Math 341: Probability
Seventeenth Lecture (11/10/09)

Steven J Miller
Williams College

Steven.J.Miller@williams.edu
http://www.williams.edu/go/math/sjmiller/public_html/341/

Bronfman Science Center
Williams College, November 10, 2009

Summary for the Day

Summary for the day

- Complex Analysis:
  - Definitions.
  - Accumulation point theorem.
- Integral Transforms.
  - Laplace and Fourier.
  - Schwartz space and Inversion.
  - Complex Analysis Theorem.
- Central Limit Theorem:
  - Statement and standardization.
  - Poisson example.
  - Proof with MGFs.

Complex Analysis

Holomorphic = Analytic

Holomorphic, analytic

Let U be an open subset of $\mathbb{C}$, and let f be a complex function.

- We say f is holomorphic on U if f is differentiable at every point $z \in U$.
- We say f is analytic on U if f has a series expansion that converges and agrees with f on U. This means that for any $z_0 \in U$, for z close to z_0 we can choose a_n's such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Holomorphic = Analytic

Holomorphic equals Analytic

Let f be a complex function and U an open set. Then f is holomorphic on U if and only if f is analytic on U, and the series expansion for f is its Taylor series.

- If f is differentiable once, it is infinitely differentiable and f agrees with its Taylor series expansion!
- Very different than what happens in the case of functions of a real variable.

Limit points

Limit or accumulation point

We say z is a limit (or an accumulation) point of a sequence $\{z_n\}_{n=0}^{\infty}$ if there exists a subsequence $\{z_{n_k}\}_{k=0}^{\infty}$ converging to z.

Limit points

Limit or accumulation point

We say z is a limit (or an accumulation) point of a sequence $\{z_n\}_{n=0}^{\infty}$ if there exists a subsequence $\{z_{n_k}\}_{k=0}^{\infty}$ converging to z.

- If $z_n = 1/n$, then 0 is a limit point.
- If $z_n = \cos(\pi n)$ then there are two limit points, namely 1 and -1. (If $z_n = \cos(n)$ then *every* point in $[-1, 1]$ is a limit point of the sequence, though this is harder to show.)

Limit points

Limit or accumulation point

We say z is a limit (or an accumulation) point of a sequence $\{z_n\}_{n=0}^{\infty}$ if there exists a subsequence $\{z_{n_k}\}_{k=0}^{\infty}$ converging to z.

- If $z_n = (1 + (-1)^n)^n + 1/n$, then 0 is a limit point. We can see this by taking the subsequence $\{z_1, z_3, z_5, z_7, \ldots\}$; note the subsequence $\{z_0, z_2, z_4, \ldots\}$ diverges to infinity.

- Let z_n denote the number of distinct prime factors of n. Then every positive integer is a limit point!

Limit points

Limit or accumulation point

We say z is a limit (or an accumulation) point of a sequence $\{z_n\}_{n=0}^{\infty}$ if there exists a subsequence $\{z_{n_k}\}_{k=0}^{\infty}$ converging to z.

- If $z_n = n^2$ then there are no limit points, as $\lim_{n\to\infty} z_n = \infty$.
- z_0 any odd, positive integer, set

$$z_{n+1} = \begin{cases} 3z_n + 1 & \text{if } z_n \text{ is odd} \\ z_n/2 & \text{if } z_n \text{ is even.} \end{cases}$$

Conjectured that 1 is always a limit point.

Accumulation points and functions

Theorem

Let f be an analytic function on an open set U, with infinitely many zeros $z_1, z_2, z_3, \ldots$. If $\lim_{n\to\infty} z_n \in U$, then f is identically zero on U. In other words, if a function is zero along a sequence in U whose accumulation point is also in U, then that function is identically zero in U.

Accumulation points and functions

Consider $h(x) = x^3 \sin(1/x)$:

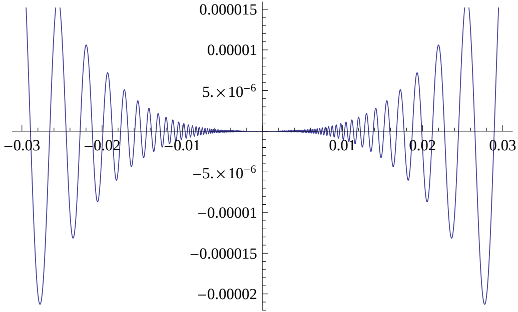

Figure: Plot of $x^3 \sin(1/x)$.

Show $x^3 \sin(1/x)$ is *not* complex differentiable. It will help if you recall $e^{i\theta} = \cos\theta + i\sin\theta$, or $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$.

Integral Transforms

Laplace and Fourier Transform

General framework: Given $K(t,s)$, consider

$$g(s) = \int_{-\infty}^{\infty} f(t)K(t,s)dt.$$

Laplace and Fourier Transform

Laplace Transform

Let $K(t, s) = e^{-ts}$. The Laplace transform of f, denoted $\mathcal{L}f$, is given by

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)e^{-st}dt.$$

Given a function g, its inverse Laplace transform, $\mathcal{L}^{-1}g$, is

$$\begin{aligned}(\mathcal{L}^{-1}g)(t) &= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} e^{st} g(s) ds \\ &= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-T}^{T} e^{(c+i\tau)t} g(c+i\tau) i d\tau.\end{aligned}$$

Laplace and Fourier Transform

Fourier Transform

Let $K(x,y) = e^{-2\pi ixy}$. The Fourier transform of f, denoted $\mathcal{F}f$ or $\widehat{f}$, is

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx,$$

where $e^{i\theta} = \cos\theta + i\sin\theta$. The inverse Fourier transform of g, $\mathcal{F}^{-1}g$, is

$$(\mathcal{F}^{-1}g)(x) = \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}\,dy.$$

Other books define the Fourier transform differently, sometimes using $K(x,y) = e^{-ixy}$ or $K(x,y) = e^{-ixy}/\sqrt{2\pi}$.

Laplace and Fourier Transform

- Laplace and Fourier transforms are related. Let $s = 2\pi iy$ and consider functions $f(x)$ which vanish for $x \leq 0$. See the Laplace and Fourier transforms are equal.
- Given a function f we can compute its transform. What about the other direction?

Schwartz Space

Schwartz space

The Schwartz space, $\mathcal{S}(\mathbb{R})$, is the set of all infinitely differentiable functions f such that, for any non-negative integers m and n,

$$\sup_{x\in\mathbb{R}} \left|(1+x^2)^m \frac{d^n f}{dx^n}\right| < \infty,$$

where $\sup_{x\in\mathbb{R}} |g(x)|$ is the smallest number B such that $|g(x)| \leq B$ for all x (think 'maximum value' whenever you see supremum).

Inversion Theorem

Inversion Theorem for Fourier Transform

Let $f \in \mathcal{S}(\mathbb{R})$, the Schwartz space. Then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(y) e^{2\pi i x y} dy.$$

$f, g \in \mathcal{S}(\mathbb{R})$ with $\widehat{f} = \widehat{g}$ then $f(x) = g(x)$.

- Interplay useful in probability: MGF is an integral transform of the density:$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(t) dt$.
- If $f(x) = 0$ for $x \leq 0$, this is the Laplace transform . Take $t = -2\pi i y$ then it is the Fourier transform. Related to the characteristic function $\phi(t) = \mathbb{E}[e^{itX}]$.

Key Results from Complex Analysis

Theorem

Assume the MGFs $M_X(t)$ and $M_Y(t)$ exist in a neighborhood of zero (i.e., there is some δ such that both functions exist for $|t| < \delta$). If $M_X(t) = M_Y(t)$ in this neighborhood, then $F_X(u) = F_Y(u)$ for all u. As the densities are the derivatives of the cumulative distribution functions, we have $f = g$.

Key Results from Complex Analysis

Theorem

Let $\{X_i\}_{i \in I}$ be a sequence of random variables with MGFs $M_{X_i}(t)$. Assume there is a $\delta > 0$ such that when $|t| < \delta$ we have $\lim_{i \to \infty} M_{X_i}(t) = M_X(t)$ for some MGF $M_X(t)$, and all MGFs converge for $|t| < \delta$. Then there exists a unique cumulative distribution function F whose moments are determined from $M_X(t)$ and for all x where $F_X(x)$ is continuous, $\lim_{n \to \infty} F_{X_i}(x) = F_X(x)$.

Key Results from Complex Analysis

Theorem: X and Y continuous random variables on $[0, \infty)$ with continuous densities f and g, all of whose moments are finite and agree, and

1. $\exists C > 0$ st $\forall c \leq C$, $e^{(c+1)t} f(e^t)$ and $e^{(c+1)t} g(e^t)$ are Schwartz functions.
2. The (not necessarily integral) moments

$$\mu'_{r_n}(f) = \int_0^\infty x^{r_n} f(x) dx \quad \text{and} \quad \mu'_{r_n}(g) = \int_0^\infty x^{r_n} g(x) dx$$

agree for some sequence of non-negative real numbers $\{r_n\}_{n=0}^\infty$ which has a finite accumulation point (i.e., $\lim_{n\to\infty} r_n = r < \infty$).

Then $f = g$ (in other words, knowing all these moments uniquely determines the probability density).

Application to equal integral moments

Return to the two densities causing trouble:

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2\pi x^2}}\, e^{-(\log^2 x)/2} \\ f_2(x) &= f_1(x)\left[1 + \sin(2\pi \log x)\right]. \end{aligned}$$

Application to equal integral moments

Return to the two densities causing trouble:

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2\pi x^2}}\, e^{-(\log^2 x)/2} \\ f_2(x) &= f_1(x)\left[1 + \sin(2\pi \log x)\right]. \end{aligned}$$

- Same integral moments: $e^{k^2/2}$.
- Have the correct decay.
- Using complex analysis (specifically, contour integration), we can calculate the $(a+ib)^{\text{th}}$moments:

$$\text{For } f_1: \quad e^{(a+ib)^2/2}$$

$$\text{For } f_2: \quad e^{(a+ib)^2/2} + \frac{i}{2}\left(e^{(a+i(b-2\pi))^2/2} - e^{(a+i(b+2\pi))^2/2}\right).$$

Application to equal integral moments

Return to the two densities causing trouble:

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2\pi x^2}}\, e^{-(\log^2 x)/2} \\ f_2(x) &= f_1(x)\,[1 + \sin(2\pi \log x)]\,. \end{aligned}$$

- No sequence of real moments having an accumulation point where they agree.
- a^{th}moment of f_2 is

$$e^{a^2/2} + e^{(a-2i\pi)^2/2}\left(1 - e^{4ia\pi}\right),$$

  and this is never zero unless a is a half-integer.
- Only way this can vanish is if $1 = e^{4ia\pi}$.

Central Limit Theorem

Normalization of a random variable

Normalization (standardization) of a random variable

Let X be a random variable with mean μ and standard deviation σ, both of which are finite. The normalization, Y, is defined by

$$Y := \frac{X - \mathbb{E}[X]}{\mathrm{StDev}(X)} = \frac{X - \mu}{\sigma}.$$

Note that

$$\mathbb{E}[Y] = 0 \quad \text{and} \quad \mathrm{StDev}(Y) = 1.$$

Statement of the Central Limit Theorem

Normal distribution

A random variable X is normally distributed (or has the normal distribution, or is a Gaussian random variable) with mean μ and variance σ^2 if the density of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We often write $X \sim N(\mu, \sigma^2)$ to denote this. If $\mu = 0$ and $\sigma^2 = 1$, we say X has the standard normal distribution.

Statement of the Central Limit Theorem

Central Limit Theorem

Let $X_1, \dots, X_N$ be independent, identically distributed random variables whose moment generating functions converge for $|t| < \delta$ for some $\delta > 0$ (this implies all the moments exist and are finite). Denote the mean by μ and the variance by σ^2, let

$$\overline{X}_N = \frac{X_1 + \cdots + X_N}{N}$$

and set

$$Z_N = \frac{\overline{X}_N - \mu}{\sigma/\sqrt{N}}.$$

Then as $N \to \infty$, the distribution of Z_N converges to the standard normal.

Statement of the Central Limit Theorem

Why are there only tables of values of standard normal?

Statement of the Central Limit Theorem

Why are there only tables of values of standard normal?

Answer: normalization. Similar to log tables (only need one from change of base formula).

MGF and the CLT

Moment generating function of normal distributions

Let X be a normal random variable with mean μ and variance σ^2. Its moment generating function satisfies

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

In particular, if Z has the standard normal distribution, its moment generating function is

$$M_Z(t) = e^{t^2/2}.$$

MGF and the CLT

Moment generating function of normal distributions

Let X be a normal random variable with mean μ and variance σ^2. Its moment generating function satisfies

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

In particular, if Z has the standard normal distribution, its moment generating function is

$$M_Z(t) = e^{t^2/2}.$$

Proof: Complete the square.

Poisson Example of the CLT

Example

Let $X, X_1, \ldots, X_N$ be Poisson random variables with parameter λ. Let

$$\overline{X}_N = \frac{X_1 + \cdots + X_N}{N}, \quad Y = \frac{\overline{X} - \mathbb{E}[\overline{X}]}{\mathrm{StDev}(\overline{X})}.$$

Then as $N \to \infty$, Y converges to having the standard normal distribution.

Poisson Example of the CLT

Example

Let $X, X_1, \ldots, X_N$ be Poisson random variables with parameter λ. Let

$$\overline{X}_N = \frac{X_1 + \cdots + X_N}{N}, \quad Y = \frac{\overline{X} - \mathbb{E}[\overline{X}]}{\mathrm{StDev}(\overline{X})}.$$

Then as $N \to \infty$, Y converges to having the standard normal distribution.

Moment generating function: $M_X(t) = \exp(\lambda(e^t - 1))$.