Math 341: Probability
Fourth Lecture (9/22/09)

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Clicker Questions

Gambler's Ruin

Question

You start off with \$13; if a fair coin lands heads you receive \$1, else you lose \$1. What is the probability you reach \$64 before you reach \$0?

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- (c) About 10%
- (d) About 15%
- (e) About 20%
- (f) About 25%
- (g) About 50%
- (h) About 80%
- (i) About 90%

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Lemma

If $N = 2^n$, then the probability is $\frac{k}{N}$.

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Lemma

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Conjecture

The probability is $\frac{k}{N}$ for any positive integers $k \leq N$.

Challenge problem: can you prove this conjecture *elementarily* for general N?

Section 2.1
Random Variables

Definition

Random Variables

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A random variable is a function X from the sample space Ω to the real numbers with the property that $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for each x.

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Example: Ω: tosses of a fair coin five times, $\mathcal{F} = 2^{\Omega}$, the set of all subsets of Ω, and let $X(\omega)$ denote the number of heads in ω. As there are $2^5 = 32$ elements, there are 2^{32} or about 4,000,000,000 elements in $\mathcal{F}$. Each element of $\mathcal{F}$ is a subset of Ω, and each subset of Ω is an element of $\mathcal{F}$. If we write $F = \{\omega_1, \ldots, \omega_k\}$ for an element of $\mathcal{F}$, then $\mathbb{P}(F) = \sum_{i=1}^{k} \mathbb{P}(\omega_i)$. A straightforward computation shows that X has the desired property; this is clear as all subsets of Ω are in $\mathcal{F}$! If $x = 1$ then $\{\omega \in \Omega : X(\omega) \leq 1\} = \{TTTTT, TTTTH, TTTHT, TTHTT, THTTT, HTTTT\}$. If instead we took $x = 4$, then the set would be all outcomes except $HHHHH$.

Distribution Function

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The distribution function of a random variable $X : \Omega \to \mathbb{R}$ is the function $F : \mathbb{R} \to [0, 1]$ given by $F(x) = \mathbb{P}(X \leq x)$. In other words, it's the probability of observing a value of X of at most x.

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Example: Consider the previous problem concerning five tosses of a fair coin. We have $F(0) = 1/32$, $F(1) = 6/32$, $F(2) = 16/32$, $F(3) = 26/32$, $F(4) = 31/32$ and $F(5) = 32/32$. Our function is supposed to be defined for all real x, so what we really have is the following: $F(x) = 0$ if $x < 0$, $F(x) = 1/32$ if $0 \leq x < 1$, $F(x) = 6/32$ if $1 \leq x < 2$, and so on.

Discrete and Continuous Random Variables

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Discrete Random Variables

A random variable X is discrete if it takes values in a countable subset $\{x_1, x_2, \ldots\}$ of $\mathbb{R}$. It has probability mass function $f : \mathbb{R} \to [0, 1]$ given by $f(x) = \mathbb{P}(X = x)$.

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Example: Toss a fair coin until the first head is obtained. Then $\Omega = \{H, TH, TTH, \ldots\}$. Let X be the number of tosses needed to obtain the first head. Then X is discrete, taking on the values $\{1, 2, 3, \ldots\}$, with the probability X equals n just $1/2^n$.

Definitions (cont)

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Example: Let $\Omega = [0, 1]$ and let $\mathcal{F}$ be the σ-field generated by the open intervals. (This is the standard σ-field.) Let $X(\omega)$ equal ω^2. If we let Y be uniformly distributed on $[0, 1]$, then we see $\mathbb{P}(X \leq x)$ is the same as $\mathbb{P}(Y \leq \sqrt{x})$, which is just $\sqrt{x}$. We are therefore looking for f so that $\sqrt{x} = \int_0^x f(u)du$ for $0 \leq x \leq 1$. Differentiating both sides gives $\frac{1}{2}x^{-1/2} = f(x)$ (note the integral is $\mathfrak{F}(x) - \mathfrak{F}(0)$ with $\mathfrak{F}$ any anti-derivative of f; differentiating yields the claim as $\mathfrak{F}' = f$). We see that for our random variable X, we may take $f(u) = 1/2\sqrt{u}$ for $0 < u \leq 1$ and 0 otherwise.