theory, linear algebra.

Office hours / feedback

- TBD and when I'm in my office (schedule online)

Other

- Webpage: numerous handouts, additional comments each day (mix of review and optional advanced material).
- Probability Lifesaver: opportunity to help write a solution key, lots of worked examples.
- Gather and analyze some data set of interest.
- PREPARE FOR CLASS! Must do readings before each class.

Being Prepared

Never know when an opportunity presents itself....

S. J. Miller at the Sarnak 61st Dinner

(copyright C. J. Mozzochi, Princeton N.J)

Being Prepared

- Your Job:
  - ◇ Be prepared for class: do reading, think about material.
  - ◇ Come to me, the TAs and each other with questions.
- My/TAs Job:
  - ◇ Provide resources, guiding questions.
  - ◇ Be available.

Other: Advice from Jeff Miller

- Party less than the person next to you.

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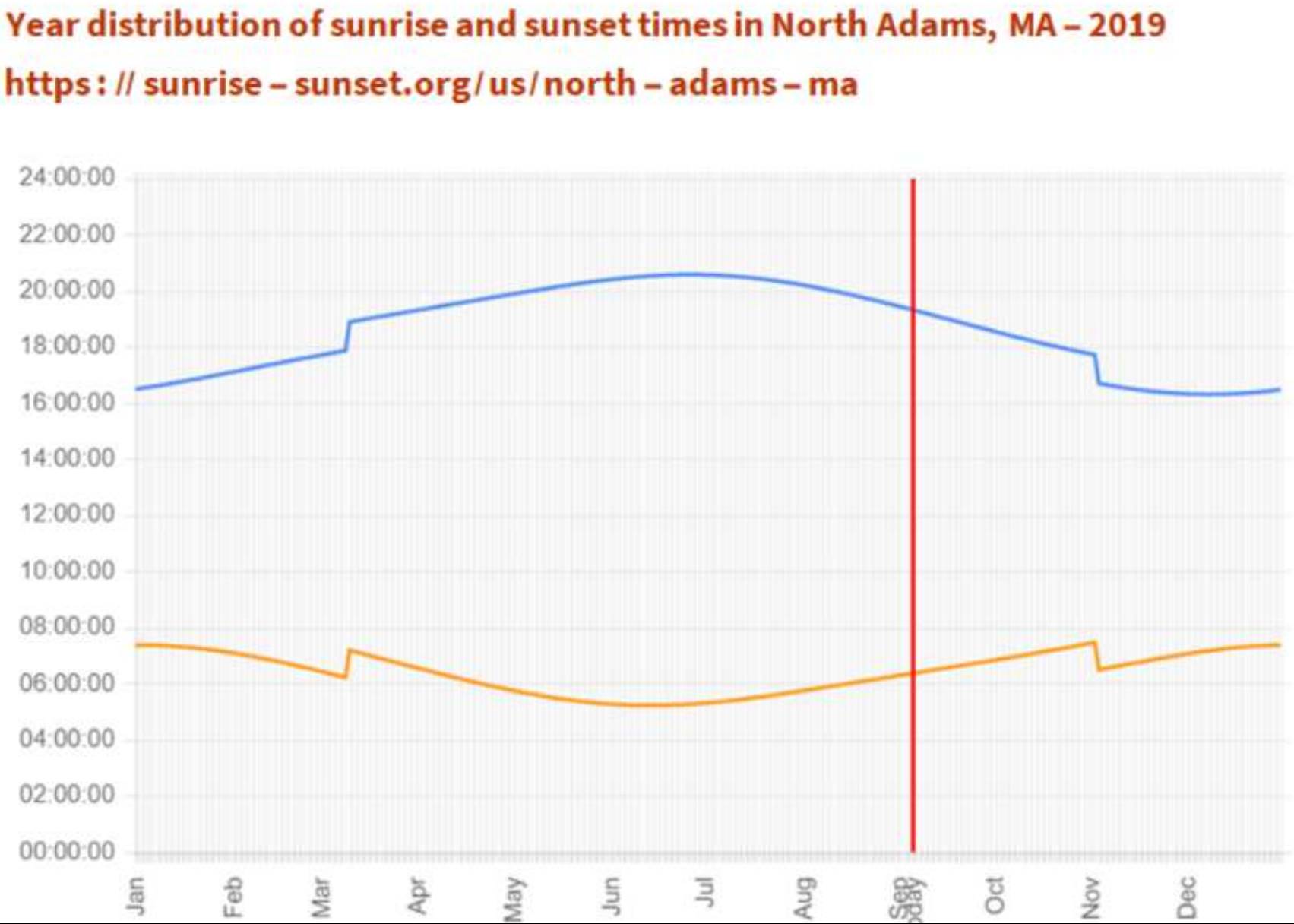
Happy to do practice interviews, adjust deadlines....


Year distribution of sunrise and sunset times in North Adams, MA – 2019
https : // sunrise – sunset.org / us / north – adams – ma
24:00:00
22:00:00
20:00:00
18:00:00
16:00:00
14:00:00
12:00:00
10:00:00
08:00:00
06:00:00
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Jan
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Sep
Oct
Nov
Dec

Who America is rooting for in the Super Bowl:

Gambling

Football Wager

2007: Friend of a favorite student bet $500 at 1000:1 odds on Patriots going undefeated and winning the Superbowl.

Football Wager

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Football Wager

2008: In third quarter, Pats leading, Vegas offers to buy back the bet at 300:1, told no....

WHAT WAS THE BETTOR'S MISTAKE?

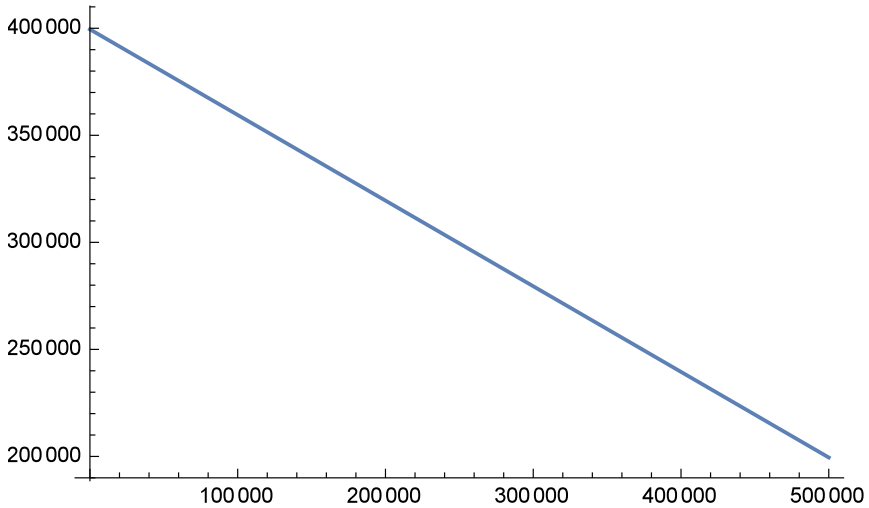
Hedging

Pats win with probability p, Giants $q = 1 - p$.

Bet \$1 bet on Giants, if they win get \$$x$.
Already bet \$500 on Patriots, now bet \$$B$ on the Giants.

Expected Winning:

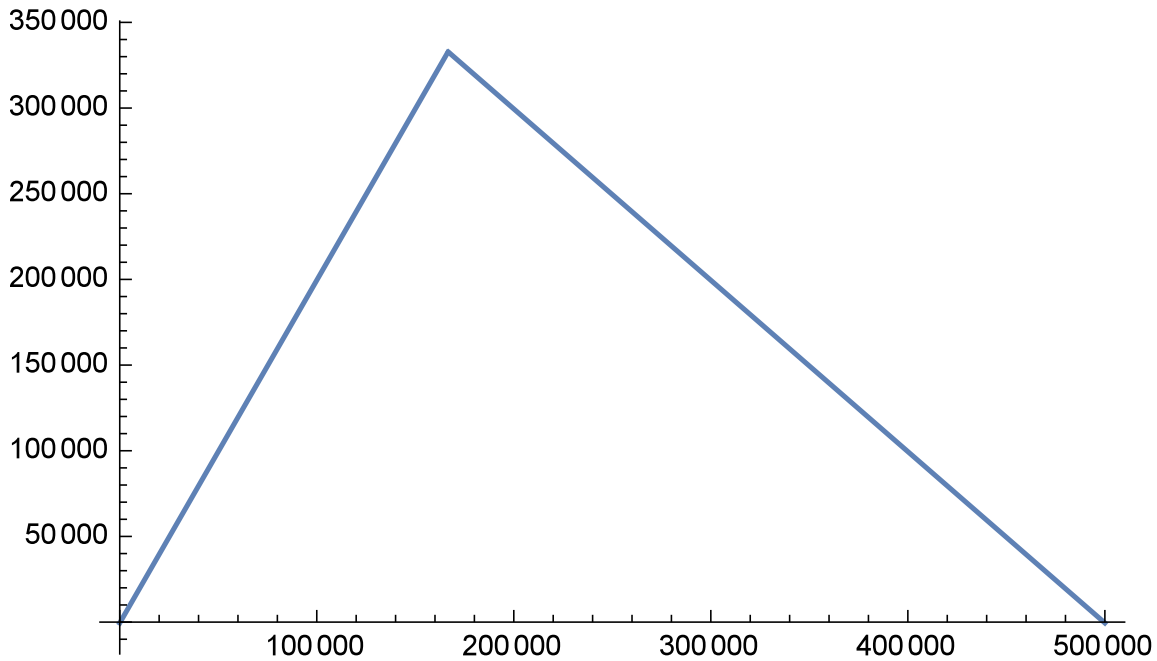
$$f(p, x, B) = p \cdot 500000 + (1 - p)Bx - 500 - B.$$



Guaranteed Winnings

By hedging can ensure some winnings:

$$g(p, x, B) = \min(500000, Bx) - 500 - B.$$

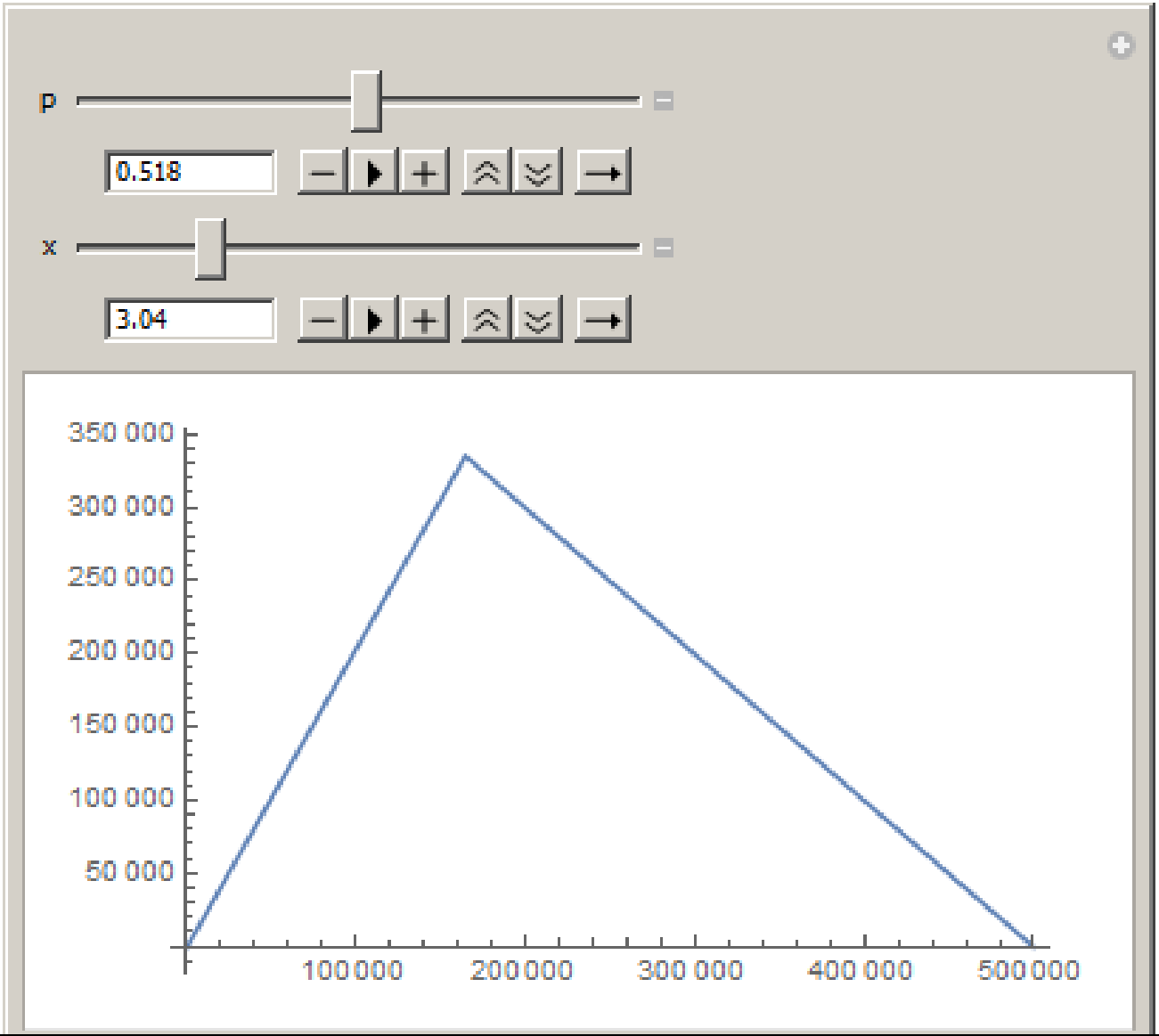


Here $p = .8, x = 3$.

Mathematica Code

```
f[p_, x_, B_] := 500 000 p + (1 - p) B x - 500 - B
g[p_, x_, B_] := Min[500 000, B x] - 500 - B
Plot[f[.8, 3, B], {B, 0, 500 000}]
Plot[g[.8, 3, B], {B, 0, 500 000}]
Manipulate[Plot[g[p, x, B], {B, 0, 500 000}], {p, 0, 1}, {x, 1, 10}]
```

Mathematica Code



Sabermetrics Club at Williams....

http://fivethirtyeight.com/features/

a-head-coach-botched-the-end-of-the-super-bowl-and-it-wasnt-pete-carroll/

Clicker Problems

Birthday Problem I

Birthday Problem

How large must N be for there to be at least a 50% probability that two of the N people share a birthday?

Birthday Problem I

Birthday Problem

How large must N be for there to be at least a 50% probability that two of the N people share a birthday?

- (A) 11 people
- (B) 22 people
- (C) 33 people
- (D) 44 people
- (E) 90 people
- (F) 180 people
- (G) 365 people
- (H) 500 people.

Birthday Problem I

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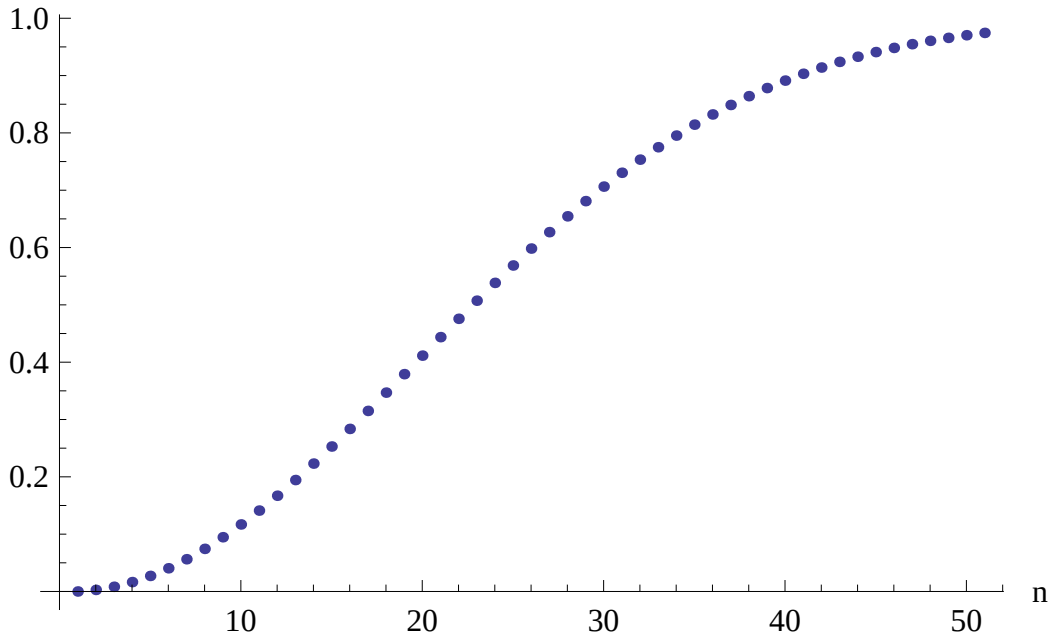

Birthday Problem II

How large must N be for there to be at least a 50% probability that two of N Plutonians share a birthday?

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- (A) 110 people
- (B) 220 people
- (C) 330 people
- (D) 440 people
- (E) 1,000 people
- (F) 5,000 people
- (G) 10,000 people
- (H) 20,000 people
- (I) more than 30,000 people.

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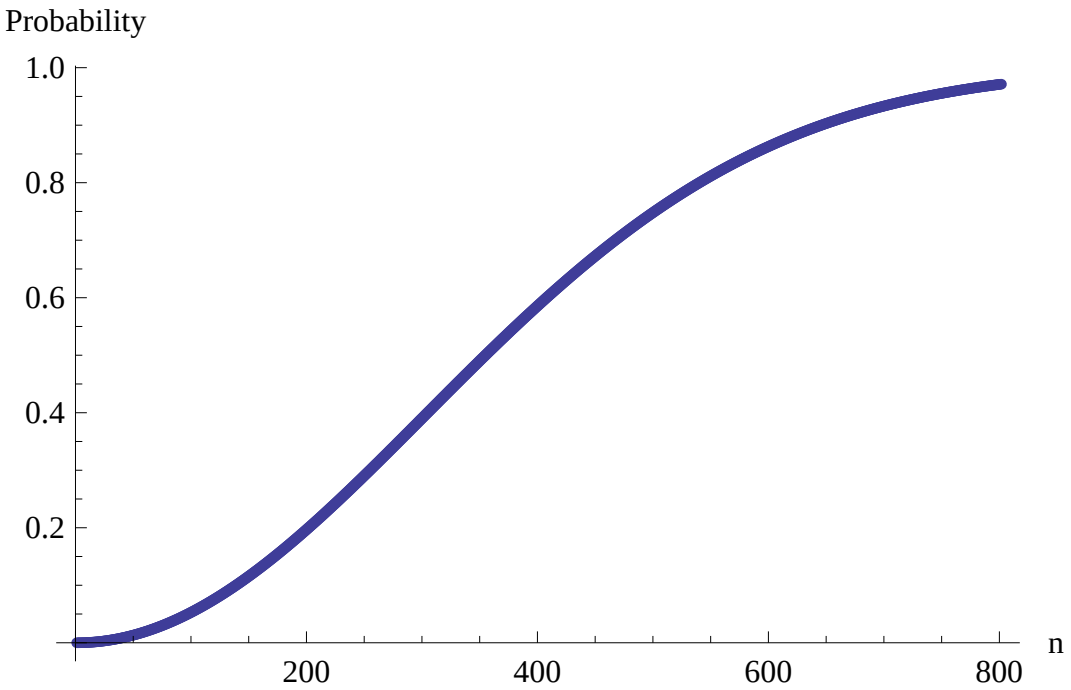

Voting: Democratic Primaries

During the Democratic primaries in 2008, Clinton and Obama received exactly the same number of votes in Syracuse, NY. How probable was this?

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During the Democratic primaries in 2008, Clinton and Obama received exactly the same number of votes in Syracuse, NY. How probable was this? (Note: they each received 6001 votes.)

- (A) 1 / 10
- (B) 1 / 100
- (C) 1 / 1,000
- (D) 1 / 10,000
- (E) 1 / 100,000
- (F) 1 / 1,000,000 (one in a million)
- (G) 1 / 1,000,000,000 (one in a billion).

Voting: Democratic Primaries (continued)

Syracuse University mathematics Professor Hyune-Ju Kim said the result was less than one in a million, according to the Syracuse Post-Standard, which quoted the professor as saying, "It's almost impossible." Her comments were reprinted widely, as the Associated Press picked up the story. (Carl Bialik, WSJ, 2/12/08)

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Far greater than 1/137! What's going on?

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Far greater than 1/137! What's going on?

Prof. Kim's calculation ... was based on the assumption that Syracuse voters were likely to vote in equal proportions to the state as a whole, which went for Ms. Clinton, its junior senator, 57%-40%. .... Prof. Kim said she had little time to make the calculation, so she made the questionable assumption ... for simplicity.

From Shooting Hoops to the Geometric Series Formula

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p.
- **Magic** always gets basket with probability q.

Let x be the probability **Bird** wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

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Break into cases:

- **Bird** wins on 1^{st} shot: p.
- **Bird** wins on 2^{nd} shot: $(1-p)(1-q)\cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.

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- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Solving the Hoop Game

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- **Bird** wins on nth shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird wins}) \\ &= p+rp+r^2p+r^3p+\cdots \\ &= p\left(1+r+r^2+r^3+\cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

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Have

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$$x = \text{Prob}(\textbf{Bird wins}) = p + (1 - p)(1 - q)x$$

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Thus

$$(1 - r)x = p \quad \text{or} \quad x = \frac{p}{1 - r}.$$

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$$x = \text{Prob}(\textbf{Bird wins}) = p + (1 - p)(1 - q)x = p + rx.$$

Thus

$$(1 - r)x = p \quad \text{or} \quad x = \frac{p}{1 - r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}.$$

Lessons from Hoop Problem

◇ Power of Perspective: Memoryless process.

◇ Can circumvent algebra with deeper understanding! (Hard)

◇ Depth of a problem not always what expect.

◇ Importance of knowing more than the minimum: connections.

◇ Math is fun!