professor phone: 617 835 3982

Sep 10, 2009

Combinatorics

$n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$

Choose r of n, order counts

$$n(n-1)\ldots(n-(r-1)) \cdot \frac{(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} = {}_nP_r$$

Choose r from n, order doesn't matter : $\binom{n}{r} = nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$

Pascal

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \qquad \begin{array}{l} (x+y)^0 \\ (x+y)^1 \\ (x+y)^2 \\ (x+y)^3 \\ (x+y)^4 \end{array}$$

$$\Rightarrow (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$\Uparrow$

Binomial Theorem.

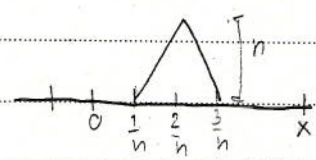
September 15, 2009

Finitive additive : disjoint union then $P(\bigcup_{i=1}^{n} A) = \sum_{i=1}^{n} P(A_i)$

$\rightarrow$ countably additive if the $\{A_i\}$ pairwise disjoint implies

$P(\bigcup_{i=1}^{\infty} A) = \sum_{i=1}^{n} P(A)$

Ex $f_n(x) =$



$$\int_{-\infty}^{\infty} f_n(x)\,dx = 1 \quad (\text{area})$$

but $f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \{\because n \to \infty\ (x) \Rightarrow \text{area} \to 0\}$

$\Downarrow$

So, $\int_{-\infty}^{\infty} f(x)\,dx = 0$

$\Downarrow$

That's why : $A = \bigcup_{i=1}^{n} A \Rightarrow P(A) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} A)$

$B = \bigcap_{i=1}^{n} B \Rightarrow P(B) = \lim_{n\to\infty} P(\bigcap_{i=1}^{n} B)$

Date : ○○○ Clicker Question

Conj : 40% all # 5 prime , 20% twin primes

2, 3, 5, 7 ↑ ↑ twin primes | 11, 13, 17, 19 ↑ ↑ twin primes

* Inclusion - Exclusion *

Ex primes $p_1, p_2, \ldots, p_r$ $N = 5$

square-free number: $N - \sum_{p^2 \le N} \left\lfloor \frac{N}{p^2} \right\rfloor + \sum_{p_1 < p_2 \le N} \left\lfloor \frac{N}{p_1^2 p_2^2} \right\rfloor - \sum_{p_i < p_j < p_k < N} \left\lfloor \frac{N}{p_i^2 p_j^2 p_k^2} \right\rfloor$

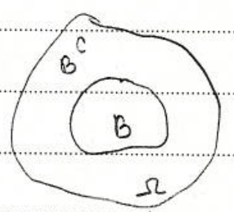
1 - 3 + 3 + 1

36 → 4, 9 5^2 → 25

Conditional Probability

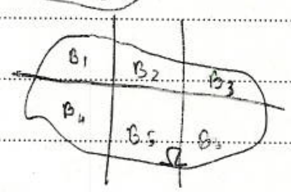
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Partition : A family of events $B_1, \ldots, B_n$ is a partition of Ω if the $\{B_i\}$'s are disjoint and $\bigcup_{i=1}^{n} B_i = \Omega$



$\Rightarrow \Omega = B \cup B^c$

or



Uncountable partition is hard to do!

Partition Lemmas

$$\underbrace{B_1 \cup (B_2 \cup \ldots \cup B_n)}_{B_1^c} = \Omega$$

$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$; $0 < P(B) < 1$

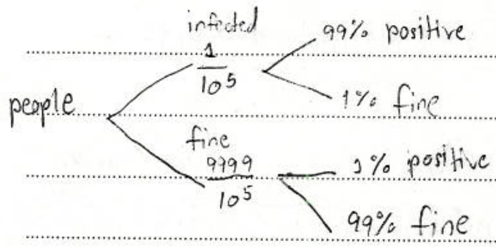
Proof

$P(A|B)P(B) = P(A \cap B)$ or $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$; $0 < P(B) < 1$

So, we get $P(A) = \dfrac{P(A \cap B)}{P(B)} \cdot P(B) + \dfrac{P(A \cap B^c)}{P(B^c)} \cdot P(B^c)$

$= P(A \cap B) + P(A \cap B^c) = P(A)$ #

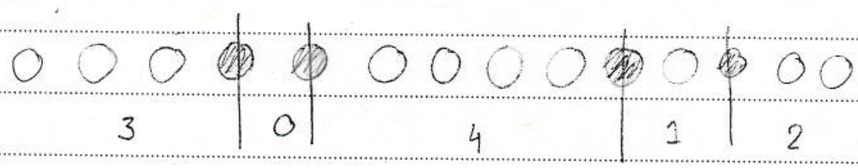
Clicker Q.



$$P(ill|+) = \frac{P(+|ill)P(ill)}{P(+)} = \frac{P(+|ill)P(ill)}{P(+|ill)P(ill) + P(+|healthy)P(healthy)}$$

$$= \frac{\frac{99}{100}(10^{-5})}{\left(\frac{99}{100} \cdot 10^{-5}\right) + \underbrace{\frac{1}{100}(1 - 10^{-5})}_{\text{it's easier to improve this}}}$$

Cookie Problems => 10 identical cookies, 5 students



$\Rightarrow \dbinom{10+5-1}{5-1}$

$x_1 + \ldots + x_5 = 10$

Question prob of winning the lotteries (50 numbers and cannot be repeated)

- $P(win) = \dfrac{1}{\binom{50}{6}}$

Date : ○○○ **Problem Set 1**

Review: 2^{nd} prob on Combinatorics

n people in line, randomly shuffle

1 2 3	1 3 2
2 1 3	3 1 2
2 3 1	3 2 1

A_i event i is in the right spot $\Rightarrow P(A_i) = \frac{1}{n} = \frac{1\cdot(n-1)!}{n!}$

A_{ij} " i, j " ——— " $(i \neq j) \Rightarrow P(A_{ij}) = \frac{1}{n(n-1)} = \frac{1\cdot 1\cdot(n-2)!}{n!}$

$A = \bigcup_{i=1}^{n} A_i$

n	P(n)
1	1
2	1/2
3	2/3
4	

$$P(A) = \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_{ij}) + \sum_{i<j<k} P(A_{ijk}) - \sum_{i<j<k<l} P(A_{i,j,k,l}) + \dots$$

1 7 6 4 5 3 2

1 2 3 4 5 6 7

$$\Longrightarrow \left[= 3 - 3 + 1 + 0\right]$$

$$= \binom{n}{1}\frac{1}{n} - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3}\frac{1}{n(n-1)(n-2)} - \dots$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + (-1)^{n+1}\frac{1}{n!}$$

$$= -1\left(-1 + \frac{1}{2!} - \frac{1}{3!} + \dots\right)$$

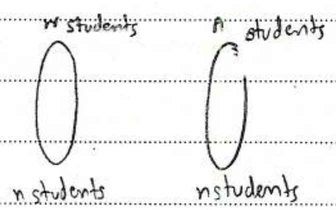
$$= 1 - \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots\right)$$

$$\boxed{\text{Taylor}: e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

$$= 1 - e^{-1}$$

$$= 1 - \frac{1}{e} \quad \underline{\text{Ans}}$$

Ex



$\Rightarrow \exists\ n!$ ways of matching them all

$= \frac{1 \cdot (n-1)!}{n!}$

$P(A_{ij}) = \frac{1}{n(n-1)}$

$= \frac{1 \cdot 1 \cdot (n-2)!}{n!}$

(12/22) Prove that $P(\bigcap_{i}^{n} A_i) = \sum_{i}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \sum_{i<j<k} P(A_i \cup A_j \cup A_k) - \dots (-1)^n P(A_1 \cup A_2 \dots \cup A_n)$

If $n=1$, $P(A_1) = P(A_1)$ True ✓✓

Induction: Prove than $n+1$ term is true.

Assume $\bigcap_{i=1}^{n} A_i = B \Rightarrow B_{n+1} = B_n \cap A_{n+1}$

$n=2$: $P(X \cap Y) = P(X) + P(Y) - P(X \cap Y)$ substitute

$P(B_n \cap A_{n+1}) = P(B) + P(A_{n+1}) - \underline{P(B \cup A_{n+1})}$

need result from 2 n n+1

Note: $B \cup A_{n+1} = (A_1 \cap \dots \cap A_n) \cup A_{n+1}$

$= \underline{(A_1 \cup A_{n+1}) \cap (A_2 \cup A_{n+1}) \cap \dots \cap (A_n \cup A_{n+1})}$

intersection of n set

(3/8) R R W W ✫ ✫

R_1 R_2 W_1 W_2 $✫_1$ $✫_2$

ways to place things $= 6!$

2 R on 2 W, 2 W on 2 ✫...

$$\frac{2 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{6!} = \frac{80}{720}$$

Sep 17, 2009

Independence

A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$

a family $\{A_i\}_{i \in I}$ is independent if

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i) \text{ for any } J \subset I$$

Ex $\Omega = \{1, \dots, 6\}^3 \Rightarrow \# \Omega = 216 = 6^3$

$F = 2^{\Omega} \Rightarrow \{\{1,2,4\}, \{4,4,1\}, \{6,6,6\}\} \in F$

every subset of omega

$\# F = 2^{216} = (2^{10})^{21.6} \approx (10^3)^{21.6} \approx 10^{65}$

Date : ○○○

$\mathbb{P}(\{a,b,c\}) = \frac{1}{6^3}$

$\Omega = \{1, \dots, 6\}^2$ (2 rolls)

Ex A: 1^{st} roll is a 1 $\quad \mathbb{P}(A) = 1/6$

B: 2^{nd} roll is a 2 $\quad \mathbb{P}(B) = 1/6$

C: sum of first two rolls $\quad \mathbb{P}(C) = 1/6$ (นับดูก็ได้)

pairwise independent

$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{36}$ (independent)

$P(A \cap C) = P(A) \cdot P(C) = \frac{1}{36}$

$P(B \cap C) = P(B) \cdot P(C) = \frac{1}{36}$

BUT $P(A \cap B \cap C) = 0 \neq P(A) \cdot P(B) \cdot P(C)$ $\Rightarrow$ three events as a whole = not independent

Roulette Problem (clicker question)

$\underbrace{\left(1 - \tfrac{1}{32}\right)}_{} \underbrace{\left(1 - \tfrac{1}{32}\right)}_{} \cdots \underbrace{\left(1 - \tfrac{1}{32}\right)}_{}$ $\Rightarrow$ block them in group.

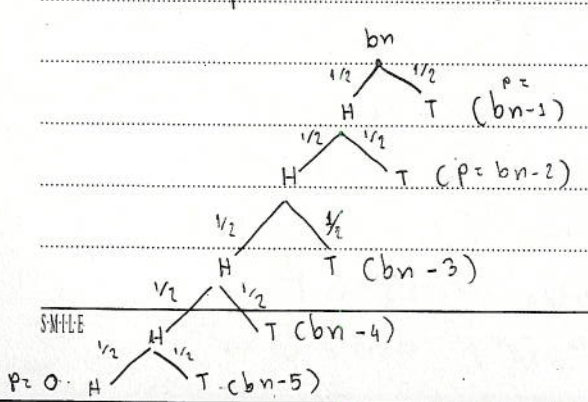
no 5 consecutive head

$P(\text{no five consecutive head}) = \left(1 - \frac{1}{32}\right)^{20} = \left(\frac{31}{32}\right)^{20}$

lower bound because there's chance ~~that 5 consec. heads are between~~ blocks

Ex a_n = prob 5 consecutive H in n tosses $\rightarrow$ will be going to 1 fairly fast

b_n = prob don't have 5 consecutive H in n tosses $\rightarrow$ will be going to zero fairly fast



$b_n = \frac{1}{2} b_{n-1} + \frac{1}{4} b_{n-2} + \frac{1}{8} b_{n-3} + \frac{1}{16} b_{n-4} + \frac{1}{32} b_{n-5}$

$b_5 = \frac{31}{32}$ มาจาก $1 - \frac{1}{32}$ → no head

$b_0 = \dots = b_4 = 1$

From Fibonacci : $a_{n+1} = a_n + a_{n-1}$

Use Divine Inspiration:

<u>Guess</u> $a_n = r^n$ plug in $a_{n+1} = a_n + a_{n-1}$

$r^{n+1} = r^n + r^{n-1}$

$r^n(r^2 - r - 1) = 0$

$r = 0$ or $r = \frac{1 \pm \sqrt{5}}{2}$

General Solution : $a_n = \underset{\uparrow \frac{1}{\sqrt{5}}}{C_1} \left(\frac{1+\sqrt{5}}{2}\right)^n + \underset{\uparrow -\frac{1}{\sqrt{5}}}{C_2} \left(\frac{1-\sqrt{5}}{2}\right)^n$ ⇒ so with the large n, $\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0$

$b_n = \sum_{k=1}^{5} C_k \cdot r_k^n$

<u>September 22, 2009</u>

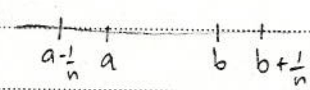
Probability Space (Ω, F, P). A random variable is a function X from the sample space Ω to <u>real number</u> with the property that $\{\omega \in \Omega : X(\omega) \leq x\} \in F$ for each x.
↳ to be able to find the average value

$F = 2^{\Omega}$ if Ω finite or countable

$\Omega = [0,1], [0,1]^k, \mathbb{R}, \mathbb{R}^k$

$F = \langle (a,b) \rangle$
↳ open interval

<u>Ex</u> $\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = [a,b]$



Ex Ω is $[0,1]$, $X(w) = w$

$\{w \in \Omega : X(w) \leq x\} = [0, x]$

$Y(w) = w^2$

$\{w \in \Omega : Y(w) \leq x\} = [0, \sqrt{x}]$

Indicator Function

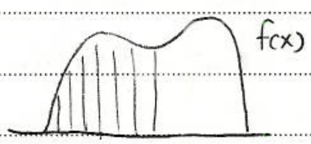
X is an indicator function for event A if $X_A(w) = \begin{cases} 1 & w \in A \\ 0 & w \notin A \end{cases}$

Ex Let $X_i(w) = \begin{cases} 1 & i^{th} \text{ toss H} \\ 0 & i^{th} \text{ toss T} \end{cases}$

$$X = \sum_{i=1}^{5} X_i$$

Distribution Function

The distribution fn of a random variable $X : \Omega \to \mathbb{R}$ is the function $F : \mathbb{R} \to [0,1]$ given by $F(x) = \mathbb{P}(X \leq x)$. In other words, it's the probability of observing a value of X of at most x.



Lemma

(1) as $x \to \infty$, $F(x) \to 1$

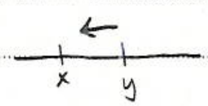
as $x \to -\infty$, $F(x) \to 0$

(2) If $x \leq y$, then $F(x) \leq F(y)$

(3) Right continuous :

$\lim_{y \downarrow x} F(y) = F(x)$

as y goes down to x



Proof (3)

From Lemma 1.3.5 : $A_1 \subset A_2 \subset \ldots$

$B_1 \supset B_2 \supset \ldots$

$\lim_{n \to \infty} \mathbb{P}(\bigcup_{A=1}^{n} A_n) = \mathbb{P}(\bigcup_{n=1}^{\infty} A_n)$

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right)$$

$$B_n = \left(-\infty, x + \tfrac{1}{n}\right)$$

<u>Discrete Random Variables</u>

A random variable X is discrete if it takes values in a countable subset $\{x_1, x_2, \dots\}$ of $\mathbb{R}$. It has probability mass function $f : \mathbb{R} \to [0,1]$ given by $f(x) = \mathbb{P}(X = x)$

Ω = 5 tosses fair coin

$X(\omega) = \#H$ in ω

$\mathbb{P}(X = 0) = \frac{1}{32}$ $\Downarrow$ choosing no head

$\mathbb{P}(X = 1) = \frac{5}{32} = \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4$ $\Downarrow$ choosing one coin to be a head

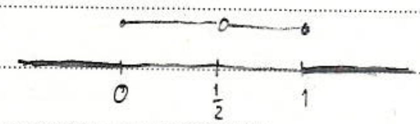
$\mathbb{P}(X = 1) = \binom{5}{1} p(1-p)^4$ => if the coin is biased.

$\mathbb{P}(X = n) = (1-p)^{n-1} p$ => for biased coin

<u>Continuous Random Variable</u> [Probability density function of X]

A random variable X is continuous if its distribution function can be written as $F(x) = \int_{-\infty}^{\infty} f(u)\,du$ for some integrable function f

$$f(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ \cancel{2} & \cancel{x = \frac{1}{2}} \quad 1 \text{ if } x = \frac{1}{2} \\ 1 & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



<u>Fundamental theorem of calculus</u> => the area under the curve : $\int_a^b f(x)\,dx = F(b) - F(a)$

' → [0,1]
value

Date :

Ex P_n be probability of winning when start at n, $0 \le n \le N$ $\left[\begin{matrix} H & \text{receives} & 1 \\ T & \text{lose} & 1 \end{matrix} \right]$

Boundary Conditions: $P_0 = 0$, $P_1 = 1$

$0 < n < N$ $\qquad P_n = \underset{(H)}{\frac{1}{2} P_{n+1}} + \underset{(T)}{\frac{1}{2} P_{n-1}}$

$$P_{n+1} = 2P_n - P_{n-1}, \quad P_0 = 0, \; P_N = 1$$

By Divine Inspiration:

Guess r^n : $r^{n+1} = 2r^n - r^{n-1}$

$r^{n-1}(r^2 - 2n + 1) = 0$

$r = 1, 1$

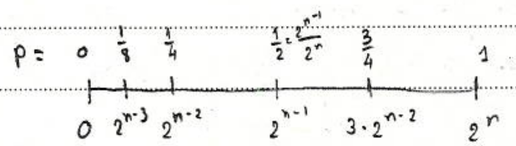
Guess nr^n : $(n+1)r^{n+1} = 2nr^n - (n-1)r^{n-1}$ $\Rightarrow$ solution $cr^n = \begin{cases} 0 & n=0 \\ 1 & n=N \end{cases}$

Solve

General Solution : $P_n = C_1 + C_2 n$

Ex Start off w/ \$k. H receive 1, T lose 1. What is the prob. you reach \$N before you reach \$0.

Lemma If $N = 2^n$, then the prob you win is $\frac{k}{N}$



$\Rightarrow$ Good Proof

Conjecture The probability is $\frac{k}{N}$ for any positive integers $k \le N$.

Aside:

$\pi(x) \underset{=}{:=} \#\{\text{primes } p \le x\}$ (is defined to be)

Conj : $\pi(x) \sim \frac{x}{\log x}$ $\quad$ Prime Number Theorem

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad Re(s) > 1$$

Riemann Hypothesis : all non trival zeros have real part $\frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \frac{6}{\pi^2} = \prod_{p} \left(1 - \frac{1}{p^2}\right)$

September 24, 2009

X or (Ω, F, P)

$F(x) = P(X \leq x)$

Joint Distribution

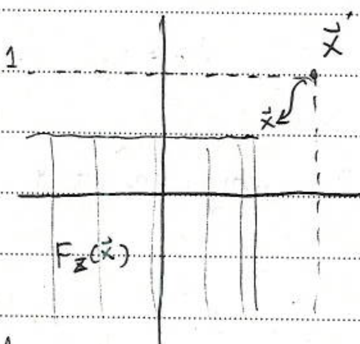
Joint Distribution of $\vec{x} = (x_1, \ldots, x_n)$ on (Ω, F, P) is the fn $F_{\vec{x}} : \mathbb{R}^n \to [0,1]$ given by $F_{\vec{x}}(\vec{x}) = P(\vec{X} \leq \vec{x})$ for $\vec{x} \in \mathbb{R}^n$ where $\vec{x} \leq \vec{y}$ means each $x_i \leq y_i$ and $\{\vec{X} \leq \vec{x}\} = \{\omega \in \Omega : \vec{X}(\omega) \leq \vec{x}\}$

Satisfies the expected properties

① $\lim_{x_1, \ldots, x_n \to \infty} F_{\vec{x}}(\vec{x}) = 0$, $\lim_{x_1, \ldots, x_n \to \infty} F_{\vec{x}}(\vec{x}') = 1$

②

③



Ex Toss coin 10 times, First toss has probability $\frac{1}{2}$ of head.

For k^{th} toss, probability is $\frac{\# \text{Heads to date} (+10)}{\# \text{tosses} \quad (+11)}$

Ex Look at other people : 3 ppl => 8 ways $= 2^3$ to look at

prob (win) $= \frac{2}{2^3} = \frac{1}{4}$ => win when look at ppl and they don't look bac

4 ppl => $3^4 = 81$ ways

10 ppl => 9^{10}

11 ppl => 10^{11}

person one can look at (n-1) person

person 2 have (n-1) choices

n ppl? => count # ways to win $\quad$ 1 2 3 .... n $\quad P = \binom{n-1}{1}\binom{n-2}{1}$

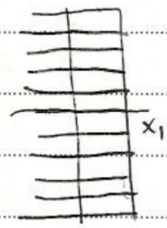
↓ ↓ ↓

2 3 4

prob b/c 3 can look at 1 or the rest

<u>Marginal</u>

$$F_{\vec{X}}(X_1, X_2) = \text{Prob}\,(\vec{X} \le (x_1, x_2))$$
$$= \int_{u_1=-\infty}^{x_1} \int_{u_2=-\infty}^{x_2} f(u_1, u_2)\, du_1\, du_2$$



$$\lim_{x_2 \to \infty} F_{\vec{X}}(x_1, x_2)$$

<u>Ex</u> $X \sim \text{Exp}(X)$

$$\mathbb{P}(X < x) = \int_0^x e^{-t/x} \cdot \frac{dt}{\lambda} \Rightarrow f(x) = \begin{cases} e^{-x/\lambda} \cdot \frac{1}{\lambda} & 0 < x \\ 0 & \text{otherwise} \end{cases}$$

$$P(X \le x) = 1 - e^{-x/\lambda}$$

<u>Ex</u> $Y = X^2$

Distribution fn of $Y = P(Y \le y) = P(X^2 \le Y)$

$= P(X \le \sqrt{y})$ => assume that X is not negative

$= 1 - e^{-\sqrt{y}/\lambda}$

Density by derivatives : $\dfrac{P(Y \le y+h) - \mathbb{P}(Y \le y)}{h} = \dfrac{\text{Prob}\,(y \le Y \le y+h)}{h}$

$$g(y) = e^{-\sqrt{y}/\lambda} \xrightarrow{\text{derivative}} = \frac{1}{2} y^{-\frac{1}{2}} \frac{1}{\lambda}$$

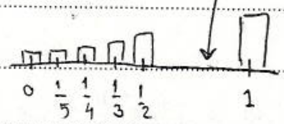
$$\frac{d}{dy} F_X(\sqrt{y}) = F_X'(\sqrt{y}) \cdot (\sqrt{y})'$$
$$= f(\sqrt{y}) \cdot \frac{1}{2} y^{-\frac{1}{2}}$$
$$= e^{-\sqrt{y}/\lambda} \cdot \frac{1}{2} y^{-1/2} \cdot \frac{1}{\lambda}$$

September 29. 2009

Probability Mass Function

$X_n = \frac{1}{n}$

$f(x_n)$ $\frac{1}{2^n}$



$F\left(\frac{2}{3}\right) = \sum_{x_i \le \frac{2}{3}} f(x_i) = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}$

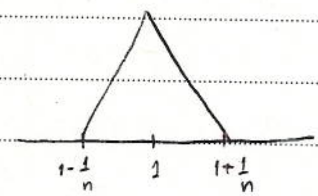
$F(y) = \frac{1}{2}$ if $\frac{1}{2} < y < 1$

$F(1) = 1$

$F(1) - \lim_{y \to x^-} F(y) = 1 - \frac{1}{2} = \frac{1}{2}$

Probability Density Function

$F(x) = \int_{-\infty}^{x} f(u)\,du$



Standard Property

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$

negative

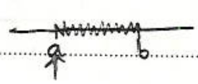
$\mathbb{P}(a \le x \le b) = \int_a^b f(x)\,dx$

Ex $\mathbb{P}(X \le b) = F(b)$ assume $b > a$

$\mathbb{P}(X \le a) = F(b)$

$\mathbb{P}(a < X \le b) = F(b) - F(a)$

less than



$= \int_{-\infty}^{b} f(u)\,du - \int_{-\infty}^{a} f(u)\,du$

$= \int_a^b f(u)\,du$

$F(a) - \lim_{x \to a^-} F(x) = \int_{-\infty}^{a} f(u)\,du - \lim_{x \to a^-} \int_{-\infty}^{x} f(u)\,du$

$= \lim_{x \to a^-} \int_x^a f(u)\,du$

It will go to zero if f is continuous or piecewise continuous

f is bounded

Ex σ field $\mathcal{F} = \{\phi, \Omega\}$

If Ω is discrete (finite, cantable), $\mathcal{F} = 2^{\Omega}$

If Ω is continuous ($[0,1]$, $[0,1]^n$, $\mathbb{R}^n$), $\mathcal{F}$ = <open intervals>

contain $\subsetneq 2^{[0,1]}$ but not equal to

Date : ○○○

Ex X is $Exp(\lambda)$

$$f(x) = \begin{cases} e^{-x/\lambda} \cdot \frac{1}{\lambda} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

} it's piecewise continuous but not continuous at zero.

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-x/\lambda} & \text{if } x \geq 0 \end{cases} \Leftarrow \boxed{\begin{aligned} F(x) &= \int_0^x e^{-u/\lambda} \cdot \frac{1}{\lambda}\, du \quad 0<x \\ &= 1 - e^{-x/\lambda} \end{aligned}}$$

Fundamental theorem of calculus: $\int_a^b f(u)\,du = F(b) - F(a)$

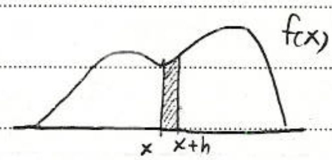
Ex Y is X^2

Cumulative Distribution Function

$$\begin{aligned} \Leftarrow \mathbb{P}(Y \leq y) &= \mathbb{P}(X^2 \leq y) \\ &= \mathbb{P}(X \leq \sqrt{y}) \\ &= F(\sqrt{y}) \\ &= 1 - e^{-\sqrt{y}/\lambda} \end{aligned}$$

$\Rightarrow$ in general it should be $\mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y})$ but X cannot be negative so we write it $\mathbb{P}(X \leq \sqrt{y})$

Calculate Density Function



$F' = f$

$$\begin{aligned} \mathbb{P}(x \leq X \leq x+h) &= F(x+h) - F(x) \\ &= \int_x^{x+h} f(u)\,du \\ &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \cdot \frac{1}{h} \int_x^{x+h} f(u)\,du = f(x) \end{aligned}$$

Note : $f(u) = f(x) + f'(x)(u-x) + \ldots$

$$\int_x^{x+h} f(u)\,du = \underbrace{f(x)}_{\text{treat as a constant}} \cdot h + f'(x) \cdot \frac{h^2}{2} + \ldots$$

Note 2 : $f(u) = f(x) + f'(c)(x-u)$ for some u and x ⇒ mean value theorem

$$f'(c) = \frac{f(u) - f(x)}{u-x}$$

Now let's call $G(y) = 1 - e^{-\sqrt{y}/\lambda}$

$$g(y) = + e^{-\sqrt{y}/\lambda} \cdot \frac{1}{2\lambda\sqrt{y}}$$ ⇐ Distribution Function of y

$G(y) = F(\sqrt{y}) = F(h(y)) \quad h(y) = \sqrt{y}$

$$g(y) = G'(y) = F'(\sqrt{y}) = F'(h(y)) \cdot h'(y)$$

$$= f(h(y)) \cdot \frac{1}{2}\sqrt{y}$$

$$= e^{-\sqrt{y}/\lambda} \cdot \frac{1}{2\lambda y}$$

THE SAME ???

Notation

F_X is cumulative distribution function of X

f_X is density of X

F_Y is cumulative distribution function of Y

f_Y is density of Y.

Theorem : If X is a continuous random variable with continuous density f_X and CDF (cumulative distribution function) F_X, then $f_X = F_X'$

Ex If x and y be two real numbers, $e^x \cdot e^y = e^{x+y}$

Let A and B be two 341×341 matrices, $e^A \cdot e^B \neq e^{A+B}$

$= e^{A+B+\frac{1}{2}[A,B] + \frac{1}{12}\{ \text{———} \}}$

Date :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \left[\text{also } \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right]$$

$$e^x \cdot e^y = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \sum_{m=0}^{\infty} \frac{y^m}{m!}$$

$$e^{x+y} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \frac{k! \cdot x^l \cdot y^{k-l}}{l!\,(k-l)!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} x^l \cdot y^{k-l}$$

$$= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

EQUAL

Size / Growth / Decay of functions

① x^r v.s. e^x $\quad r \geq 0$ and $x \to \infty$

Let $n > r$ and n is an integer

$$\frac{x^n}{n!} \to \infty$$

② $\log x$ v.s. $\underset{\text{win}}{\underset{\Downarrow}{x^r}}$ $\quad 0 < r < 1$ and $x \to \infty$

Let $x = e^{(y/r)}$

$\frac{y}{r}$ v.s. e^y $\Rightarrow$ $\frac{y}{r}$ will go slower than e^y

③ $\lim_{x\to 0} \underbrace{x}_{\text{will go to } 0} \cdot \underbrace{\log x}_{\text{will go to } -\infty}$ $\quad$ write $x = \frac{1}{n}$

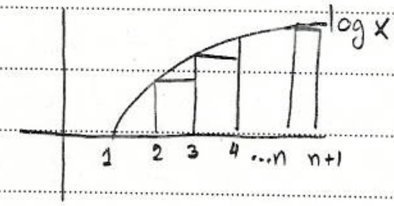
$$= \lim_{n\to\infty} \frac{1}{n}(-\log n) = \lim_{n\to\infty} \left(\frac{-\log n}{n}\right) = 0$$

because $\log n$ will go faster than n.

Pore Mathematician's Stirling

$$n! \approx n^n \cdot e^{-n} \sqrt{2\pi n} \left(1 + \text{error of size } \frac{1}{(2n)} + \dots\right)$$

Proof $\log(n!) = \sum_{k=1}^{n} \log k$



$$\int_1^n \log t \, dt < \log n! < \int_1^{n+1} \log t \, dt \Leftarrow$$

$$< \int_1^n \log t \, dt + \log(n+1)$$

Note that $\int_1^n \log t \, dt = [t \log t - t] \Big|_1^n$

$$= n \log n - n + 1$$

$$\log n! \approx n \log n - n \pm \log n$$

$$n! \approx n^n \cdot e^{-n} \text{ (at most } n)$$

October 1, 2009

Outcomes: H, T, E

Toss n times

$$p(H_n = h, T_n = t, E_n = e) = \begin{cases} 0 & \text{, otherwise} \\ \dfrac{n!}{h!\,t!\,e!} \left(\dfrac{1}{3}\right)^n & \text{, } h+t+e=n \end{cases}$$

Ex MISSISSIPPI

ways to rearrange them $= \dfrac{11!}{4!\,4!\,2!\,1!} = \dbinom{11}{4\ 4\ 2\ 1}$

$$P(H_n = h, O_n = n-h) = \binom{n}{h} \left(\frac{1}{3}\right)^h \left(\frac{2}{3}\right)^{n-h}$$

$$p(T_{n-h} = t, E_{n-h} = n-h-t) = \binom{n-h}{t} \left(\frac{1}{2}\right)^t \left(\frac{1}{2}\right)^{n-h-t}$$

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$$p(H_n=h, T_n=t, E_n=e) = p(H_n=h \mid T_n+E_n = n-h) \cdot p(T_{n-h}=t, E_{n-h}=n-h-t)$$

$$= \binom{n}{h}\left(\frac{1}{3}\right)^h\left(\frac{2}{3}\right)^{n-h} \cdot \binom{n-h}{t}\left(\frac{1}{2}\right)^t\left(\frac{1}{2}\right)^{n-h-t}$$

$$= \frac{n!}{h!\,\cancel{(n-h)!}} \cdot \frac{\cancel{(n-h)!}}{t!\,(n-h-t)!}\left(\frac{1}{3}\right)^n \cdot \cancel{2^{n-h}} \cdot \cancel{\left(\frac{1}{2}\right)^{n-h}}$$

Marginals

$$* \sum_{t=0}^{n}\sum_{e=0}^{n-t} p(H_n=h, T_n=t, E_n=e) = \sum_{t=0}^{n}\sum_{e=0}^{n-t} \frac{n!}{h!\,t!\,e!}\left(\frac{1}{3}\right)^n$$

$$= \sum_{t=0}^{n} \frac{n!}{t!}\left(\frac{1}{3}\right)^n \sum_{e=0}^{n-t} \frac{1}{\underbrace{(n-t-e)!}_{h!}\,e!}$$

$$= \sum_{t=0}^{n} \frac{n!}{t!\,(n-t)!}\left(\frac{1}{3}\right)^n \sum_{e=0}^{n-t} \frac{(n-t)!}{(n-t-e)!\,e!}$$

$$= \left(\frac{1}{3}\right)^n \sum_{t=0}^{n}\binom{n}{t} \underbrace{\sum_{e=0}^{n-t}\binom{n-t}{e} \cdot 1^e \cdot 1^{n-t}}$$ binomial coefficient

$$= \left(\frac{1}{3}\right)^n \sum_{t=0}^{n}\binom{n}{t}(1+1)^{n-t}$$

$$= \left(\frac{1}{3}\right)^n \underbrace{\sum_{t=0}^{n}\binom{n}{t} 2^{n-t} \cdot 1^t}$$

$$= \left(\frac{1}{3}\right)^n (1+2)^n$$

$$= 1 \qquad \checkmark\checkmark\checkmark \quad \underline{\text{Ans}}$$

Ex

$$S_n = 1 + r + \dots + r^n$$

$$rS_n = r + \dots + r^n + r^{n+1}$$

$$(1-r)S_n = 1 - r^{n+1}$$

$$\therefore S_n = \frac{1-r^{n+1}}{1-r}$$

Ex $1+2+4+\dots = -1$

Ex $1+2+3+4+5+\dots = -\frac{1}{12}$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r} \quad \text{if } |r|<1$$

Ex Probabilistic Proof

	M	N
hit	p	q
miss	$1-p$	$1-q$

Define : $r = (1-p)(1-q)$

x = probability that M wins

$0 < p, q < 1$

$$X = p + (1-p)(1-q)p + (1-p)^2(1-q)^2 p + \dots$$

$$X = p + rp + r^2 p + \dots$$

$$X = p(1 + r + r^2 + \dots)$$

$$\frac{X}{p} = 1 + r + r^2 + \dots$$

$$X = p + rX$$

$$(1-r)X = p$$

$$\frac{X}{p} = \frac{1}{1-r}$$

Proof of Geometric formula

หลอกเรื่อง

① $\sqrt{a+\sqrt{a+\sqrt{\dots}}} = x$

$a + x = x^2$

② $\int \cos^2 x \, dx = \frac{1}{2}\int \cos^2 x \, dx + \text{known}$

Binomial

$\text{Bin}(n, p)$ $P(X = k) = f_X(t) = \binom{n}{k} p^k (1-p)^{n-k}$

tosses

probability of success

Poisson

Define $0^0 = 1$

$\text{Poiss}(\lambda)$ $P(X = k) = f_X(k) = \dfrac{\lambda^k \cdot e^{-\lambda}}{k!}$

probability distribution $\displaystyle\sum_{k=0}^{\infty} \frac{\lambda^k \cdot e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

$$= e^{-\lambda} \cdot e^{\lambda} = 1$$

$$\frac{\lambda^k}{k!} \approx \frac{\lambda^k}{k^k \cdot e^{-k}\sqrt{2\pi k}} = \left(\frac{e\lambda}{k}\right)^k \cdot \frac{1}{\sqrt{2\pi k}}$$

Date :

<u>Exponential</u> : Exp (λ)

$$f_X(x) = \begin{cases} e^{-x/\lambda} \cdot \frac{1}{\lambda} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

<u>Uniform</u> : Unif (a,b)

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{other} \end{cases}$$

<u>(3.2) (4.2) Independence</u>

$$P(A \cap B) = P(A) \cdot P(B) \quad -*$$

$$P(A \mid B) \cdot \cancel{P(B)} = P(A) \cdot \cancel{P(B)}$$

$$P(A \mid B) = P(A) \quad -*$$

- <u>Discrete</u> : X, Y random variables

Events $A = \{X = x\}$

$B = \{Y = y\}$

X, Y independent if events A_x and B_y are independent for all x and y.

- <u>Continuous</u> $A_x = \{X \leq x\}$

$B_y = \{Y \leq y\}$

<u>Ex</u> Prob p of H, $1-p$ of tails

Toss N times

$X = \#H$, $Y = \#T = N - X$ → independent

Assume N is Poiss (λ)

Expectation (3.3 / 4.3)

X random variable

↳ continuous

$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

↳ discrete

$= \sum_{i=1}^{\infty} x_i f_X(x_i)$

μ = mean, average value = expected value

Bin (1, p) = Bernouilli (p)

X = # heads in 1 toss

$$f_X(n) = \begin{cases} 1/2 & \text{if } n=0 \\ 1/2 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

$E[X] = 0 \cdot \underbrace{\tfrac{1}{2}}_{p(H)} + 1 \cdot \tfrac{1}{2}$

$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$

Bin (100, p)

X = # heads in 100 tosses

$$f_X(n) = \begin{cases} \binom{100}{n} p^n (n-p)^{100-n} & 0 \leq n \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

$E[X] = \sum_{n=0}^{100} n \cdot \binom{100}{n} \cdot p^n (1-p)^{100-n}$

$= 100p$

Date : 10/6/09

Sections 3.3 & 4.3 : Expectation

X random variable with density / mass function f_X, then expected value is

- Discrete Case : $E(X) = \sum_x x f_X(x)$ if sum converges absolutely
- Continuous case : $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$ if integral converges absolutely

X = # heads

$P(H) = p$, $P(T) = 1-p$

$E(X) = 1 \cdot p + 0(1-p) = p$

Cauchy Distribution

$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$

like $\int_A^{\infty} \frac{1}{x^2}$ when x is big => so, this integral will converge.

$E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi} \cdot \frac{1}{1+x^2}\,dx = 0$ because it's symmetric

look at it as $\lim_{A\to\infty} \int_{-A}^{A} \frac{1}{\pi}\frac{1}{1+x^2}\,dx = 0$

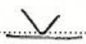
$$\lim_{A\to\infty} \int_{-A}^{2A} \frac{1}{\pi}\frac{1}{1+x^2}\,dx = \lim_{A\to\infty} \frac{1}{\pi} \underbrace{\int_{A}^{2A} \frac{2x}{1+x^2}\,dx}_{\frac{1}{2x}}$$

Moments

Let X be random variable. We define

- k^{th} moment : $m_k := E[X^k]$ (if converges absolutely.) Assume X has a finite mean, which denote by μ (so $\mu = E[X]$). We define
- k^{th} centered moment : $\sigma_k := E[(X-\mu)^k]$ (if converges absolutely)

Taylor series $f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + \dots$

Note $g(x) = |x|$ 

↳ no derivative, so don't have Taylor series expansion.

What if f is infinitely differentiable?

- No!

$$f(0) = f(0) + f'(0)\cdot 0 + f''(0)\cdot\frac{0^2}{2!} + f'''(0)\cdot\frac{0^3}{3!} + \dots$$

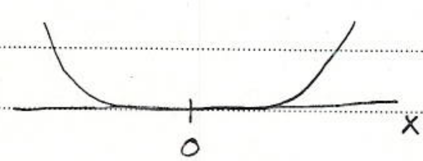
$$f'(x) = f'(0) + f''(0)\cdot x + f'''(0)\cdot\frac{x^2}{2!} + \dots$$

$$f''(x) = f''(0) + f''(0)\cdot x + \dots$$

Ex $g(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(h)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h}$ USE L'HOSPITAL RULE

$g'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$



ite

Ex $f(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2}\,dx$

Second Moment : $\sigma_2 = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\pi}\cdot\frac{1}{1+x^2}\,dx$

Date :

Ex $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

$$E[X^2] \neq E[X]^2$$

Ex 52 cards, draw 7. On average, how many kings?

$$E[X] = \sum_{k=0}^{4} k \cdot \text{Prob}(X = b)$$

$$= \sum_{k=0}^{4} \frac{k \cdot \binom{4}{k}\binom{48}{7-k}}{\binom{52}{7}} \Rightarrow \text{prob. of having exactly 4 kings}$$

$$= \frac{72037840}{133784560} = \frac{7}{13}$$

thirteen possible outcome

Ex $X_1, X_2, \ldots, X_7$

$$X_k = \begin{cases} 1 & \text{if } k^{th} \text{ card is a king} \\ 0 & \text{otherwise} \end{cases}$$

Theorem: Expectation is linear.

$$E[aX + bY] = aE[X] + bE[Y]$$

> **Aside** $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ ⇒ derivative doesn't equal because it's not finite.

$$P(X_k = 1) = \frac{4}{52}$$

$$P(X_k = 0) = \frac{48}{52}$$

$$E[X_k] = 1 \cdot \frac{4}{52} + 0 \cdot \frac{48}{52}$$

$\mathbb{X} = \mathbb{X}_1 + \mathbb{X}_2 + \dots + \mathbb{X}_7$

$E[\mathbb{X}] = \sum_{k=1}^{7} E[X_k] = 7 \cdot \frac{4}{52} = \frac{28}{52} = \frac{7}{13}$

Proof: $f(x,y)$

$E[a\mathbb{X} + b\mathbb{Y}] = \iint (ax+by) f(x,y)\, dx dy$

$= a\iint x f(x,y)\, dx dy + b \iint y f(x,y)\, dx dy$

$= a\int_{-\infty}^{\infty} x \underbrace{\left[\int_{-\infty}^{\infty} f(x,y)\, dy\right]}_{\text{marginal of } x = f_{\mathbb{X}}} dx + \dots$

$= a \underbrace{\int_{-\infty}^{\infty} x f_{\mathbb{X}}\, dx}_{\text{expected value of } x} + \dots$

$= a E[X] + \dots$ ✓✓✓

Other proof of the kings problem

$E[\text{king}] = E[\text{queen}] = \dots = X$

$13X = 7$

$X = \frac{7}{13}$

October 8, 2009

① $E[aX + bY] = aE[X] + bE[Y]$ It doesn't work the other way.

② * Independent X, Y : $E[XY] = E[X] \cdot E[Y]$ if RHS holds say uncorrelated

Proof Probability Density Function

$f_{\mathbb{X},\mathbb{Y}}(x,y)$

$P(a \le \mathbb{X} \le b,\ c \le \mathbb{Y} \le d) = \int_{x=a}^{b}\int_{y=c}^{d} f_{\mathbb{X},\mathbb{Y}}(x,y)\, dx dy$ — *

Date : ○○○

$Z = XY$

$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy$ ถ้าต่อ

$\boxed{\begin{array}{l} \underline{\text{Ex}} \quad X \begin{cases} 1 & \text{prob } \frac{1}{2} \\ -1 & \text{prob } \frac{1}{2} \end{cases} \\ E[XX] \neq E[X] \cdot E[X] \end{array}}$

<u>Lemma</u> : X, Y independent $\Rightarrow f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

<u>back to</u> $*$

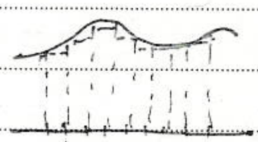
$$P(a \le X \le b,\ c \le Y \le d) = \int_{x=a}^{b} \int_{y=c}^{d} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{x=a}^{b} f_X(x)\,dx \cdot \int_{y=c}^{d} f_Y(y)\,dy$$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx \cdot \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

<u>Recall</u> : Fubini : $\iint f(x,y)\,dx\,dy = \int f_X\,dx \cdot \int f_Y\,dy$



$$= E[X] \cdot E[Y]$$

✓✓✓ <u>True</u> when X, Y are independent.

S·M·I·L·E

not linear

Standard Deviation $= \sqrt{Var(X)} = \sigma_X$

<u>Variance</u> : $Var(X) = \sigma_X^2 = E[(x-\mu)^2]$

<u>Lemma</u>: $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y)$ if uncorrelated

Let $Z = aX + bY$

$E[Z] = aE[X] + bE[Y]$

$Var(Z) = E[(Z - E(Z))^2]$

<u>Proof</u>

$Covariance(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var X_i + 2\sum_{1\le i<j\le n} Covariance(X_i, X_j)$$

<u>Ex</u> Stocks X, Y

$E[X] = E[Y] = \mu > 0$

$Var(X) = \sigma^2 = Var(Y)$

<u>assume</u> X, Y independent (uncorrelated) $\Rightarrow$ not very reasonable but assume this for now

<u>Assume</u> $0 \le a, b \le 1$ and $a + b = 1$

Invest \$a in X, \$b in Y

$E[aX + bY] = aE[X] + bE[Y]$

$= a\mu + b\mu = \mu$

$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y)$

$= (a^2+b^2)\sigma^2 \longrightarrow$ If $a = \frac{1}{2}, b = \frac{1}{2}$, $Var = \frac{\sigma^2}{2}$ (minimum risk $\Rightarrow$ use calculus)

<u>Ex</u> $\{2, 3, 5, \ldots, p, \ldots\}$

n, flip at each prime. At prime p, with probability $\frac{1}{p}$. I say it is p-good. <u>Question</u>: how many goods do I get?

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Fix n, Play game :

$$\mathbb{X}_p = \begin{cases} 1 & \text{prob. } 1/p \\ 0 & \text{prob. } 1 - 1/p \end{cases}$$

$$\mathbb{X} = \mathbb{X}_2 + \mathbb{X}_3 + \mathbb{X}_5 + \dots$$

$$E[\mathbb{X}] = E[\mathbb{X}_2] + E[\mathbb{X}_3] + \dots$$

$$E[\mathbb{X}_p] = 1 \cdot \frac{1}{p} + 0\left(1 - \frac{1}{p}\right)$$

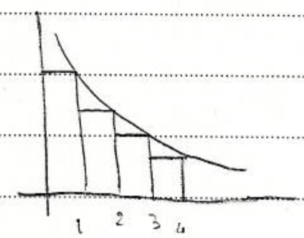
$$E[\mathbb{X}_p] = \frac{1}{p}$$

$$E[\mathbb{X}] = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots$$

$$= \sum_{p \leq n} \frac{1}{p} \approx \log\log n$$

$\Downarrow$

Harmonic series $\displaystyle\sum_{m \leq n} \frac{1}{m} \approx \log n$



คิดมาจาก $\displaystyle\sum_{m \leq n} \frac{1}{m} \approx \int_1^n \frac{1}{x}\,dx$

$$\approx \log n$$

Ex $\mathrm{Var}(\mathbb{X}_1 + \dots + \mathbb{X}_\ell)$ $\quad$ mean $= \frac{1}{p} = \mu$

$$\mathrm{Var}(\mathbb{X}_p) = E\left[\left(\mathbb{X}_p - \tfrac{1}{p}\right)^2\right]$$

$$\mathrm{Var}(1) = \left(1 - \frac{1}{p}\right)^2 \cdot \frac{1}{p} + \left(\frac{1}{p}\right)^2 \left(1 - \frac{1}{p}\right)$$

$$= \frac{1}{p}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p} + \frac{1}{p}\right)$$

$$= \frac{1}{p}\left(1 - \frac{1}{p}\right) = \frac{1}{p} - \frac{1}{p^2}$$

Aside

3 6 9 12 15 18 21 24 27 30 33 36 => with the numbers that is divisible by 3, $\frac{1}{5}$ of these will be divisible by 5.

$$Var(X) = \sum_{p \le n} Var(X_p)$$

$$= \sum_{p \le n} \left(\frac{1}{p} - \frac{1}{p^2}\right)$$

$$= \sum_{p \le n} \frac{1}{p} - \sum_{p \le n} \frac{1}{p^2} \approx \log\log n$$

Standard Deviation = $\sqrt{\log\log n}$ #

Aside Calculus : $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converge} & p > 1 \\ \text{diverge} & p \le 1 \end{cases}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Differentiating Identities

Binomial: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$

If $x = p$; $y = 1-p$

$\sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$ = expected number of heads

apply $x \cdot \frac{d}{dx}$ on both side => $x \cdot n(x+y)^{n-1} = x \sum_{k=0}^{n} k \binom{n}{k} x^{k-1} y^{n-k}$ take derivative

$$n(x+y)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^k y^{n-k}$$

$$np = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[H] = \sum k \, Prob(H)$$

$$Var(X) = E[(X-\mu)^2]$$

$$= E[X^2] - (E[X])^2$$

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Geometric Series

$$1 + x + x^2 + \dots = \frac{1}{1-x} = (1-x)^{-1}$$

$$\sum_{k=0}^{\infty} x^k = (1-x)^{-1} \quad \text{Apply } x\frac{dy}{dx}$$

$$x \sum_{k=0}^{\infty} k \cdot x^{k-1} = x(1-x)^{-2}$$

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}$$

If $x = p$

$$\sum_{k=0}^{\infty} k \cdot p^k = \frac{p}{(1-p)^2}$$

Flip coin, prob p Heads. First H is on toss k has prob $(1-p)^{k-1} \cdot p$

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p = \frac{p}{1-p} \sum_{k=1}^{\infty} k(1-p)^k$$

$$= \frac{p}{1-p} \sum_{k=0}^{\infty} k \cdot (1-p)^k = \frac{p}{1-p} \frac{(1-p)}{(1-(1-p))^2} = \frac{1}{p}$$

October 14, 2009

Coupon Problem

Bernoulli : Bern (p)

$$\text{Prob}(X = n) = \begin{cases} p & n=1 \\ 1-p & n=0 \\ 0 & \text{otherwise} \end{cases}$$

Binomial : Bin (n,p)