Math 341: Probability
Twelfth Lecture (10/22/09)

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Summary for the Day

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- Cauchy-Schwartz Inequality:
  - ◇ Application to covariance.
  - ◇ Analysis of technique.
- Clicker questions:
  - ◇ Three hats.
- Review:
  - ◇ Questions from the class.
  - ◇ Applications and preview.

Cauchy-Schwartz Inequality

Standard Formulation

Cauchy-Schwartz Inequality

$$\left(\int_{-\infty}^{\infty} |f(x)g(x)|dx\right)^2 \leq \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)$$

Proof:

- Use most important inequality in math: $|z|^2 \geq 0$.
- Mechanics:

$$0 \leq \int_{-\infty}^{\infty} (af(x) + bg(x))^2 \, dx.$$

- Alternate proof: just 'guess' a good value of a and b.

General Formulation

Cauchy-Schwartz Inequality

$$\left(\int \cdots \int_A \left| f(\vec{x}) g(\vec{x}) \right| d\vec{x} \right)^2 \leq \left(\int \cdots \int_A \left| f(\vec{x}) \right|^2 d\vec{x} \right) \left(\int \cdots \int_A \left| g(\vec{x}) \right|^2 d\vec{x} \right)$$

with

$$\vec{x} = (x_1, \ldots, x_n), \quad d\vec{x} = dx_1 \cdots dx_n.$$

Application: $\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\, \mathbb{E}[Y^2]$.

Application to Covariance and Correlations I

Covariance

The covariance of X and Y is defined to be

$$\mathrm{CoVar}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)].$$

Algebra yields

$$\mathrm{CoVar}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Note that

$$\mathrm{CoVar}(X, X) = \mathrm{Var}(X).$$

Application to Covariance and Correlations II

Correlation Coefficient

Define the correlation coefficient by

$$\rho(X, Y) = \frac{\mathrm{CoVar}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}.$$

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Key input: $f_{X,Y}(x,y) = \sqrt{f_{X,Y}(x,y)} \cdot \sqrt{f_{X,Y}(x,y)}$.

Allows us to write $\mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$ as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \mu_X)\sqrt{f_{X,Y}(x,y)} \cdot (y - \mu_Y)\sqrt{f_{X,Y}(x,y)}dxdy.$$

This is a very powerful, common technique (see also the Cramer-Rao inequality).

Clicker Questions

Three Hat Problem

Problem Statement

3 mathematicians equally likely to have white or black hat. Each sees color of other hats, but not own. On three, each says ‘white’, ‘black’, or stays silent. If all speaker correct win $1 million; if even one is wrong lose $1 million. What is their winning percentage if play optimally?

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- (a) About 12.5% (i) I've seen this problem before.
- (b) About 25%
- (c) About 37.5%
- (d) About 50%
- (e) About 62.5%
- (f) About 75%
- (g) About 87.5%
- (h) About 100%

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Relevance for us: dependent random variables.

- Joint density function, with marginals $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 0) = 1/2$.
- Note $\mathbb{E}[X_i] = 1/2$.
- Application: Error correcting codes.