Math 341: Probability
Eighth Lecture (10/6/09)

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Independence

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Independence of events

Two events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

As $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$, if $\mathbb{P}(B) > 0$ this is equivalent to $\mathbb{P}(A|B) = \mathbb{P}(A)$, or that knowledge of one happening does not affect knowledge of the other happening.

Independence (continued)

Independence of random variables

Two random variables X and Y are independent if for all x, y:

- Discrete case: events $\{X = x\}$ and $\{Y = y\}$ are independent.
- Continuous case: events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent.

Non-trivial example (from book): Toss a coin with probability p of heads N times, where N is a Poisson random variable with parameter λ. Then the number of heads and the number of tails are independent random variables.

Main result

Key Lemma

Let $g, h : \mathbb{R} \to \mathbb{R}$ and assume X and Y are independent random variables. Then $g(X)$ and $h(Y)$ are independent.

- The proof involves real analysis, specifically properties of the σ-fields.
- Assume g, h continuous and strictly increasing (so g^{-1}, h^{-1} exist) and X, Y continuous random variables.
- Then $\{g(X) \leq a\}$ and $\{h(Y) \leq b\}$ are the same as $\{X \leq g^{-1}(a)\}$ and $\{Y \leq h^{-1}(b)\}$.
- As latter two sets are independent (due to independence of X, Y), we see $g(X)$ and $h(Y)$ independent.

Sections 3.3 & 4.3: Expectation

Definition

Expectation (mean value, average)

X random variable with density / mass function f_X, then expected value is

- Discrete case: $\mathbb{E}[X] := \sum_x x f_X(x)$ if sum converges absolutely.
- Continuous case: $\mathbb{E}[X] := \int_{-\infty}^{\infty} x f_X(x) dx$ if integral converges absolutely.

Notation:

- Often use integral notation for both.
- Set $\mathbb{E}[g(X)]$ equal to $\int_{-\infty}^{\infty} g(x) f_X(x) dx$ if exists.

Definition (continued)

Moments

Let X be a random variable. We define

- k^{th} moment: $m_k := \mathbb{E}[X^k]$ (if converges absolutely).

Assume X has a finite mean, which we denote by μ (so $\mu = \mathbb{E}[X]$). We define

- k^{th} centered moment: $\sigma_k := \mathbb{E}[(X - \mu)^k]$ (if converges absolutely).

Definition (continued)

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- Be alert: Some books write μ'_k for m_k and μ_k for σ_k.
- Call σ_2 the variance, write it as σ^2.
- Note $\sigma^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Clicker Questions

Prime divisors

Number of prime divisors

Let N be a large number. If we choose an integer of size approximately N, on average about how many distinct prime factors do we expect N to have (as $N \to \infty$)?

It might be useful to recall the Prime Number Theorem: The number of primes at most x is about $x / \log x$.

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It might be useful to recall the Prime Number Theorem: The number of primes at most x is about $x / \log x$.

- (a) At most 10.
- (b) Around $\log \log \log N$.
- (c) Around $\log \log N$.
- (d) Around $\log N$.
- (e) Around $\log N \log \log N$.
- (f) Around $(\log N)^2$.
- (g) This is an open question.

Fermat Primes

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If $F_n = 2^{2^n} + 1$ is prime, we say F_n is a Fermat prime. About how many Fermat primes are there less than x as $x \to \infty$?

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- (a) 5
- (b) 10
- (c) Between 11 and 20.
- (d) Between 21 and 100.
- (e) $\log\log\log x$.
- (f) $\log\log x$.
- (g) $\log x$.
- (h) More than $\log x$.
- (i) This is an open problem.

$3x+1$ Problem

$3x+1$: Iterating to the fixed point

Define the $3x+1$ map by $a_{n+1} = \frac{3a_n+1}{2^k}$ where $2^k || 3a_n + 1$. Choose a large integer N and randomly choose a starting seed a_0 around N. About how many iterations are needed until we reach 1 (equivalently, about how large is the smallest n such that $a_n = 1$)?

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$3x + 1$: Iterating to the fixed point

Define the $3x + 1$ map by $a_{n+1} = \frac{3a_n+1}{2^k}$ where $2^k || 3a_n + 1$. Choose a large integer N and randomly choose a starting seed a_0 around N. About how many iterations are needed until we reach 1 (equivalently, about how large is the smallest n such that $a_n = 1$)?

There is a constant C so that the answer is about

- (a) At most 10.
- (b) Around $C \log\log\log N$.
- (c) Around $C \log\log N$.
- (d) Around $C \log N$.
- (e) Around $C \log N \log\log N$.
- (f) Around $C(\log N)^2$.
- (g) This is an open question.