Math 344: The Math of Sports: MWF 10-10:50am: Spring 2023: Williams College

Professor Steven Miller (sjm1 AT williams.edu), Wachenheim 339

My Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/

Course Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/344Sp23/

Slides:
https://web.williams.edu/Mathematics/sjmiller/public_html/344Sp23/Math344Sp23LectureNotes.pdf

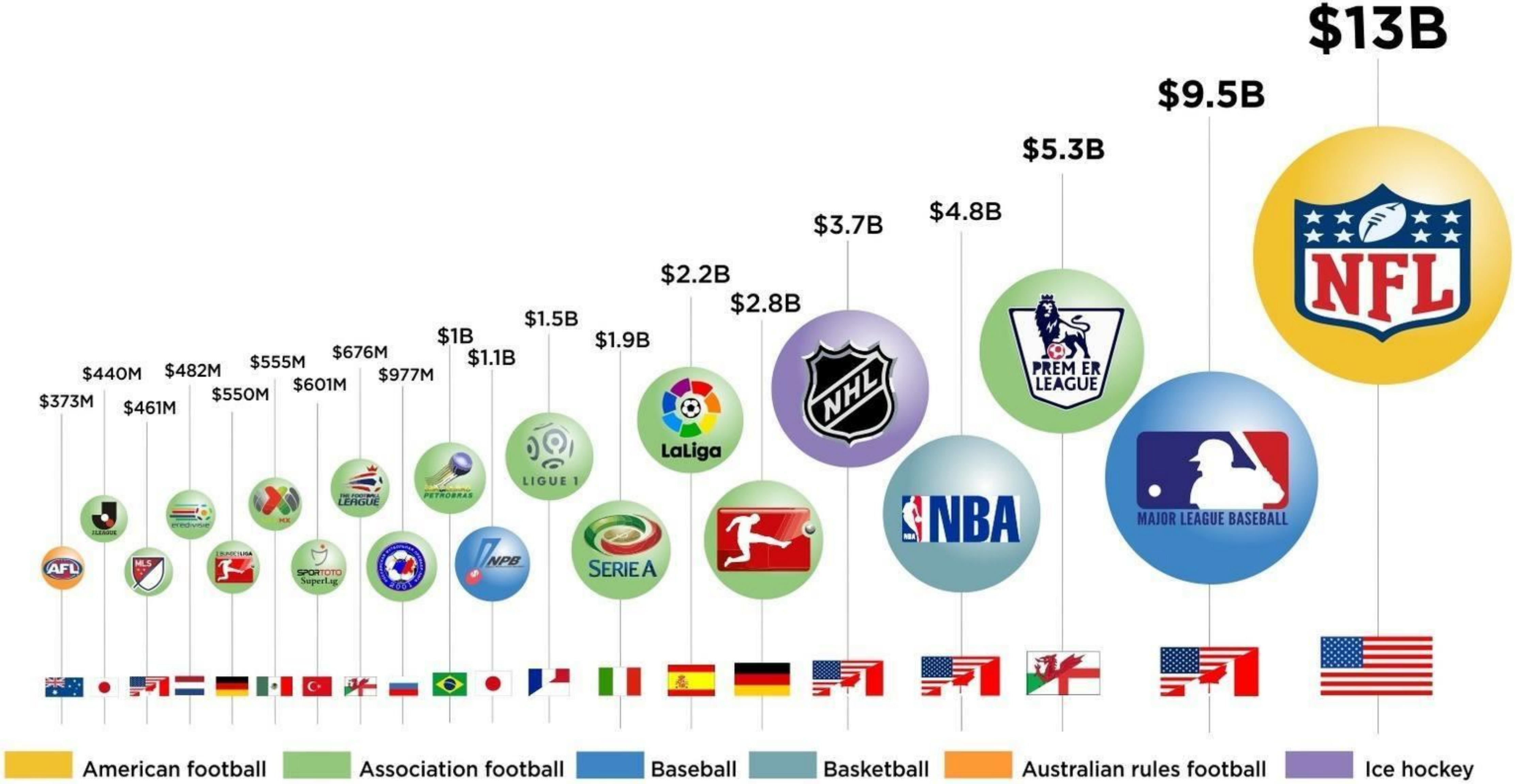
Other: Advice from Jeff Miller

- Party less than the person next to you.
- Take advantage of office hours / mentoring.
- Learn to manage your time: no one else wants to.

Happy to do practice interviews, adjust deadlines....

Lecture 1: https://youtu.be/eO6BxVDEOhQ

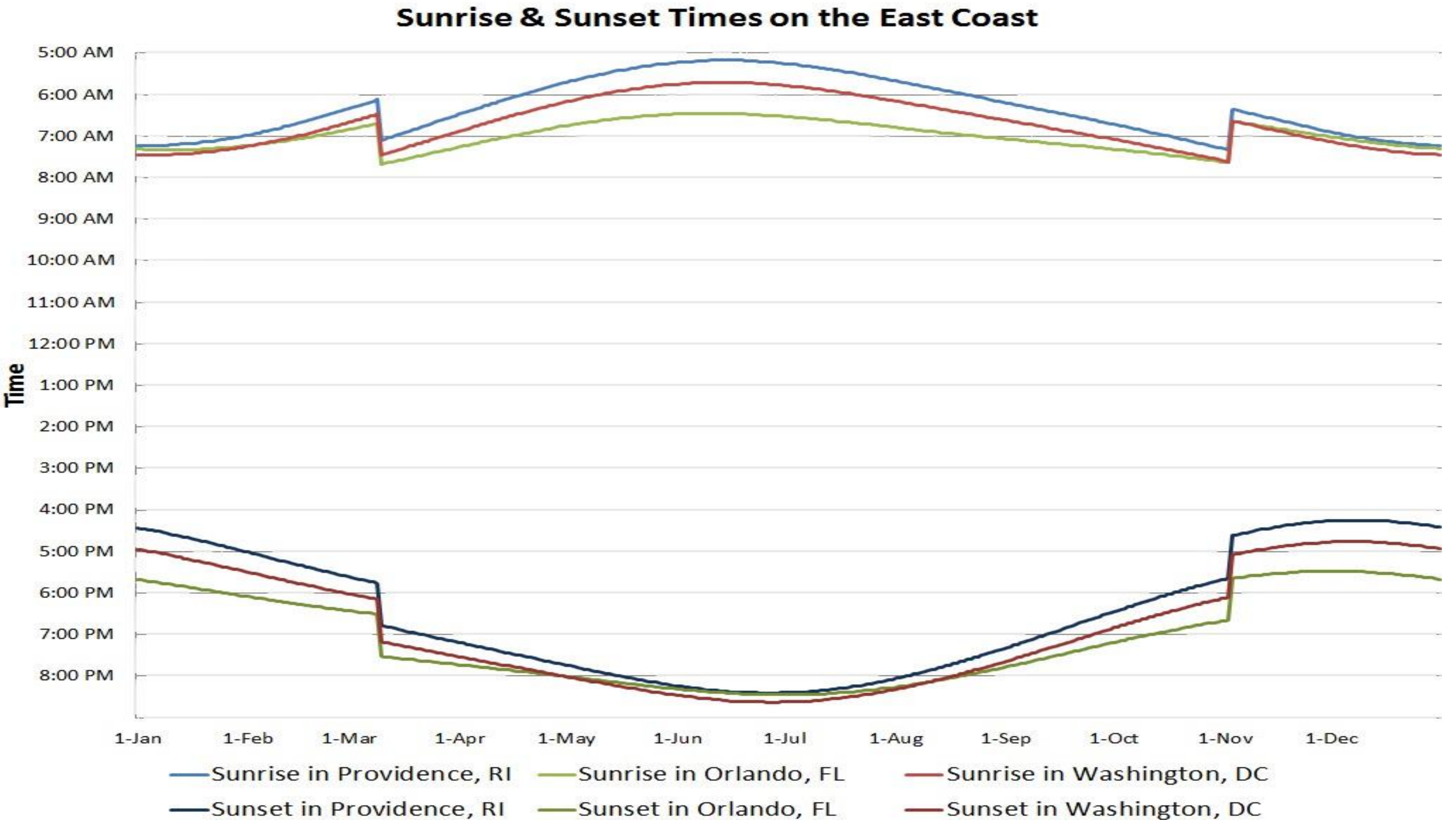
Top Professional Sports Leagues by Revenue



Source:http://howmuch.net/articles/sports-leagues-by-revenue

howmuch net

Who America is rooting for in the Super Bowl:


Sunrise & Sunset Times on the East Coast
Time
5:00 AM
6:00 AM
7:00 AM
8:00 AM
9:00 AM
10:00 AM
11:00 AM
12:00 PM
1:00 PM
2:00 PM
3:00 PM
4:00 PM
5:00 PM
6:00 PM
7:00 PM
8:00 PM
1-Jan
1-Feb
1-Mar
1-Apr
1-May
1-Jun
1-Jul
1-Aug
1-Sep
1-Oct
1-Nov
1-Dec
Sunrise in Providence, RI
Sunrise in Orlando, FL
Sunrise in Washington, DC
Sunset in Providence, RI
Sunset in Orlando, FL
Sunset in Washington, DC

Building Intuition: The $\log 5$ Method

Assume team *A* wins *p* percent of their games, and team *B* wins *q* percent of their games. Which formula do you think does a good job of predicting the probability that team *A* beats team *B*?

$$\frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}$$

$$\frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq}$$

Building intuition: *A* wins *p* percent, *B* wins *q* percent

$$\frac{p+pq}{p+q+2pq}, \quad \frac{p+pq}{p+q-2pq}$$

$$\frac{p-pq}{p+q+2pq}, \quad \frac{p-pq}{p+q-2pq}$$

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$

$$\frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$$

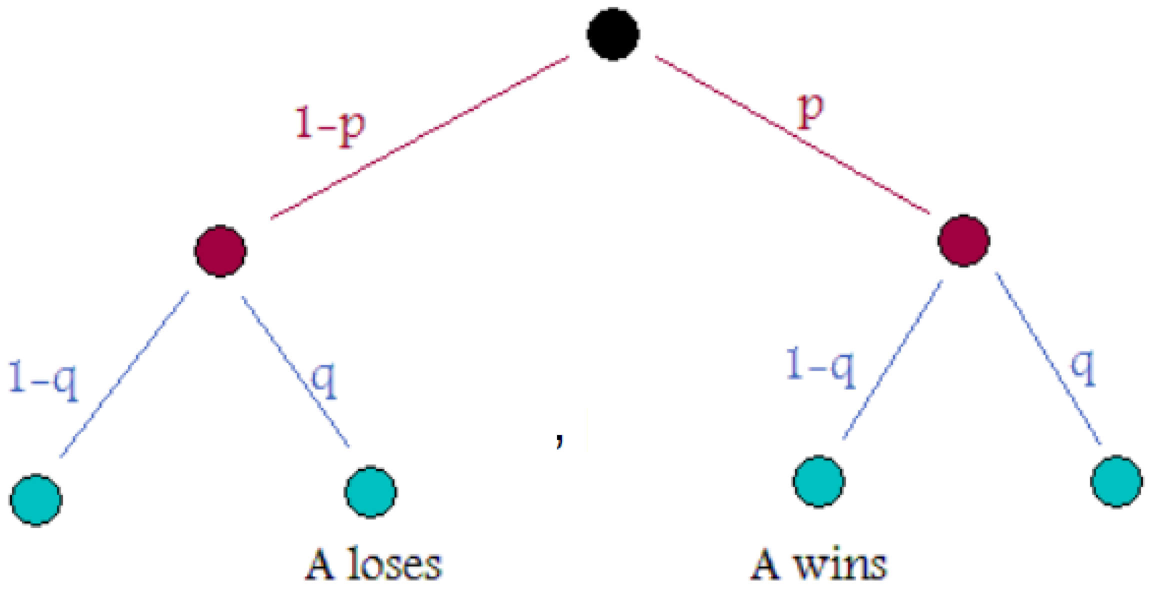
$$\frac{4\cancel{9}}{\cancel{9}8} = \frac{1}{2}$$

$$\frac{1\cancel{2}}{\cancel{2}4} = \frac{1}{4}$$

Consider special cases:

1. $\text{Prob}(A \text{ beats } B) + \text{Prob}(B \text{ beats } A) = 1$.
2. If $p = q$ then the probability *A* beats *B* is 50%.
3. If $p = 1$ and $q \neq 0, 1$ then *A* always beats *B*.
4. If $p = 0$ and $q \neq 0, 1$ then *A* always loses to *B*.
5. If $p > 1/2$ and $q < 1/2$ then $\text{Prob}(A \text{ beats } B) > p$.
6. If $q = 1/2$ prob *A* wins is p ($p = 1/2$ the prob *B* wins is q).

Building intuition: Sketch of proof: $\frac{p-pq}{p+q-2pq}$



- A beats B has probability $p(1-q)$.
- A and B do not have the same outcome has probability $p(1-q)+(1-p)q$.
- $\text{Prob}(A \text{ beats } B) = \frac{p(1-q)}{p(1-q)+(1-p)q} = \frac{p-pq}{p+q-2pq}$.

WHAT DO YOU MEAN?!?

Definitions

Means and averages

- Given x and y, the average or mean is the number in between
- ArithmeticMean(x,y) = (x + y) / 2.
- There is more than one mean that can be defined!
- What properties should a mean have? Assume $0 < x \leq y$.

Desired Properties

We want:

- $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
- $\text{mean}(x,x) = x$.

Does $\text{ArithmeticMean}(x,y) = (x+y)/2$ satisfy these properties?

Desired Properties

We want:

1. $x \leq \text{mean}(x,y) \leq y$. Should be "in between"
2. $\text{mean}(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Proof of (1): Since $0 < x \leq y$, we have $x + x \leq x + y \leq y + y$.
So we know $2x \leq x + y \leq 2y$. Divide everything by 2 and we get
$x \leq (x+y)/2 \leq y$ or $x \leq \text{ArithmeticMean}(x,y) \leq y$.
We proved the first result!

Desired Properties

We want:

1. $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
2. $\text{mean}(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Proof of (2):Does ArithmeticMean(x,x) equal x?
Yes! ArithmeticMean(x,x) = (x+x)/2 = 2x / 2 = x.
So the ArithmeticMean(x,y) = (x+y)/2 satisfies our two properties.

We write AM(x,y) = ArithmeticMean(x,y) = (x+y)/2 to save space.

Question

Is there another choice of mean that satisfies the two properties we wish?

We want:

1. $x \leq \text{mean}(x,y) \leq y$. Should be "in between"
2. $\text{mean}(x,x) = x$.

Thoughts?

Question

Is there another choice of mean that satisfies the two properties we wish?

We want:

1. $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
2. $\text{mean}(x,x) = x$.

Try mean(x,y) = Sqrt(x y).

- Check: Sqrt(2 * 8) = Sqrt(16) = 4 and that IS between 2 and 8.
- Check: Sqrt(1 * 100) = Sqrt(100) = 10 and that is between 1 and 100.

So maybe this is another choice of mean. Maybe it also satisfies the two properties….

Question

Try $\mathrm{mean}(x,y) = \mathrm{Sqrt}(x,y)$. Must show

1. $x \leq \mathrm{mean}(x,y) \leq y$. Should be “in between”
2. $\mathrm{mean}(x,x) = x$.

First property: Show if $0 < x \leq y$ then $x \leq \mathrm{Sqrt}(x\ y) \leq y$.

We know $x \leq y$ so $x\ x \leq x\ y \leq y\ y$

But $x^2 \leq x\ y \leq y^2$. Now take the square-root!

$\mathrm{Sqrt}(x^2) = x$ and $\mathrm{Sqrt}(y^2) = y$, so get $x \leq \mathrm{Sqrt}(x\ y) \leq y$, as claimed!

Question

Try mean(x,y) = Sqrt(x,y). Must show

1. x ≤ mean(x,y) ≤ y. Should be "in between"
2. mean(x,x) = x.

Second is easier!

We have Sqrt(x x) = Sqrt(x^2) = x. We are done!

We call this the GEOMETRIC MEAN. We write GM(x,y) = Sqrt(x y)

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For 0 < x ≤ y both satisfy x ≤ mean(x,y) ≤ y and mean(x,x) = x.

More used to the first.

Try x = 2 and y = 8:

- Get AM(2,8) = (2 + 8) / 2 = 10 / 2 = 5
- Get GM(2,8) = Sqrt(2 * 8) = Sqrt(16) = 4

$AM(1,100) = 50.5$

$GM(1,100) = 1$

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \leq y$ both satisfy $x \leq \text{mean}(x,y) \leq y$ and mean(x,x) = x.

More used to the first.

Try x = 3 and y = 12

- Then AM(3, 12) = 15/2 = 7.5
- And GM(3,12) = Sqrt(36) = 6.

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \leq y$ both satisfy $x \leq \text{mean}(x,y) \leq y$ and mean(x,x) = x.

Try x = 1 and y is VERY large….

- Then AM(1,y) = (1 + y)/2 which is APPROXIMATELY y/2
- But GM(1,y) = Sqrt(y) which is MUCH smaller if y is large.
- Note if y is small we would say (1 + y)/2 is approximately .5

CONJECTURE: GM(x,y) ??? AM(x,y)

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For 0 < x ≤ y both satisfy x ≤ mean(x,y) ≤ y and mean(x,x) = x.

Try x = 1 and y is VERY large….

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- But GM(1,y) = Sqrt(y) which is MUCH smaller if y is large.
- Note if y is small we would say (1 + y)/2 is approximately .5

CONJECTURE: GM(x,y) ≤ AM(x,y)

CONJECTURE: GM(x,y) ≤ AM(x,y)

PROOF: Consider: $0 < x \leq y$, what is true about (Sqrt(x) - Sqrt(y))^2 ? It must be positive…

- So $0 \leq$ (Sqrt(x) - Sqrt(y))^2.

Remember FOIL: $(a - b)^2 = (a - b)(a - b) = a a - a b - b a + b b$: First Outside Inside Last

- So $(a-b)^2 = a^2 - 2 a b + b^2$

We are looking at (Sqrt(x) - Sqrt(y))^2.

- $0 \leq$ (Sqrt(x) - Sqrt(y))^2 = Sqrt(x)^2 - 2 Sqrt(x) Sqrt(y) + Sqrt(y)^2.
- $0 \leq x$ - 2 Sqrt(x y) + y

Trying to get AM(x,y) = (x+y)/2 and GM(x,y) = Sqrt(x,y)

- 2 Sqrt(x,y) ≤ x + y
- Sqrt(x,y) ≤ (x+y)/2
- GM(x,y) ≤ AM(x,y).

We proved it!

Extensions

What if we had three objects: $0 < x \leq y \leq z$?

- $AM(x,y,z) = (x+y+z) / 3$
- $GM(x,y,z) = (x\ y\ z)^{1/3}$.

Is there another combination?

- $((x\ y\ +\ y\ z\ +\ x\ z) / ???\ )^{???}$

Food for thought: can you find a choice of a and b such that

- $((xy + yz + zx) / a)^b$ is a mean, so it would satisfy
- $x \leq \text{TripleMean}(x,y,z) \leq z$ and $\text{TripleMean}(x,x,x) = x$

If $x = y = z$ then $((xx + xx + xx) / a)^b = (3\ x^2 / a)^b = x$ for ALL x.

- SO b = ??? and a = ???

Extensions

What if we had three objects: $0 < x \leq y \leq z$?

- $AM(x,y,z) = (x+y+z) / 3$
- $GM(x,y,z) = (x\ y\ z)^{1/3}$.

Is there another combination? YES

- $((x\ y\ +\ y\ z\ +\ x\ z) / 3\ )^{1/2}$

If $x = y = z$ then $((xx + xx + xx) / a)^b = (3\ x^2 / a)^b = x$ for ALL x.

- SO $b = ½$ and $a = 3$

SO this is our guess….

- Try $x = 3$ and $y = 4$ and $z = 5$
- $\text{TripleMean}(3,4,5) = (\ (12 + 20 + 15) / 3\ )^{1/2} = (47/3)^{1/2}$ is approximately 3.958
- This IS a reasonable answer! It is more than 3, less than 5!

Final Thoughts

AM(x,y) = (x+y)/2   GM(x,y) = Sqrt(x y)

Test 1 Get 1 and on Test 2 get 100

- AM(1, 100) = (1 + 100)/2 = 50.5
- GM(1,100) = Sqrt(1 100) = 10

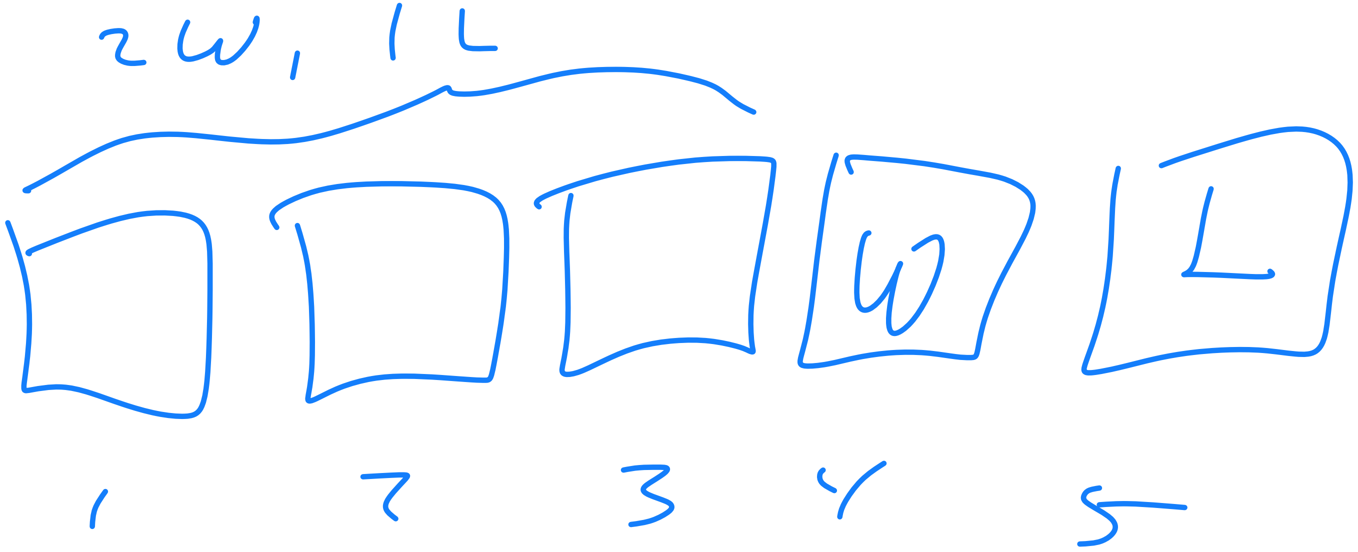
Recall

- Log(x y) = Log(x) + Log(y)
- So there is a relation between logarithms, AM and GM

Challenge: Order the cards from most expensive to least expense (do not search the web), order is from price I paid online, what is the ***metric*** for judging who does best?

1959	Topps #180	Yogi Berra Player
1964	Topps #21	Yogi Berra Manager
1991	Upper Deck #44	Michael Jordan
1999	Upper Deck #3	Wayne Gretzky
2002	Pacific #258	Tom Brady
2011	Topps #240	Tom Brady
2012	Topps #440	Tom Brady
2018	Donruss #1	Lionel Messi
2019	Score #142	Tom Brady
2021	Donruss #1	Tom Brady

Won 3 of last 4 games.... Good?



Last 3: 2/3 ≈ 67%

Last 5: 3/5 ≈ 60%

Last 4: 3/4 ≈ 75%

Math 344: Mathematics of Sports: Spring 2023: Lecture 02: What is a good statistic? https://youtu.be/OSbcaW8WA-4

Plan for the day.

- Discuss how information is presented (theme of the class!).
- Discuss how one does calculations (another theme of the term!).
- Discuss good statistics.
- Discuss presentation possibilities.

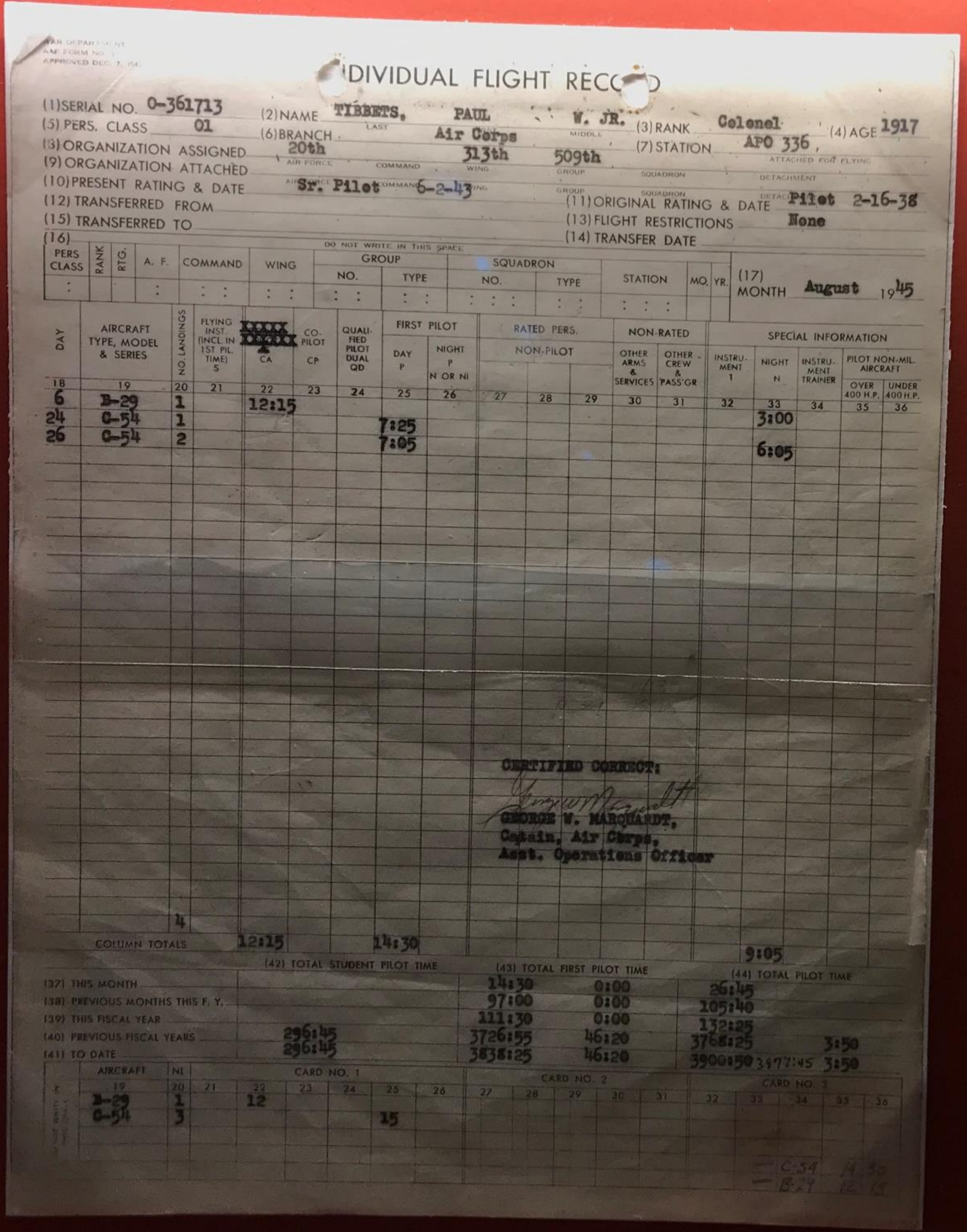
- Images from the National World War II Museum – New Orleans: https://www.nationalww2museum.org/

01

BASEBALL BAT, FIELDER'S MITT, AND SHOES

The St. Lo Collection, 1994.001.0070.1; Gift of John O' Donnell, 2002.056.001-.003

In the unfamiliar and faraway places of the Pacific, recreation was a lifesaver. Games and sports, wher they could be arranged, were critical to the mental health of young Americans overseas. Baseball traveled around the world during the war, even to Okinawa, where John O'Donnell used these shoes and this mitt.



AR DEPARTMENT
AAF FORM NO.
APPROVED DEC. 7, 194

DIVIDUAL FLIGHT RECO D

(1) SERIAL NO. 0-361713
(2) NAME TIBBETS, PAUL W. JR. (LAST / MIDDLE)
(3) RANK Colonel
(4) AGE 1917
(5) PERS. CLASS 01
(6) BRANCH Air Corps
(7) STATION APO 336, ATTACHED FOR FLYING
(8) ORGANIZATION ASSIGNED 20th (AIR FORCE) COMMAND 313th (WING) 509th (GROUP) SQUADRON DETACHMENT
(9) ORGANIZATION ATTACHED AIR FORCE COMMAND WING GROUP SQUADRON DETACHMENT
(10) PRESENT RATING & DATE Sr. Pilot 6-2-43
(11) ORIGINAL RATING & DATE Pilot 2-16-38
(12) TRANSFERRED FROM
(13) FLIGHT RESTRICTIONS None
(14) TRANSFER DATE
(15) TRANSFERRED TO
(16)

DO NOT WRITE IN THIS SPACE

PERS CLASS	RANK	RTG.	A. F.	COMMAND	WING	GROUP NO.	GROUP TYPE	SQUADRON NO.	SQUADRON TYPE	STATION	MO.	YR.
:			:	: :	: :	: :	: :	: : :	: :	: : :		

(17) MONTH August 1945

DAY 18	AIRCRAFT TYPE, MODEL & SERIES 19	NO. LANDINGS 20	FLYING INST. (INCL. IN 1ST PIL. TIME) S 21	CA 22	CO-PILOT CP 23	QUALIFIED PILOT DUAL QD 24	FIRST PILOT DAY P 25	FIRST PILOT NIGHT P N OR NI 26	RATED PERS. NON-PILOT 27	28	29	NON-RATED OTHER ARMS & SERVICES 30	NON-RATED OTHER CREW & PASS'GR 31	SPECIAL INFORMATION INSTRUMENT I 32	NIGHT N 33	INSTRUMENT TRAINER 34	PILOT NON-MIL. AIRCRAFT OVER 400 H.P. 35	UNDER 400 H.P. 36
6	B-29	1		12:15														
24	C-54	1					7:25								3:00			
26	C-54	2					7:05								6:05			
COLUMN TOTALS		4		12:15			14:30								9:05			

CERTIFIED CORRECT:
George W Marquardt
GEORGE W. MARQUARDT,
Captain, Air Corps,
Asst. Operations Officer

	(42) TOTAL STUDENT PILOT TIME	(43) TOTAL FIRST PILOT TIME		(44) TOTAL PILOT TIME		
(37) THIS MONTH		14:30	0:00	26:45		
(38) PREVIOUS MONTHS THIS F. Y.		97:00	0:00	105:40		
(39) THIS FISCAL YEAR		111:30	0:00	132:25		
(40) PREVIOUS FISCAL YEARS	296:45	3726:55	46:20	3768:25		3:50
(41) TO DATE	296:45	3838:25	46:20	3900:50	3977:45	3:50

AIRCRAFT 19	NL 20	CARD NO. 1 21	22	23	24	25	26	CARD NO. 2 27	28	29	30	31	CARD NO. 3 32	33	34	35	36
B-29	1		12														
C-54	3					15											

C-54 14:30
B-29 12:15

WAR DEPARTMENT
AAF FORM NO. 5
APPROVED DEC. 7, 194

DIVIDUAL FLIGHT RECORD

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LAST MIDDLE
(3) RANK Colonel (4) AGE 1917
(5) PERS. CLASS 01 (6) BRANCH Air Corps
(8) ORGANIZATION ASSIGNED 20th 313th 509th
AIR FORCE COMMAND WING GROUP SQUADRON
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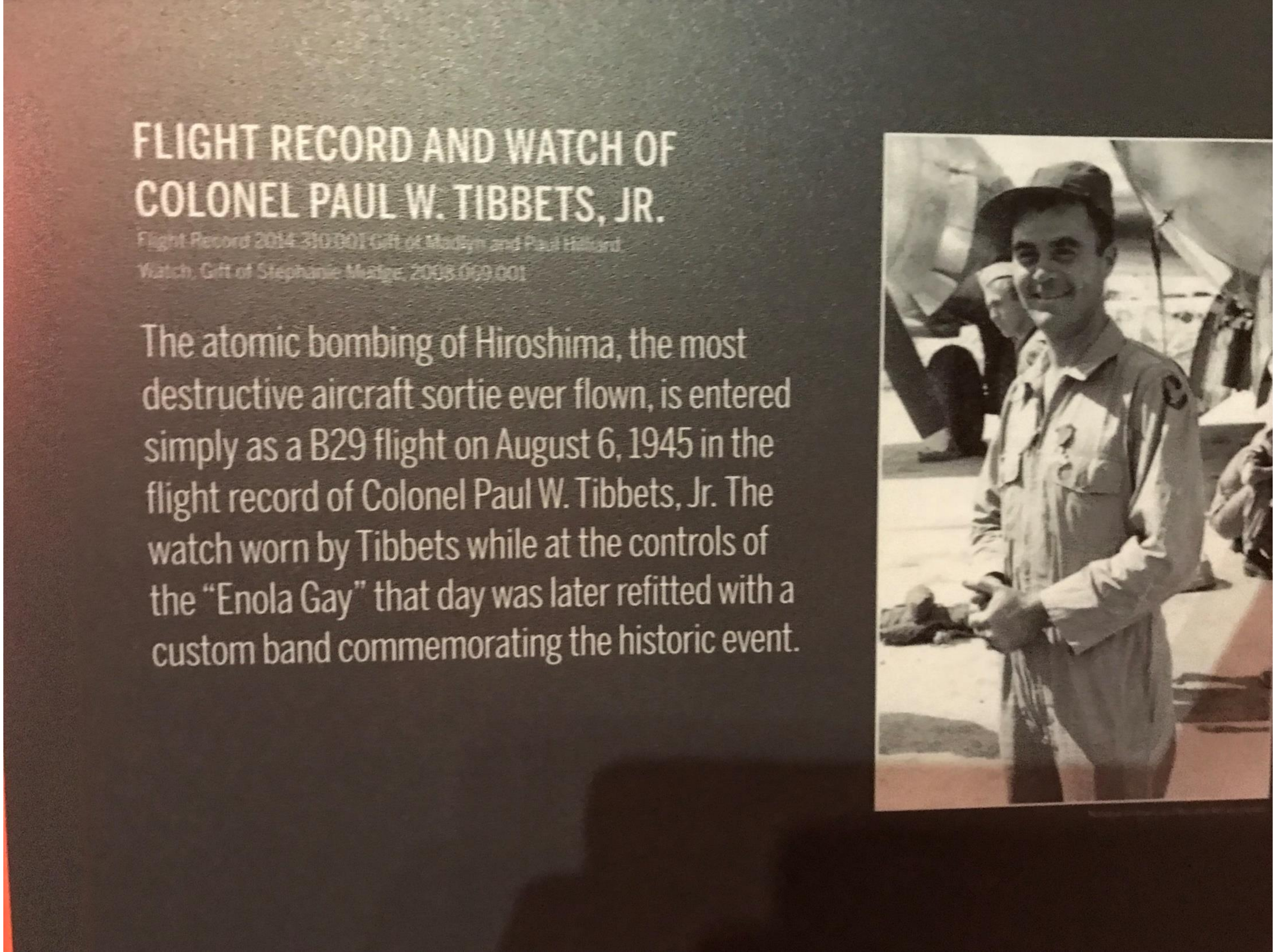
6	B-29	1		12:15														
24	C-54	1					7:25								3:00			
26	C-54	2					7:05								6:05			

Six degrees of separation….



FLIGHT RECORD AND WATCH OF COLONEL PAUL W. TIBBETS, JR.

Flight Record 2014.310.001 Gift of Madlyn and Paul Hilliard
Watch, Gift of Stephanie Mudge, 2008.069.001

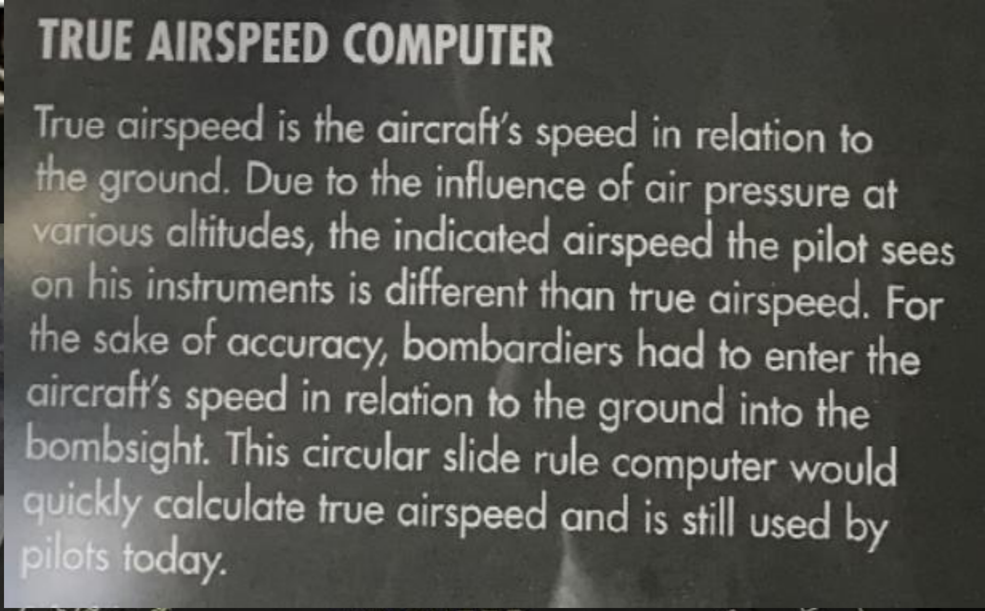
The atomic bombing of Hiroshima, the most destructive aircraft sortie ever flown, is entered simply as a B29 flight on August 6, 1945 in the flight record of Colonel Paul W. Tibbets, Jr. The watch worn by Tibbets while at the controls of the "Enola Gay" that day was later refitted with a custom band commemorating the historic event.

Oracle of Bacon: https://www.oracleofbacon.org/

1. SET PRESSURE ALTITUDE AGAINST CALIBRATED INDICATED AIRSPEED IN M. P. H. ON V_c.
2. SET INDICATOR ON AIR TEMPERATURE.
3. READ TRUE AIRSPEED IN M. P. H ON V_t SCALE UNDER INDICATOR.
4. USE NOTCHED SIDE OF INDICATOR TO MARK AIR-SPEED METER CALIBRATION ON BLANK ARC.

SPEC. NO. 27367-A
MFG. DWG. NO. D-36539
ORDER NO. (535) 42-18061-P

THE SILLCOCKS-MILLER COMPANY



TRUE AIRSPEED COMPUTER

True airspeed is the aircraft's speed in relation to the ground. Due to the influence of air pressure at various altitudes, the indicated airspeed the pilot sees on his instruments is different than true airspeed. For the sake of accuracy, bombardiers had to enter the aircraft's speed in relation to the ground into the bombsight. This circular slide rule computer would quickly calculate true airspeed and is still used by pilots today.

1. SET PRESSURE ALTITUDE AGAINST CALIBRATED INDICATED AIRSPEED IN M. P. H. ON V_c.
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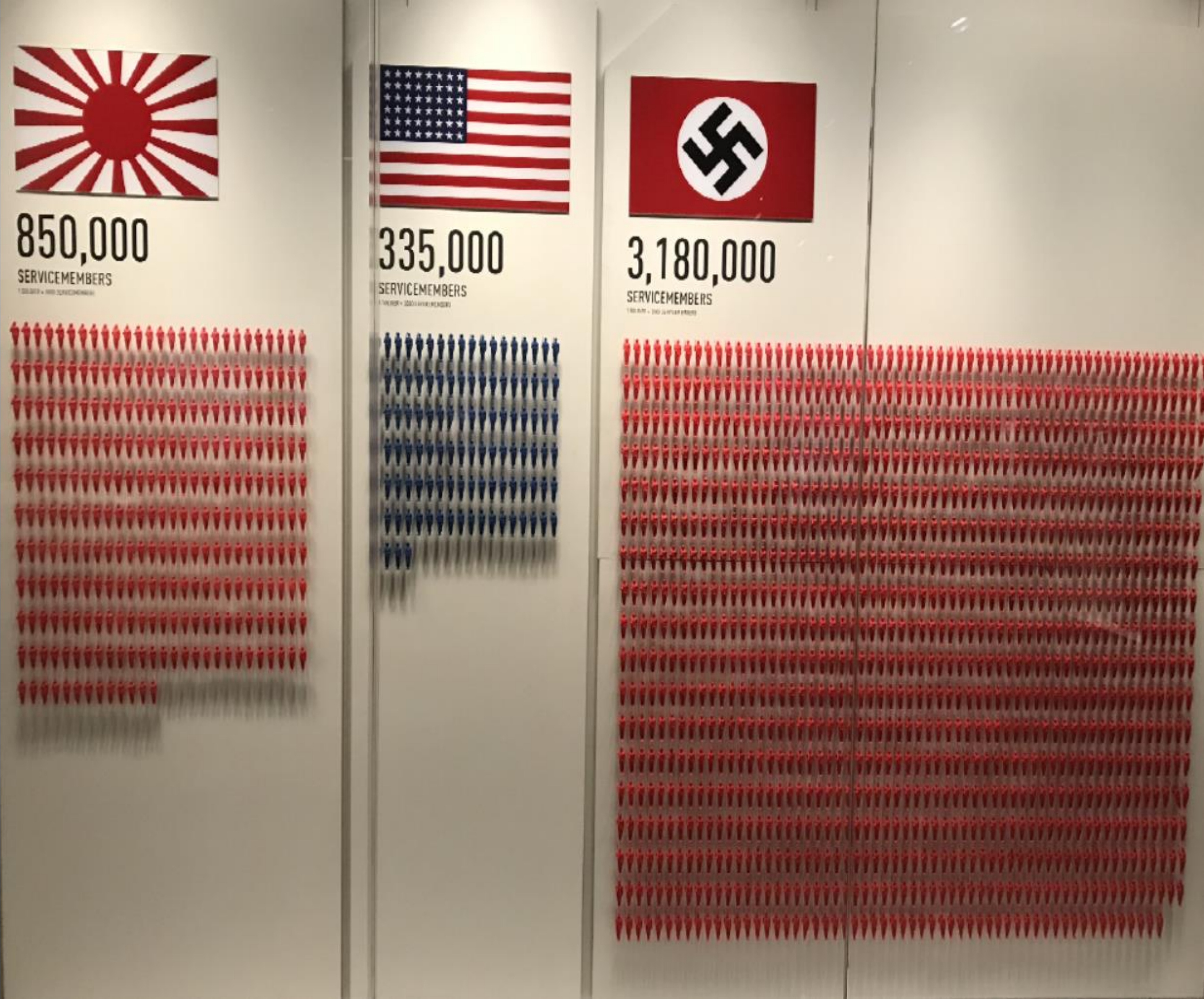
THE SILLCOCKS-MILLER COMPANY

MILITARY STRENGTH

When World War II broke out in 1939, the United States was not a great military power. The number of US service personnel was just 335,000, and the US Army was comparable in size to much smaller states like Bulgaria, Portugal, and Romania. Equipment was so scarce that only a tiny fraction of US troops had ever trained with modern weapons. By contrast, Germany had been rapidly rebuilding its military strength since 1933, and had more than three million men under arms. Japan, fighting an all-out war of conquest in China since 1937, had 850,000 men in the field. The world had become a dangerous place, and the US was dangerously unready.



What are good statistics?

What do they measure?

Do we care about that?

LIKE

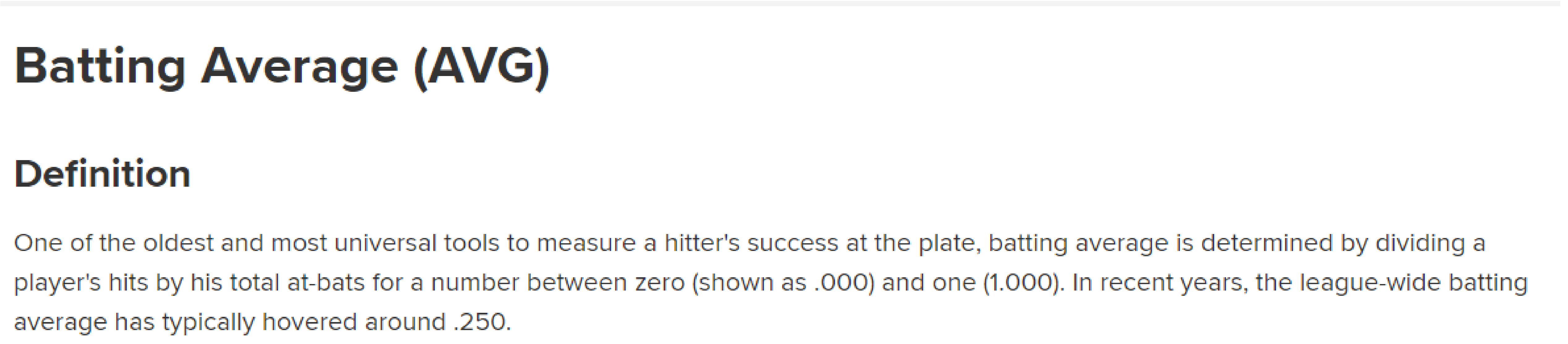
+/−

Goals/Assists per 90 minutes

DISLIKE

Per 36 (minutes, basketball)

Saves



Batting Average (AVG)

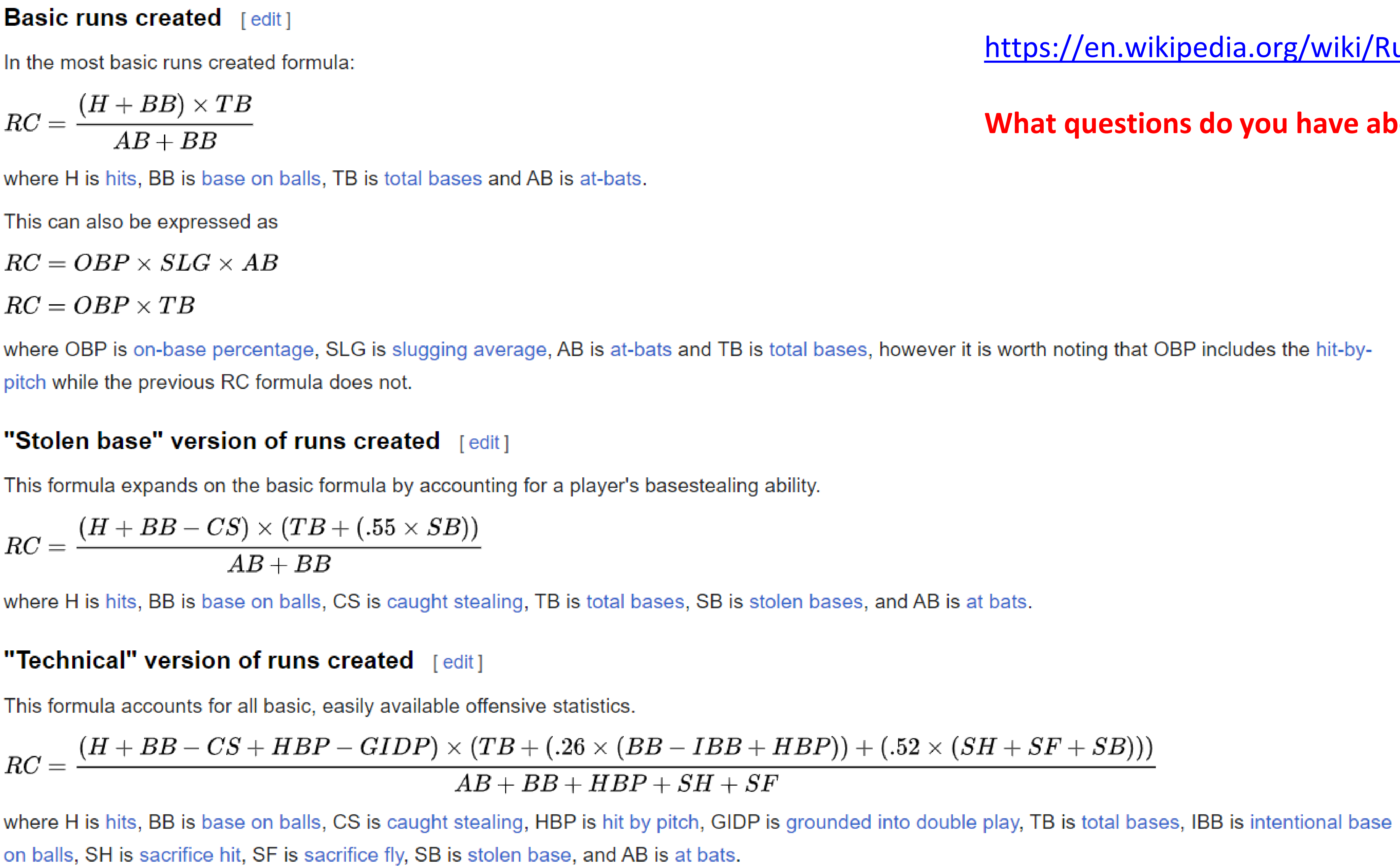
Definition

One of the oldest and most universal tools to measure a hitter's success at the plate, batting average is determined by dividing a player's hits by his total at-bats for a number between zero (shown as .000) and one (1.000). In recent years, the league-wide batting average has typically hovered around .250.

From: https://www.mlb.com/glossary/standard-stats/batting-average

What is good about this? What is bad? What do we need to know to compute it? How hard is the computation?

issue: "at-bats" vs "plate appearance"



Basic runs created [edit]

In the most basic runs created formula:

$$RC = \frac{(H + BB) \times TB}{AB + BB}$$

where H is hits, BB is base on balls, TB is total bases and AB is at-bats.

This can also be expressed as

$$RC = OBP \times SLG \times AB$$

$$RC = OBP \times TB$$

where OBP is on-base percentage, SLG is slugging average, AB is at-bats and TB is total bases, however it is worth noting that OBP includes the hit-by-pitch while the previous RC formula does not.

"Stolen base" version of runs created [edit]

This formula expands on the basic formula by accounting for a player's basestealing ability.

$$RC = \frac{(H + BB - CS) \times (TB + (.55 \times SB))}{AB + BB}$$

where H is hits, BB is base on balls, CS is caught stealing, TB is total bases, SB is stolen bases, and AB is at bats.

"Technical" version of runs created [edit]

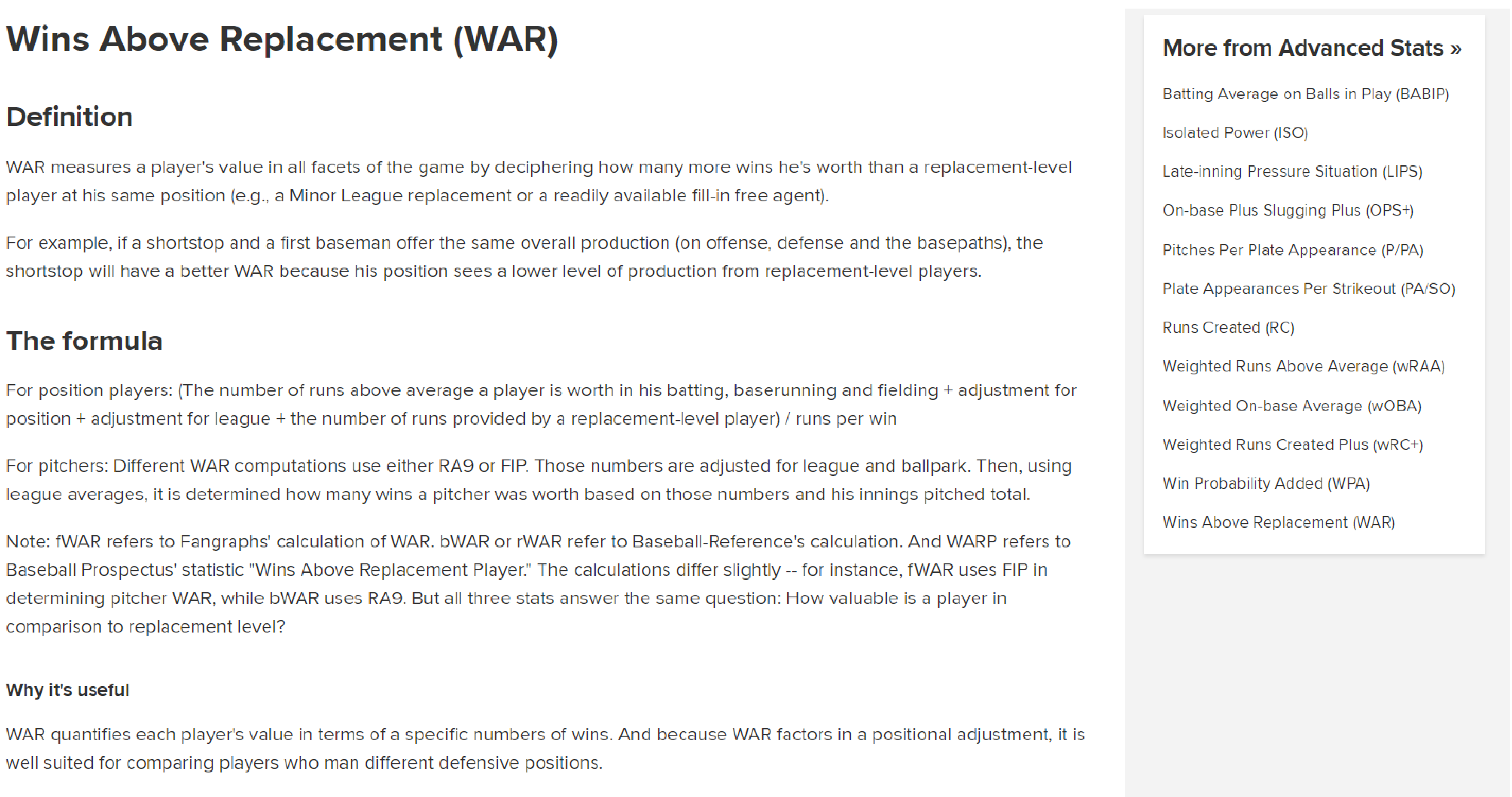
This formula accounts for all basic, easily available offensive statistics.

$$RC = \frac{(H + BB - CS + HBP - GIDP) \times (TB + (.26 \times (BB - IBB + HBP)) + (.52 \times (SH + SF + SB)))}{AB + BB + HBP + SH + SF}$$

where H is hits, BB is base on balls, CS is caught stealing, HBP is hit by pitch, GIDP is grounded into double play, TB is total bases, IBB is intentional base on balls, SH is sacrifice hit, SF is sacrifice fly, SB is stolen base, and AB is at bats.

https://en.wikipedia.org/wiki/Runs_created

What questions do you have about this?



Wins Above Replacement (WAR)

Definition

WAR measures a player's value in all facets of the game by deciphering how many more wins he's worth than a replacement-level player at his same position (e.g., a Minor League replacement or a readily available fill-in free agent).

For example, if a shortstop and a first baseman offer the same overall production (on offense, defense and the basepaths), the shortstop will have a better WAR because his position sees a lower level of production from replacement-level players.

The formula

For position players: (The number of runs above average a player is worth in his batting, baserunning and fielding + adjustment for position + adjustment for league + the number of runs provided by a replacement-level player) / runs per win

For pitchers: Different WAR computations use either RA9 or FIP. Those numbers are adjusted for league and ballpark. Then, using league averages, it is determined how many wins a pitcher was worth based on those numbers and his innings pitched total.

Note: fWAR refers to Fangraphs' calculation of WAR. bWAR or rWAR refer to Baseball-Reference's calculation. And WARP refers to Baseball Prospectus' statistic "Wins Above Replacement Player." The calculations differ slightly -- for instance, fWAR uses FIP in determining pitcher WAR, while bWAR uses RA9. But all three stats answer the same question: How valuable is a player in comparison to replacement level?

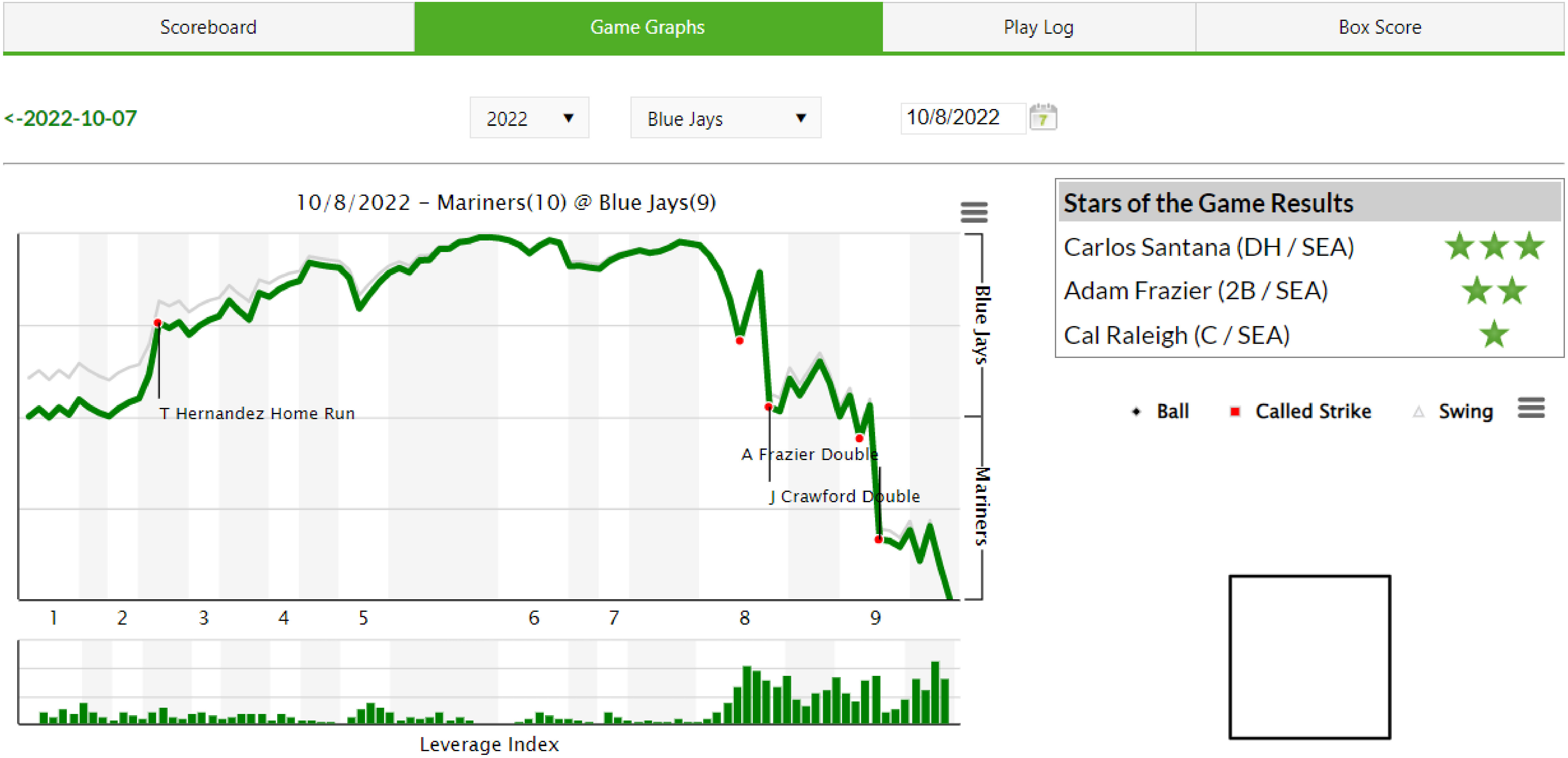
Why it's useful

WAR quantifies each player's value in terms of a specific numbers of wins. And because WAR factors in a positional adjustment, it is well suited for comparing players who man different defensive positions.

More from Advanced Stats »

- Batting Average on Balls in Play (BABIP)
- Isolated Power (ISO)
- Late-inning Pressure Situation (LIPS)
- On-base Plus Slugging Plus (OPS+)
- Pitches Per Plate Appearance (P/PA)
- Plate Appearances Per Strikeout (PA/SO)
- Runs Created (RC)
- Weighted Runs Above Average (wRAA)
- Weighted On-base Average (wOBA)
- Weighted Runs Created Plus (wRC+)
- Win Probability Added (WPA)
- Wins Above Replacement (WAR)

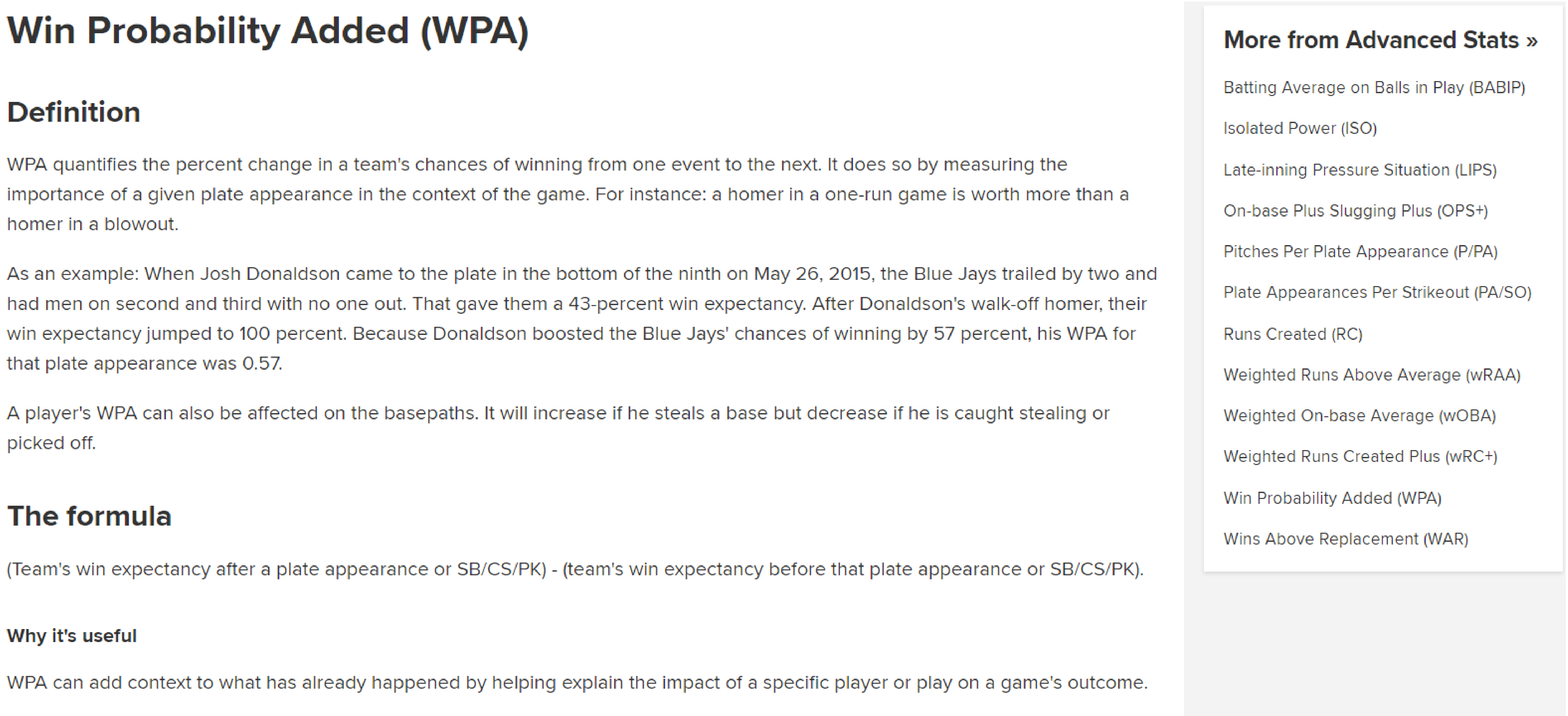
https://www.mlb.com/glossary/advanced-stats/wins-above-replacement **Same questions: is it good? How to compute?**



Stars of the Game Results

Carlos Santana (DH / SEA)	★★★
Adam Frazier (2B / SEA)	★★
Cal Raleigh (C / SEA)	★

https://www.fangraphs.com/wins.aspx?date=2022-10-08&team=Blue%20Jays&dh=0&season=2022



Win Probability Added (WPA)

Definition

WPA quantifies the percent change in a team's chances of winning from one event to the next. It does so by measuring the importance of a given plate appearance in the context of the game. For instance: a homer in a one-run game is worth more than a homer in a blowout.

As an example: When Josh Donaldson came to the plate in the bottom of the ninth on May 26, 2015, the Blue Jays trailed by two and had men on second and third with no one out. That gave them a 43-percent win expectancy. After Donaldson's walk-off homer, their win expectancy jumped to 100 percent. Because Donaldson boosted the Blue Jays' chances of winning by 57 percent, his WPA for that plate appearance was 0.57.

A player's WPA can also be affected on the basepaths. It will increase if he steals a base but decrease if he is caught stealing or picked off.

The formula

(Team's win expectancy after a plate appearance or SB/CS/PK) - (team's win expectancy before that plate appearance or SB/CS/PK).

Why it's useful

WPA can add context to what has already happened by helping explain the impact of a specific player or play on a game's outcome.

More from Advanced Stats »

- Batting Average on Balls in Play (BABIP)
- Isolated Power (ISO)
- Late-inning Pressure Situation (LIPS)
- On-base Plus Slugging Plus (OPS+)
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- Win Probability Added (WPA)
- Wins Above Replacement (WAR)

https://www.mlb.com/glossary/advanced-stats/win-probability-added

For a related stat in chess see: https://zwischenzug.substack.com/p/centipawns-suck

How should we determine course grades?

What are our objectives?

What questions should we ask?

HOMEWORK FOR NEXT CLASS:

Find a statistic from sports, email the class about it by 5pm on Tuesday, be prepared to talk about it and how you would answer our standard questions about it:

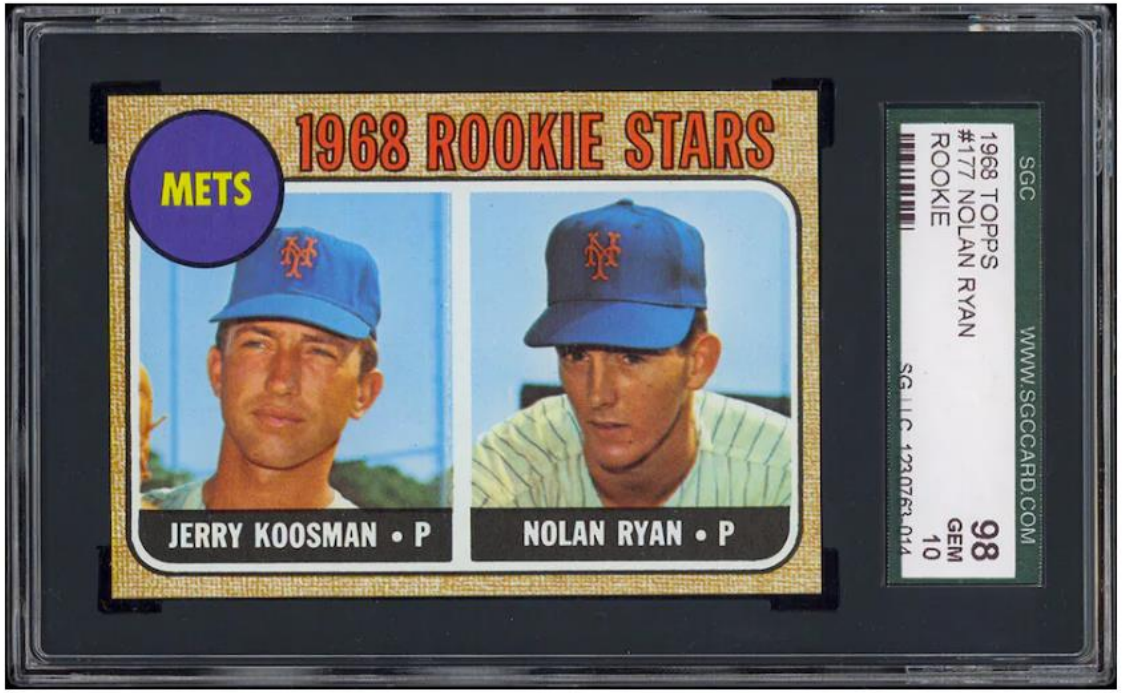
1) **Does it measure what we want to?**

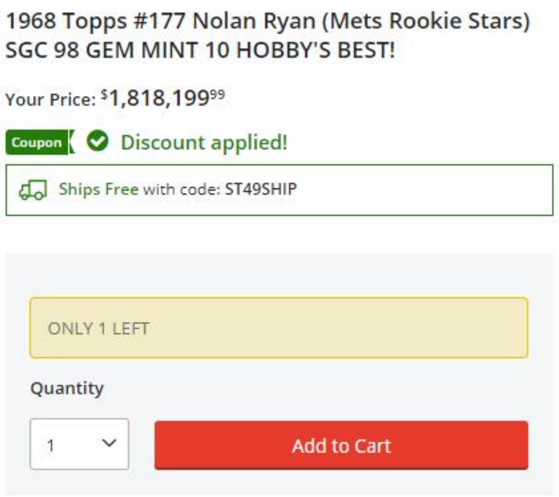
2) **How do we compute it?**

Math 344: Mathematics of Sports: Spring 2023: Lecture 03: Comparing Statistics: https://youtu.be/IZbxq8Grgkw

Plan for the day.

- Discuss various statistics claiming to measure same item (card problem).
- Discuss statistics to study.
- Discuss presentation possibilities.





1968 Topps #177 Nolan Ryan (Mets Rookie Stars)
SGC 98 GEM MINT 10 HOBBY'S BEST!

Your Price: $^{\$}$1,818,199^{99}

Coupon ✓ Discount applied!

Ships Free with code: ST49SHIP

ONLY 1 LEFT

Quantity

1

Challenge: Order the cards from most expensive to least expense (do not search the web), order is from price I paid online, what is the ***metric*** for judging who does best?

1959	Topps #180	Yogi Berra Player
1964	Topps #21	Yogi Berra Manager
1991	Upper Deck #44	Michael Jordan
1999	Upper Deck #3	Wayne Gretzky
2002	Pacific #258	Tom Brady
2011	Topps #240	Tom Brady
2012	Topps #440	Tom Brady
2018	Donruss #1	Lionel Messi
2019	Score #142	Tom Brady
2021	Donruss #1	Tom Brady

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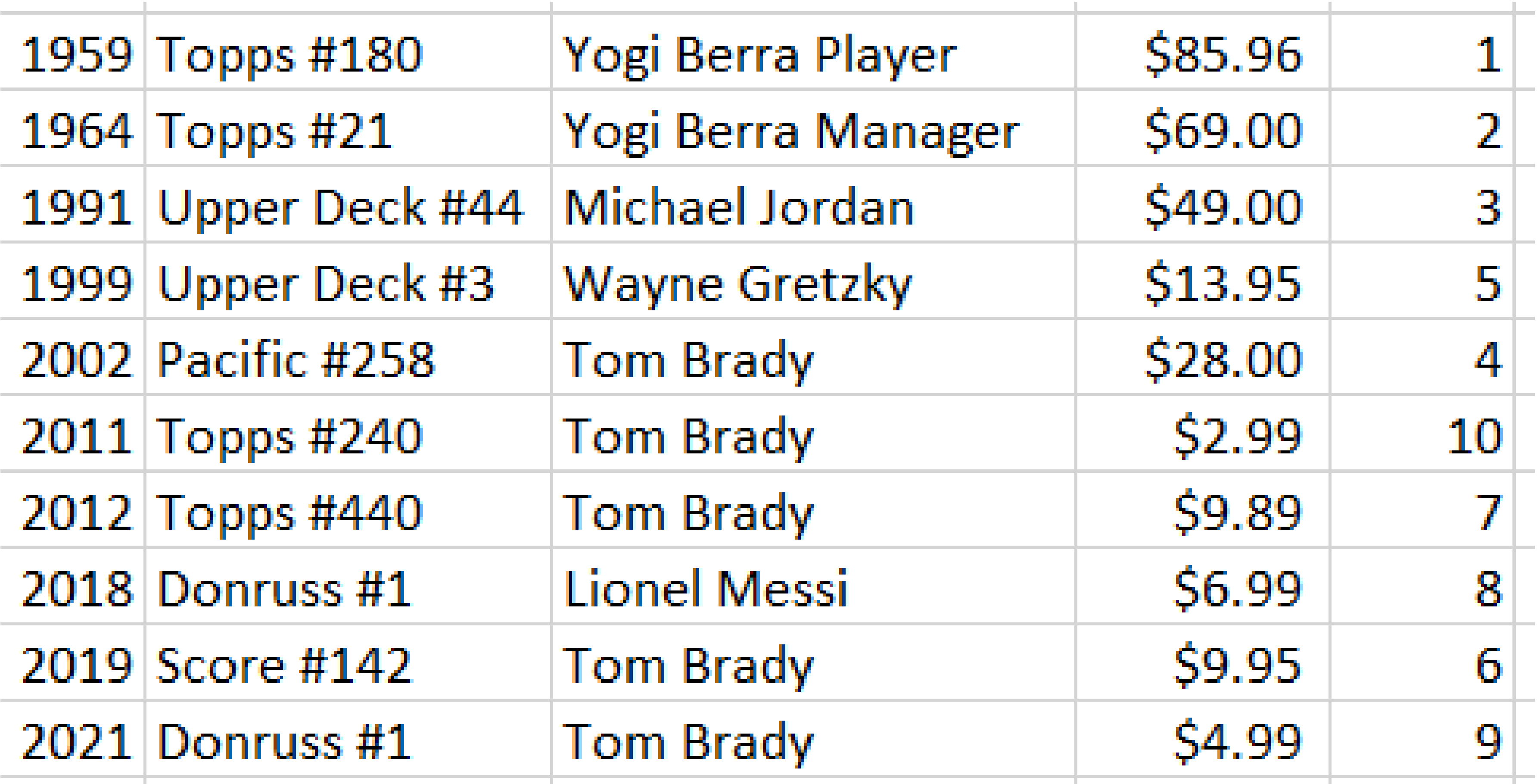
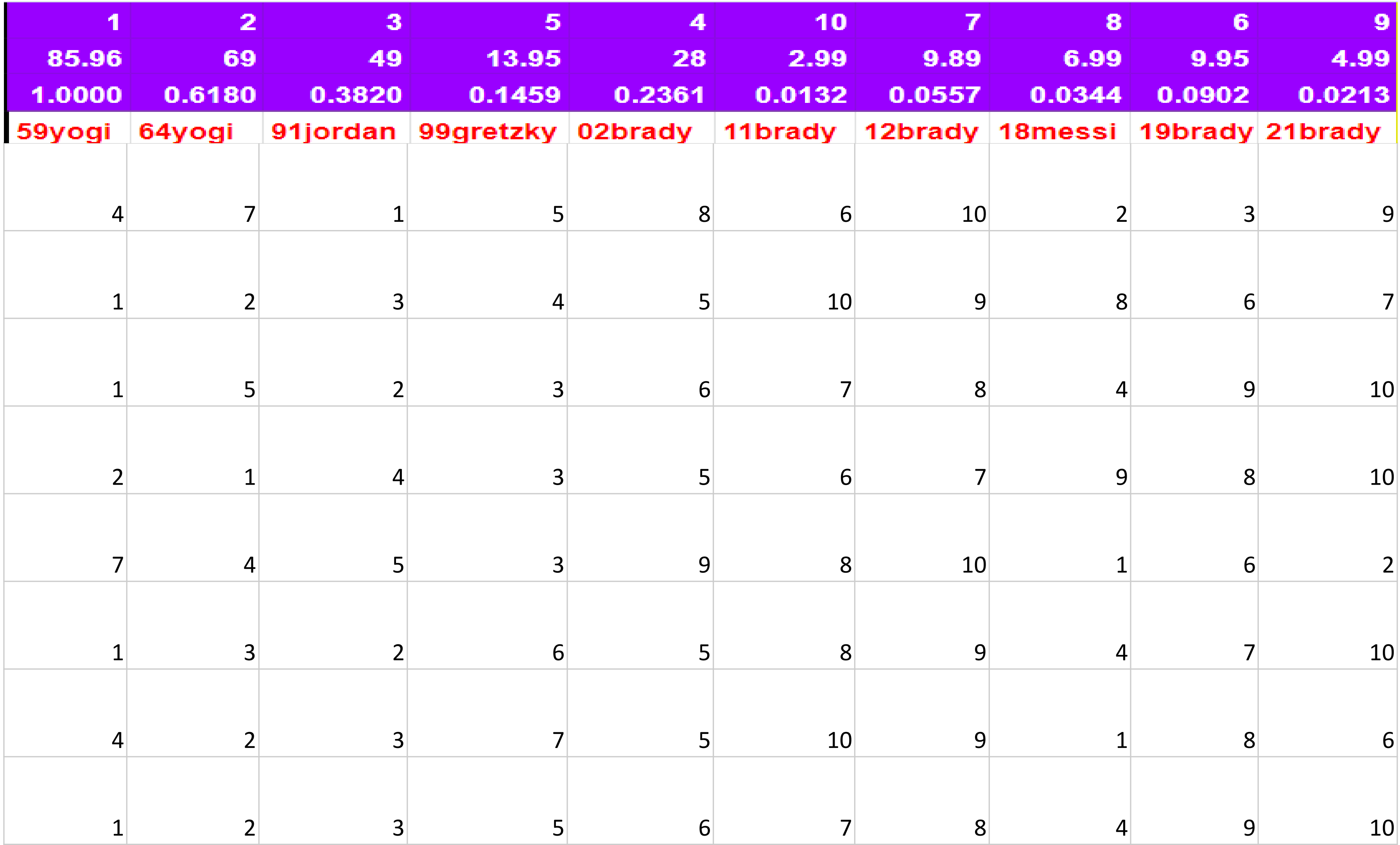


1959	Topps #180	Yogi Berra Player	$85.96	1
1964	Topps #21	Yogi Berra Manager	$69.00	2
1991	Upper Deck #44	Michael Jordan	$49.00	3
1999	Upper Deck #3	Wayne Gretzky	$13.95	5
2002	Pacific #258	Tom Brady	$28.00	4
2011	Topps #240	Tom Brady	$2.99	10
2012	Topps #440	Tom Brady	$9.89	7
2018	Donruss #1	Lionel Messi	$6.99	8
2019	Score #142	Tom Brady	$9.95	6
2021	Donruss #1	Tom Brady	$4.99	9

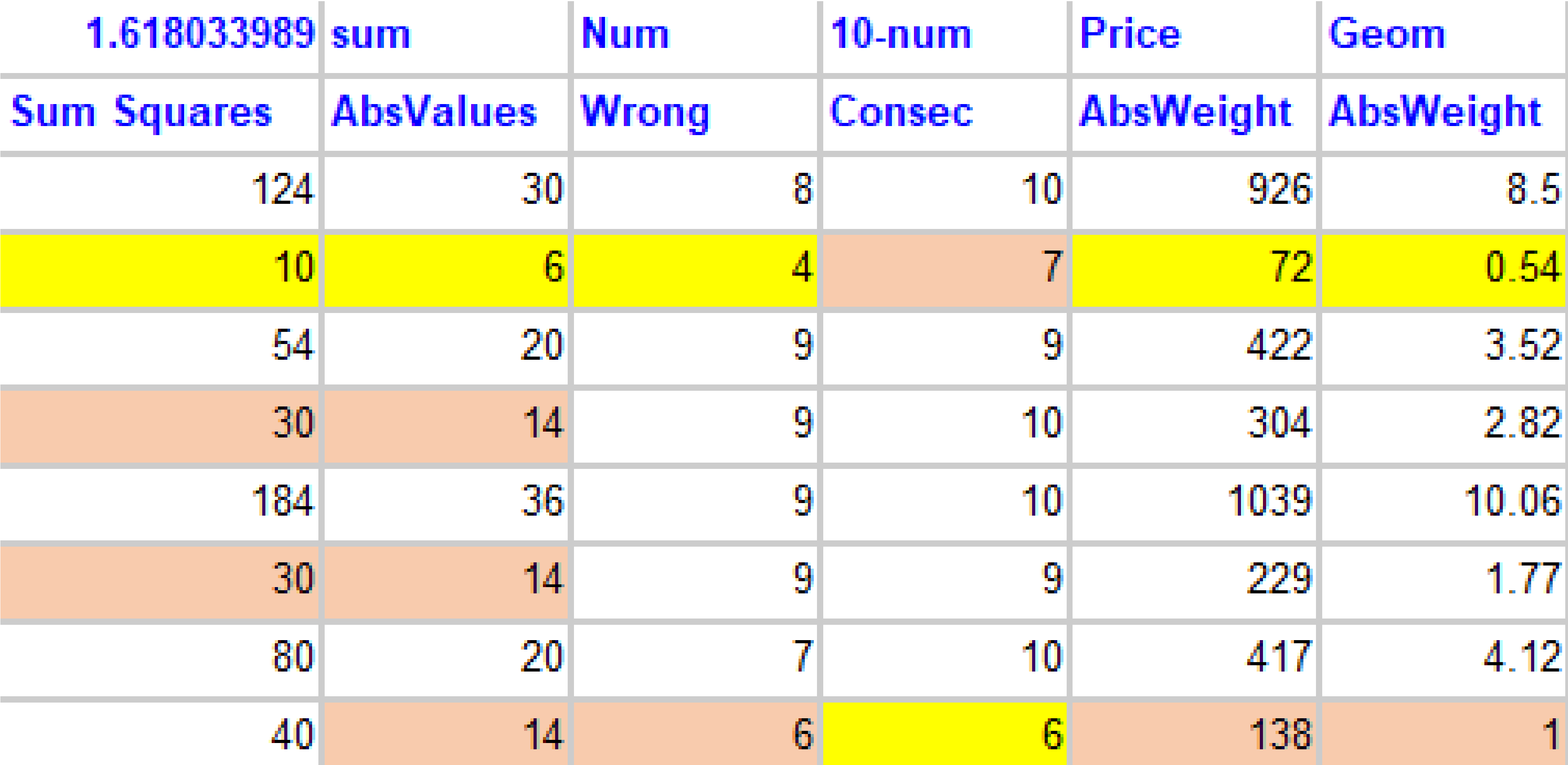
HOW SHOULD WE SCORE HOW WELL PEOPLE DO? Very different than who wins a ‘game’…

Challenge: Order the cards from most expensive to least expense (do not search the web), order is from price I paid online, what is the ***metric*** for judging who does best?



1	2	3	5	4	10	7	8	6	9
85.96	69	49	13.95	28	2.99	9.89	6.99	9.95	4.99
1.0000	0.6180	0.3820	0.1459	0.2361	0.0132	0.0557	0.0344	0.0902	0.0213
59yogi	64yogi	91jordan	99gretzky	02brady	11brady	12brady	18messi	19brady	21brady
4	7	1	5	8	6	10	2	3	9
1	2	3	4	5	10	9	8	6	7
1	5	2	3	6	7	8	4	9	10
2	1	4	3	5	6	7	9	8	10
7	4	5	3	9	8	10	1	6	2
1	3	2	6	5	8	9	4	7	10
4	2	3	7	5	10	9	1	8	6
1	2	3	5	6	7	8	4	9	10

Challenge: Order the cards from most expensive to least expense (do not search the web), order is from price I paid online, what is the ***metric*** for judging who does best?



1.618033989	sum	Num	10-num	Price	Geom
Sum Squares	AbsValues	Wrong	Consec	AbsWeight	AbsWeight
124	30	8	10	926	8.5
10	6	4	7	72	0.54
54	20	9	9	422	3.52
30	14	9	10	304	2.82
184	36	9	10	1039	10.06
30	14	9	9	229	1.77
80	20	7	10	417	4.12
40	14	6	6	138	1

Statistics brought up from the class

- **Clutch time: NBA.** Records NBA metrics when the following two conditions are met:
  (1) it is during the final 5 minutes of a game and
  (2) the score differential is 5 or lower.

This is supposed to help see how valuable players and strategies are when there is the least room for error.

- **Expected Goals (xG) and Expected Assists (xA): Soccer.** xG and xA factor in the probability that a shot will result in a goal based on the characteristics of that shot and the events leading up to it, considering the location of the shooter, the body part used to shoot, the type of pass, and the type of attack. Every shot falls between an xG of 0 (definite miss) and 1 (definite goal), using a large sample size of shots with similar characteristics to determine the probability of a goal. In this way, differing xG models often result in different xG values for the same shot.

Statistics brought up from the class

- **CORSI: Hockey.** The most common CORSI statistics for teams are CORSI For (CF), CORSI against (CA), and CORSI For % (CF %). CF measures a teams shot attempts for at even strength (shot attempts includes all shots on goal, shot attempts that missed the net, and shots that were blocked). CA measures a teams shot attempts against at even strength. CF % is equal to CF / (CF + CA). CF% rel = CFon% - CFoff% is for individual players and measures the CF% of the team when a given player is on the ice compared to when they are off the ice. CF% rel is intended to identify impactful players, and Corsi analytics attempt to indicate whether a team (or player) spends more time in the offensive or defensive zone.

- **Putting statistics: Golf.** If shoot exactly par, use putter around 2x per hole, or roughly 50% of the time. Interested in putts per hole (total putts in the round / 18) and 1st putt length. A lower putts per hole typically results in lower overall score. 1st putt length and putts per hole are related: a smaller average 1st putt length results in a decreased putts per hole. Show how well a golfer performed approaching green and on the green putting. Also consider statistics such as the number of greens, as if golfer is chipping from just off the green would likely result in having a shorter 1st putt length.

Statistics brought up from the class

- **Hitting Statistic: Volleyball.** It is calculated as the number of kills subtracted by the number of errors. This number is then divided by the total number of hitting attempts. Of course, there are issues here I may talk about in class. Furthermore, receiving error simply adds the number of "shanked" passes where a second person cannot touch the volleyball. For practice, we stat serve receive with more detail than statisticians can to get a better understanding of how well we pass off a serve and can set up the next play.

- **Time of Possession: Soccer.** Time of possession is an interesting statistic for soccer; however, it is calculated in different ways depending on the level (for higher-level games, the number of passes completed is counted and taken as a fraction of the total passes in a game because the time of possession and passes completed are correlated).

Statistics brought up from the class

- **Strokes Gained: Golf.** Measure of a player’s performance relative to the rest of the playing field. One cool aspect of this metric is that it can been measured under a variety of circumstances. For example, Strokes Gained: Putting and Strokes Gained: Off-the-Tee are sub-categories of this metric. Additionally, these player statistics need not be limited to a single tournament. We actually can evaluate an entire season’s worth of shots, comparing player performance across the entire year. For instance, Player A might have an average Strokes Gained: Putting of +0.5, meaning they are a better-than-average putter when compared to the rest of the PGA tour. Conversely, we can look at individual holes in a tournament and see which players performed best/worst over the week on that hole. Without getting too much into the math of the calculations, the general idea behind calculating Strokes Gained is to compare a single player’s performance to the tour (or specific tournament average) given a certain shot. For example, if Player A has 170 yards—from the rough—remaining to the hole, and that week, the average remaining shots needed from that scenario is 2.5, if Player A takes only 2 shots, they gained 0.5 strokes on the field. To evaluate Strokes Gained: Putting, you look at details regarding those 2 shots taken. Did Player A hit the ball to 2 feet from the pin (meaning they hit a great approach shot), or did they hit to 40 feet (and sunk a massive putt)?

Statistics brought up from the class

- **Average Centipawn Loss: Chess.** Objectively, no such thing as a "good" move that improves the player's position: any move maintains an existing advantage or concedes advantage; the ACL aims to measure how much advantage a player concedes per move, on average. Strong players tend to have a low ACL, perfect play is a 0. Based on related metric, centipawn advantage (CP): takes a position as an input, aims to tell who's winning & by how much. The centipawn loss of a move is difference in CP before and after move (i.e., amount of CP lost by that move), and the ACL averages over all moves. Many factors determine who has a winning position: material count, piece activity/coordination, king safety... Material count is the most "concrete" of them, so the CP attempts to convert all forms of advantage in terms of this. A CP evaluation of -500 might be interpreted as: "Black is up by the equivalent of 5 pawns". Doesn't mean Black up by 5 pawns' worth of material: a position where Black is down material but has a devastating attack could result in such an evaluation. No explicit formula to calculate the CP from any given position (and thus the ACL); comes from computer analysis, not necessarily meant for human interpretation. Engines of different strengths/versions can give different evaluations of same position, making the statistic somewhat arbitrary in absolute terms. Top grandmasters have an ACL of around 10 to 20. Average online chess player might have an ACL or 50-100, or even more. Beginners may have an ACL in the hundreds.

Statistics brought up from the class

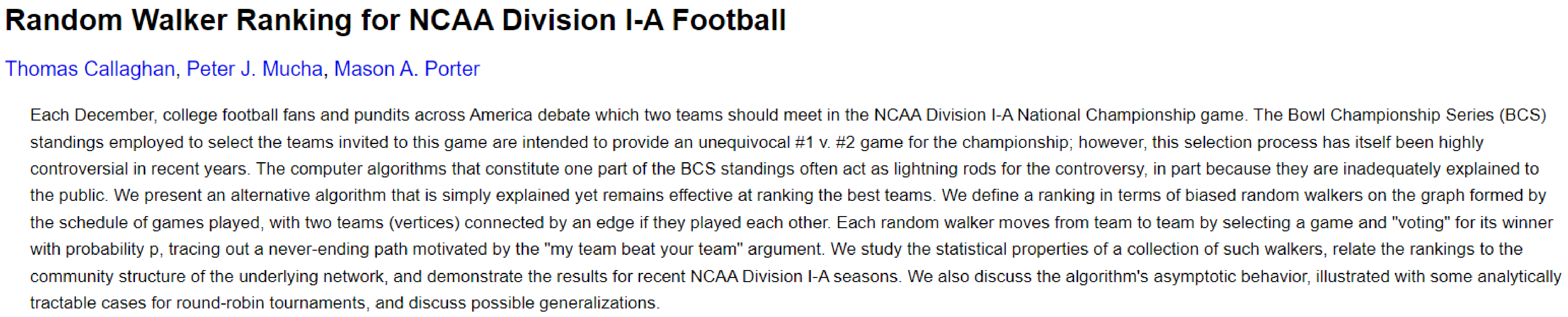
- **Quarterback Passing Rating: Football.** Measure QB using completions, yards, TDs, interceptions, on a per attempt basis. Takes these numbers, shifting/multiplying by constants, and summing them before again multiplying by a constant. Widely used since ‘70s, several problems with this metric. Doesn’t incorporate external factors: weather, pressure, and difficulty of the throws (i.e. quality of teammates and opponents). 20/22 with 300 yards, 3 TDs and no interceptions is more impressive against a top than weak defense so examining a single passer rating does not tell the full story. Fails to incorporate the negative effect of sacks on a team’s overall offensive performance. QB can be at fault for sacks due to lack of mobility or indecision, so these should be included. Finally, a QB can achieve a perfect passer rating of 158.3 without having perfect stats, as each category has a fixed maximum and minimum number. For example, a QB with a completion percentage of 77.5% can achieve a perfect passer rating, which seems counterintuitive because a QB with identical stats other than a 90% completion rate should have a higher rating. New metric designed by ESPN called Total Quarterback Rating (QBR) has gained popularity. QBR measures how a QB impacts its team’s expected points added (EPA) on each play. Then, the QB’s performance is adjusted to reflect opponent quality, pressure of the situation, difficulty of the play, and other situational factors. The adjusted EPA is normalized on a 0-100 scale.

Statistics brought up from the class

- **Free Throw Percentages: Basketball.** Interesting based on how it can drive the hype of specific moments. With referee calls and fouls always being a controversial aspect of basketball, the free throws create a moment of anticipation outside of the clock that can both get a team more points and help create valuable rewatchable content. Measuring free throws made versus total attempted will give a better indicator of how successful their throws were at the end of the game/season and could be used as a metric for the future. Unlike some other stats, the basis of being less impacted by the other team can give a better lens into player ability. However, being able to differentiate individual player FTP vs team FTP can tell a different story about how successful the team's free throws will be, especially if one player has a higher FTP which affects the average.

Possible Presentation Topics

Ranking College Football: https://arxiv.org/pdf/physics/0310148

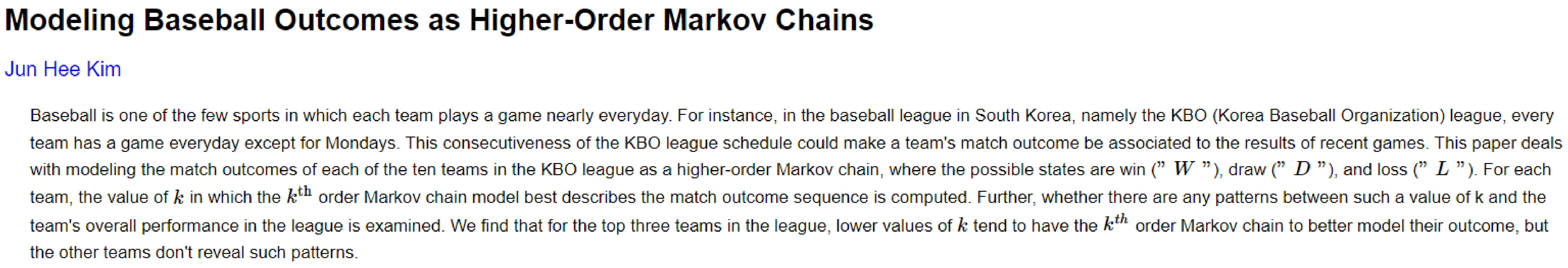


Random Walker Ranking for NCAA Division I-A Football

Thomas Callaghan, Peter J. Mucha, Mason A. Porter

Each December, college football fans and pundits across America debate which two teams should meet in the NCAA Division I-A National Championship game. The Bowl Championship Series (BCS) standings employed to select the teams invited to this game are intended to provide an unequivocal #1 v. #2 game for the championship; however, this selection process has itself been highly controversial in recent years. The computer algorithms that constitute one part of the BCS standings often act as lightning rods for the controversy, in part because they are inadequately explained to the public. We present an alternative algorithm that is simply explained yet remains effective at ranking the best teams. We define a ranking in terms of biased random walkers on the graph formed by the schedule of games played, with two teams (vertices) connected by an edge if they played each other. Each random walker moves from team to team by selecting a game and "voting" for its winner with probability p, tracing out a never-ending path motivated by the "my team beat your team" argument. We study the statistical properties of a collection of such walkers, relate the rankings to the community structure of the underlying network, and demonstrate the results for recent NCAA Division I-A seasons. We also discuss the algorithm's asymptotic behavior, illustrated with some analytically tractable cases for round-robin tournaments, and discuss possible generalizations.

Modeling Baseball Games: https://arxiv.org/pdf/1811.07259



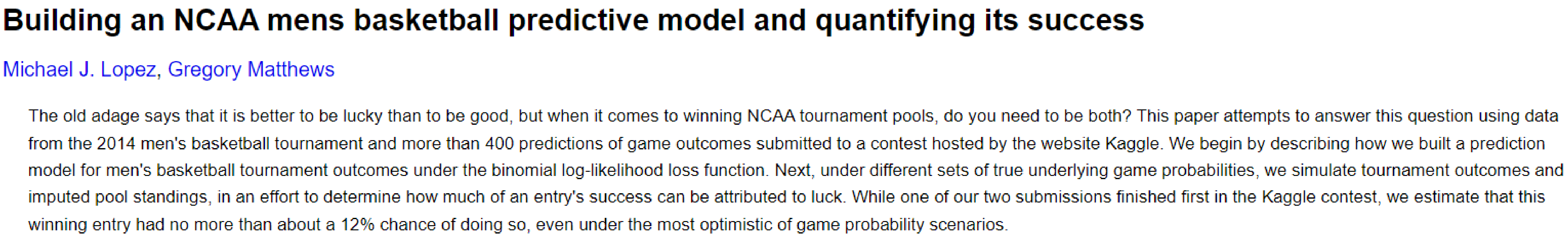
Modeling Baseball Outcomes as Higher-Order Markov Chains

Jun Hee Kim

Baseball is one of the few sports in which each team plays a game nearly everyday. For instance, in the baseball league in South Korea, namely the KBO (Korea Baseball Organization) league, every team has a game everyday except for Mondays. This consecutiveness of the KBO league schedule could make a team's match outcome be associated to the results of recent games. This paper deals with modeling the match outcomes of each of the ten teams in the KBO league as a higher-order Markov chain, where the possible states are win (” W ”), draw (” D ”), and loss (” L ”). For each team, the value of k in which the k^{th} order Markov chain model best describes the match outcome sequence is computed. Further, whether there are any patterns between such a value of k and the team's overall performance in the league is examined. We find that for the top three teams in the league, lower values of k tend to have the k^{th} order Markov chain to better model their outcome, but the other teams don't reveal such patterns.

Possible Presentation Topics

NCAA Tourney (Mens Hoops): https://arxiv.org/pdf/1412.0248



Building an NCAA mens basketball predictive model and quantifying its success

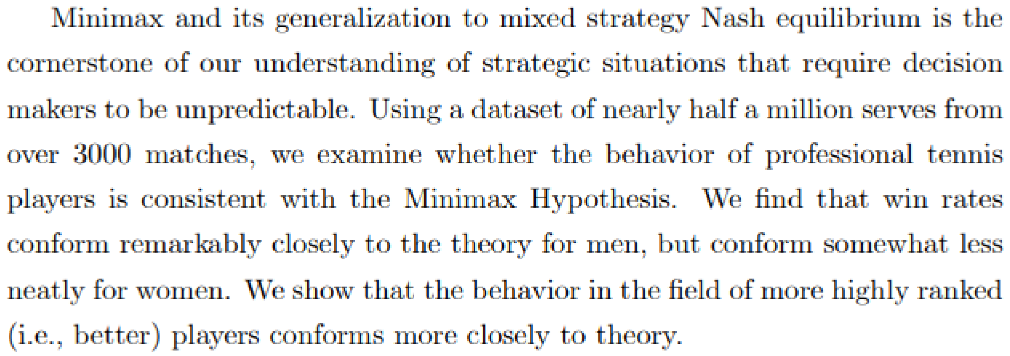
Michael J. Lopez, Gregory Matthews

The old adage says that it is better to be lucky than to be good, but when it comes to winning NCAA tournament pools, do you need to be both? This paper attempts to answer this question using data from the 2014 men's basketball tournament and more than 400 predictions of game outcomes submitted to a contest hosted by the website Kaggle. We begin by describing how we built a prediction model for men's basketball tournament outcomes under the binomial log-likelihood loss function. Next, under different sets of true underlying game probabilities, we simulate tournament outcomes and imputed pool standings, in an effort to determine how much of an entry's success can be attributed to luck. While one of our two submissions finished first in the Kaggle contest, we estimate that this winning entry had no more than about a 12% chance of doing so, even under the most optimistic of game probability scenarios.

Monte Carlo Tennis: https://cpb-us-e1.wpmucdn.com/sites.usc.edu/dist/5/476/files/2020/04/JQA2009.pdf

Game Theory in Tennis: http://www.johnwooders.com/papers/NashAtWimbledon.pdf

Nash at Wimbledon: Evidence from Half a Million Serves∗
Romain Gauriot ,Lionel Page, John Wooders



Minimax and its generalization to mixed strategy Nash equilibrium is the cornerstone of our understanding of strategic situations that require decision makers to be unpredictable. Using a dataset of nearly half a million serves from over 3000 matches, we examine whether the behavior of professional tennis players is consistent with the Minimax Hypothesis. We find that win rates conform remarkably closely to the theory for men, but conform somewhat less neatly for women. We show that the behavior in the field of more highly ranked (i.e., better) players conforms more closely to theory.

Math 344: Mathematics of Sports: Spring 2023: Lecture 04: GOAT of all GOATs: https://youtu.be/aaPCnWG0zOg

Plan for the day.

- Proving the Geometric Series Formula from a game of hoops.
- Who is the GOAT of all GOATs?
- Issues with that question.
- What can we answer?
- Paper/conference presentation possibilities….

The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

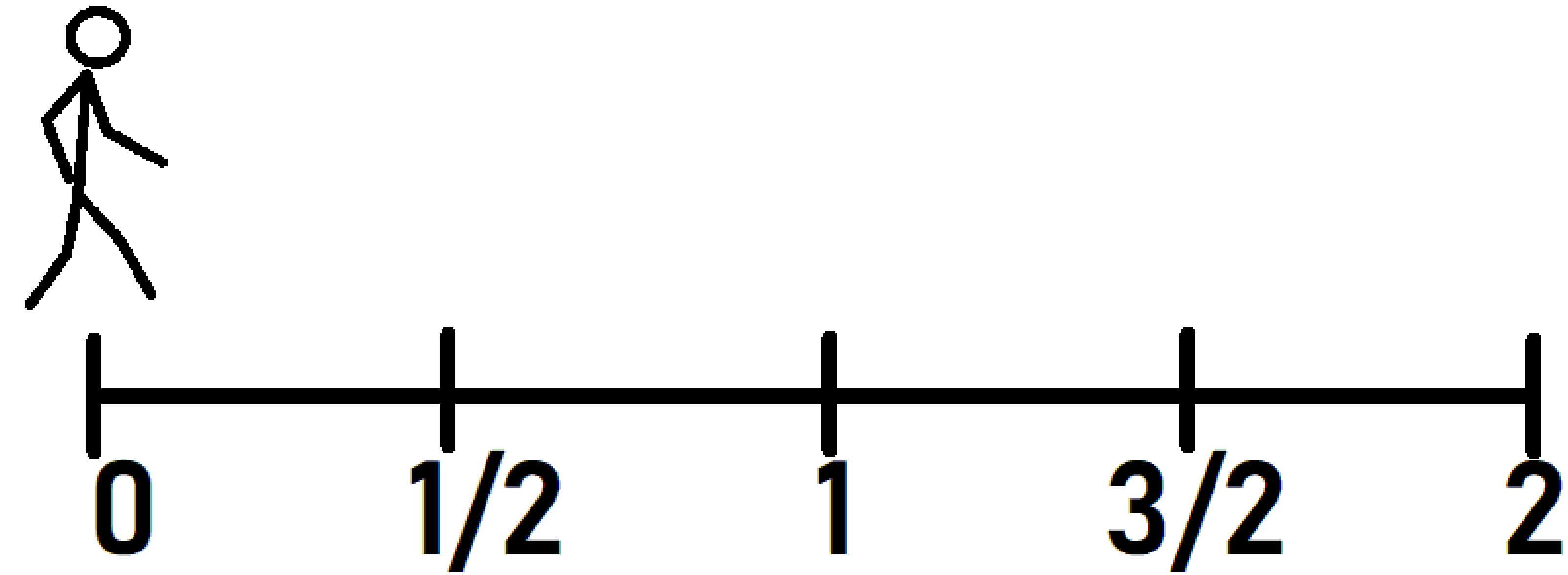
If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

This is often proved by first computing the finite sum, up to r^n, and taking a limit. Note since $|r| < 1$ that each term r^n gets small fast…..

The Geometric Series Converges if |r| < 1

$$1 + r + r2^{2}_{4} + {}^{3}_{4}r3^{4}_{9} + r4 + \cdots = \frac{1}{1-r}.$$

Why does this converge? Take r = ½. We then have 1 + ½ + ¼ + …. = $\frac{1}{1-\frac{1}{2}}$ = 2, and we can view this as we start at 0, and each step covers half the distance to 2. We thus never reach it in finitely many steps, but we cover half the ground each tir



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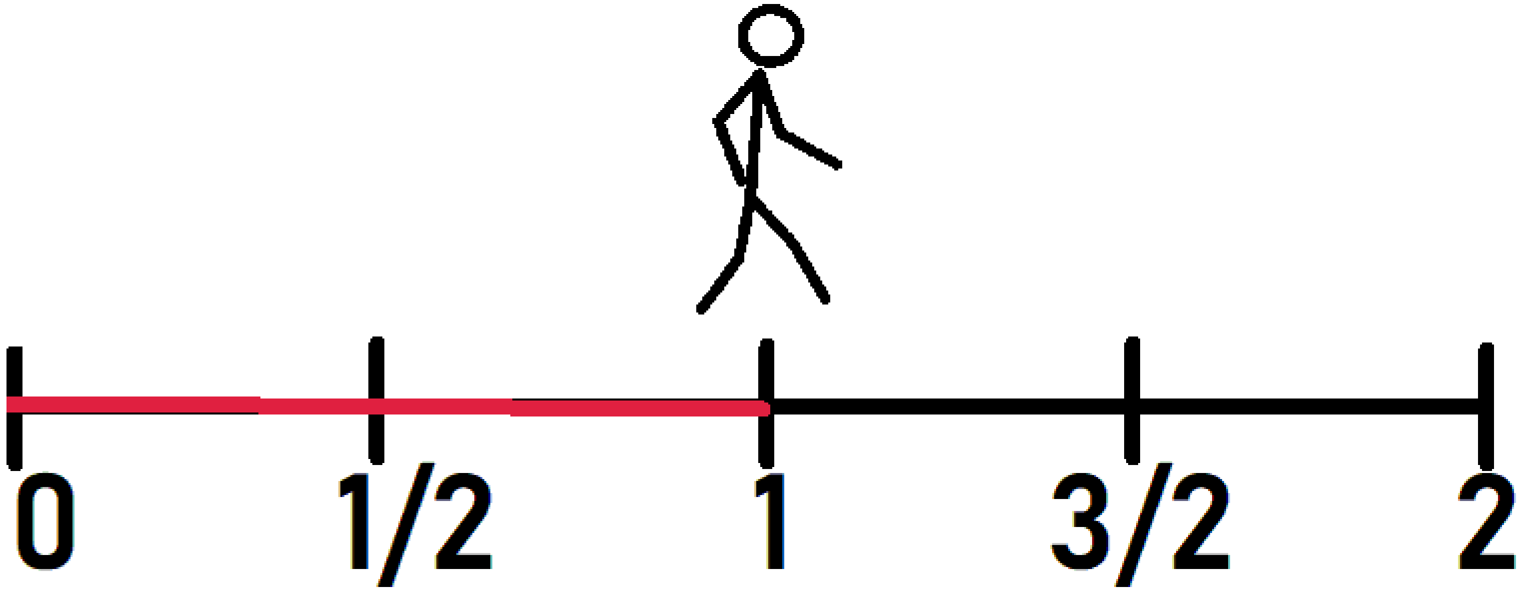

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The Geometric Series Formula

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Lemma: If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \dots + r^n = \frac{1 - r^{n+1}}{1-r}$.

Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \dots + r^n$

Then $r\,S_n = r + r^2 + r^3 + r^4 + \dots + r^n + r^{n+1}$

What should we do now?

The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

Lemma: If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$.

Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \ldots + r^n$

Then $r S_n = r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

Subtract: $S_n - r S_n = 1 - r^{n+1}$,

So $(1-r) S_n = 1 - r^{n+1}$, or S_n

The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

Lemma: If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}$.

Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \dots + r^n$

Then $r S_n = r + r^2 + r^3 + r^4 + \dots + r^n + r^{n+1}$

Subtract: $S_n - r S_n = 1 - r^{n+1}$,

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Subtract: $S_n - r S_n = 1 - r^{n+1}$,

So $(1-r) S_n = 1 - r^{n+1}$, or $S_n = \frac{1 - r^{n+1}}{1-r}$.

If we let n go to infinity, we see r^{n+1} goes to 0, so we get the infinite sum is $\frac{1}{1-r}$.

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

We will prove the Geometric Series Formula just by studying this basketball game!

Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p.
- **Magic** always gets basket with probability q.

Let x be the probability **Bird** wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.

Solving the Hoop Game

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Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1 - p)(1 - q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q) \cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q) \cdot (1-p)(1-q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

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- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
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- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q) \cdot p$.
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- **Bird** wins on nth shot:
  $(1-p)(1-q) \cdot (1-p)(1-q) \cdots (1-p)(1-q) \cdot p$.

Let $r = (1-p)(1-q)$. Then

$$\begin{aligned} x &= \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) \\ &= p + rp + r^2p + r^3p + \cdots \\ &= p\left(1 + r + r^2 + r^3 + \cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p +$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p + (1-p)(1-q) * ???$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1 - p)(1 - q)x$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p + (1 - p)(1 - q)x = p + rx.$$

Solving the Hoop Game: The Power of Perspective

Showed

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1 - p)(1 - q)x = p + rx.$$

Thus

$$(1 - r)x \ = \ p \quad \text{or} \quad x \ = \ \frac{p}{1 - r}.$$

Solving the Hoop Game: The Power of Perspective

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$$x = \mathrm{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \mathrm{Prob}(\textbf{Bird}\text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \;=\; p \quad \text{or} \quad x \;=\; \frac{p}{1-r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots \;=\; \frac{1}{1-r}.$$

Advanced Geometric Series Comments

Always carefully look at what you did, and be explicit on what you proved.

The geometric series formula is:

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

We proved this when $r = (1-p)(1-q)$, where p and q are the probabilities of making a basket for Bird and Magic. What are the ranges for p and q? We have what range of p and q?

Advanced Geometric Series Comments

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We proved this when $r = (1-p)(1-q)$, where p and q are the probabilities of making a basket for Bird and Magic. What are the ranges for p and q? We have $0 \leq p, q \leq 1$ BUT we cannot have $p=q=0$, or the game never ends. Thus we only proved the Geometric Series Formula for what range of r?

Advanced Geometric Series Comments

Always carefully look at what you did, and be explicit on what you proved.

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If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

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Lessons from Hoop Problem

- ◇ Power of Perspective: Memoryless process.
- ◇ Can circumvent algebra with deeper understanding! (Hard)
- ◇ Depth of a problem not always what expect.
- ◇ Importance of knowing more than the minimum: connections.
- ◇ Math is fun!

Classifying GOATs (like Brady, Russell and Ruth) by Measuring Their Tails.

Rick Cleary, Babson College
Steve Miller, Williams College

rcleary@babson.edu, sjm1@williams.edu

MathFest: August 5, 2021

BIG PICTURE SPORTS QUESTIONS...

-Who is the GOAT (Greatest Of All Time) in a particular sport?

-Who is GOAT of GOATs across sports?

We recognize that this is not a well posed question … but fans and media try to answer it. To do so they make mathematical and statistical arguments that lead them to particular metrics … sometimes without even realizing it!

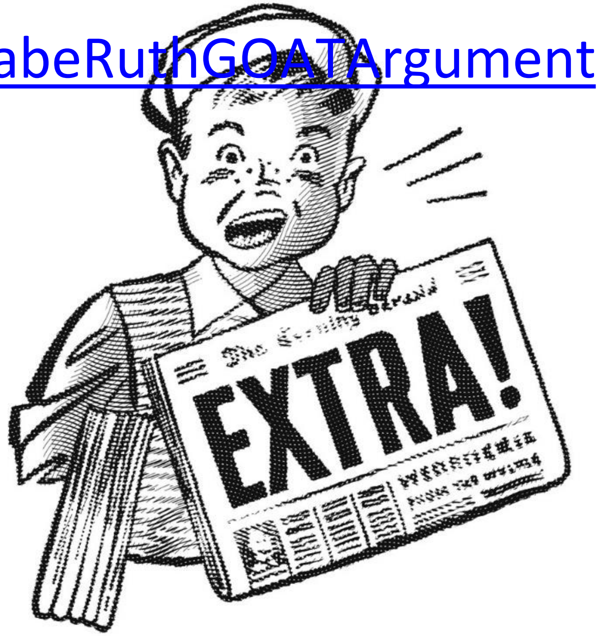
Examples abound…

- A student in the school paper at Nova Southeastern University makes the case for Tom Brady in football: TomBradyGOATArgument

- Justin Quinn in USAToday says it's Bill Russell in basketball: BillRussellGOATArgument

- And at Quora.com Mike Berard makes the case for Babe Ruth: BabeRuthGOATArgument

The Brady and Russell arguments are largely team oriented, while Ruth's case is more about his individual excellence.



OUR GOAL TODAY…

This is a ***preliminary*** report on our work to date. We want to:

-Show that metrics matter.

-Give examples of ways that assumptions lead to metrics.

-Choose explicit metrics first and use them to evaluate ‘something like’ a GOAT argument; maybe a ‘best teammate”.

Note: The ideas here can be applied in teaching many applications besides sports. Consistent with ideas in social choice, finance, economics and other fields. Can adjust the technical level to be anything from a first year seminar to a capstone project!

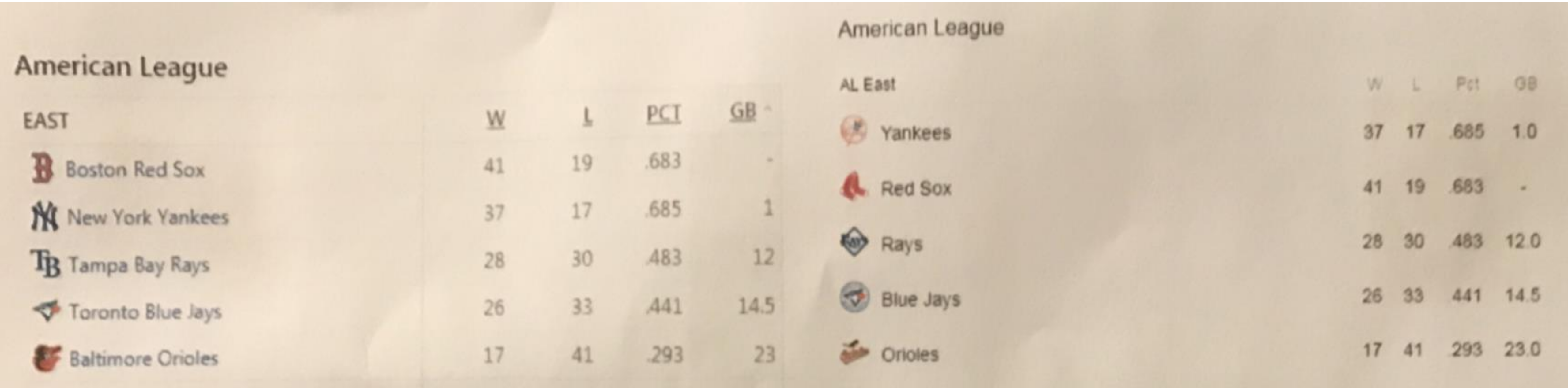
METRICS MATTER - Who’s IN first?

How can we tell who the GOAT if we can’t even decide GORN (Greatest of Right Now)!

Consider June 4, 2018. Boston Red Sox were 41-19, winning percentage .683. New York Yankees were 37-17, winning percentage .685.

ESPN *correctly* reports Red Sox in first (HURRAH!) as one “game ahead.”

GOOGLE reports Yankees in first (BOOOO!) as have higher winning percentage.



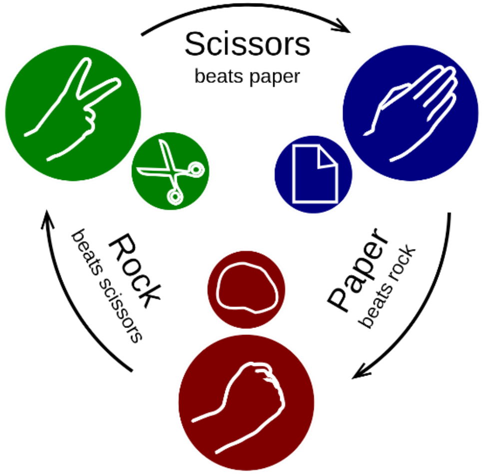
American League

EAST	W	L	PCT	GB
Boston Red Sox	41	19	.683	-
New York Yankees	37	17	.685	1
Tampa Bay Rays	28	30	.483	12
Toronto Blue Jays	26	33	.441	14.5
Baltimore Orioles	17	41	.293	23

American League

AL East	W	L	Pct	GB
Yankees	37	17	.685	1.0
Red Sox	41	19	.683	-
Rays	28	30	.483	12.0
Blue Jays	26	33	.441	14.5
Orioles	17	41	.293	23.0

Or … Cross country example



Order of finish:

A - B - B - A - A - C - C - B - C - C - C - C - C - B - A - B - A - A - B - B

“Invitational” Scoring: A wins!!

A: 1+4+5+15+17 = 42; C: 6+7+9+10+11 = 43, B: 2+3+8+14+16 = 43.   (C gets second since their sixth runner beat B’s sixth runner.)

“Dual Meet” Scoring: Three different races:

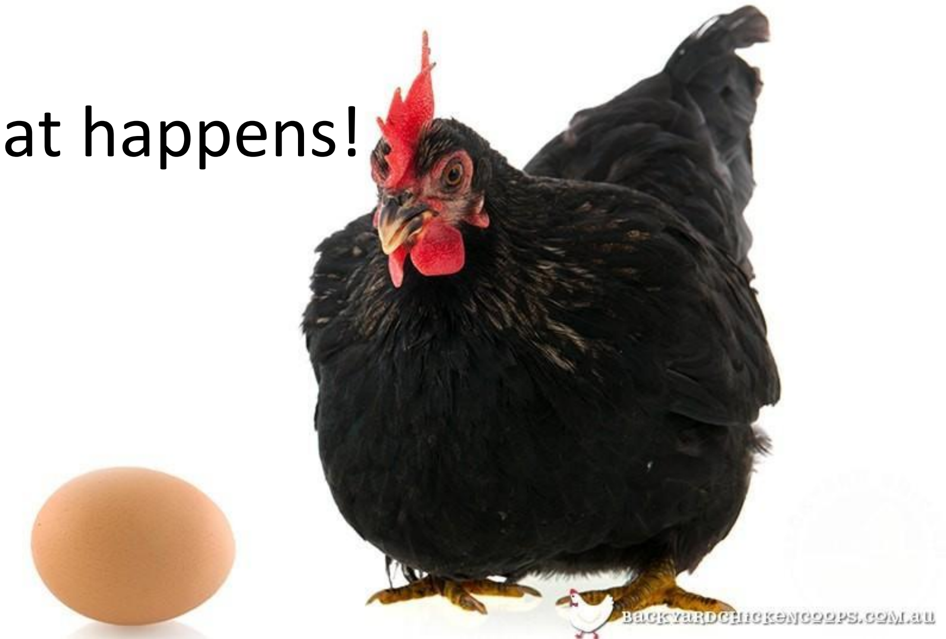
WHICH COMES FIRST? THE METRIC OR THE GOAT?

You don’t simply choose the GOAT … you choose a metric!

Scientific method … Choose the metric and see who’s the GOAT;

OR
As a fan/writer with deadline for a column… Choose the GOAT, then find a metric!

We will pick some reasonable metrics and see what happens!



Let’s have a vote!

In the chat, tell us who you think is the best candidate for GOAT of GOATs:

BRADY … NFL

RUSSELL … NBA

RUTH … MLB

OTHER … Provide Name/Sport

Let’s follow up with some data

In the chat, we had votes for the best candidate for GOAT of GOATs:

BRADY … NFL

RUSSELL … NBA

RUTH … MLB

OTHER … Provide Name/Sport

AN ALTERNATE PLACE WE CHOSE TO START: Who is the BOAT (Best Of All Teammates)? Measured by team success relative to league size/quality/playoff format. Preliminary research suggests Brady and Russell is the place to start looking.

The BOAT must … Get to the playoffs!

Under simplest assumptions, with goal of ‘make the playoffs’. Each team equally likely to qualify each year, years are independent, so a binomial model.

Brady: 18 times to playoff in 20 year career in 32 team league with 12 playoff qualifiers. Appearances ~ binomial (20, 12/32).

P(18 or more in 20 years) = **$3.02 * 10^{-9}$ = .00000000302.**

Brady vs. Russell playoff appearances

Brady: 18 times to playoff in 20 year career in 32 team league with 12 playoff qualifiers. Appearances ~ binomial (20, 12/32).

P(18 or more in 20 years) = **$3.02 * 10^{-9}$ = .00000000302.**

Russell: Thirteen for thirteen in making playoffs … but in small league where more than half of teams made playoffs!

P(13 for 13 in making playoffs) = **$6.29 * 10^{-3}$ = .00629.**

A point in Brady's favor here!

But if we do titles instead of appearances:

Under simplest assumptions: Each team equally likely to win each year, years are independent, so a binomial model:

Brady: Seven titles in 20 years in 32 team league.

X ~ binom(20, 1/32)

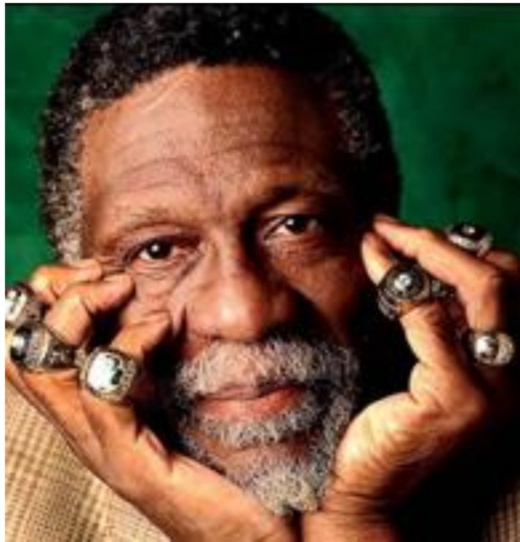
P (X at least 7) = 1.49 * 10^{-6} = .00000149.

Russell: Eleven titles in 13 years in (approx.) 10 team league.

X ~ binom(13, 1/10)

P(X at least 11) = 6.44 * 10^{-10} = .000000000644.

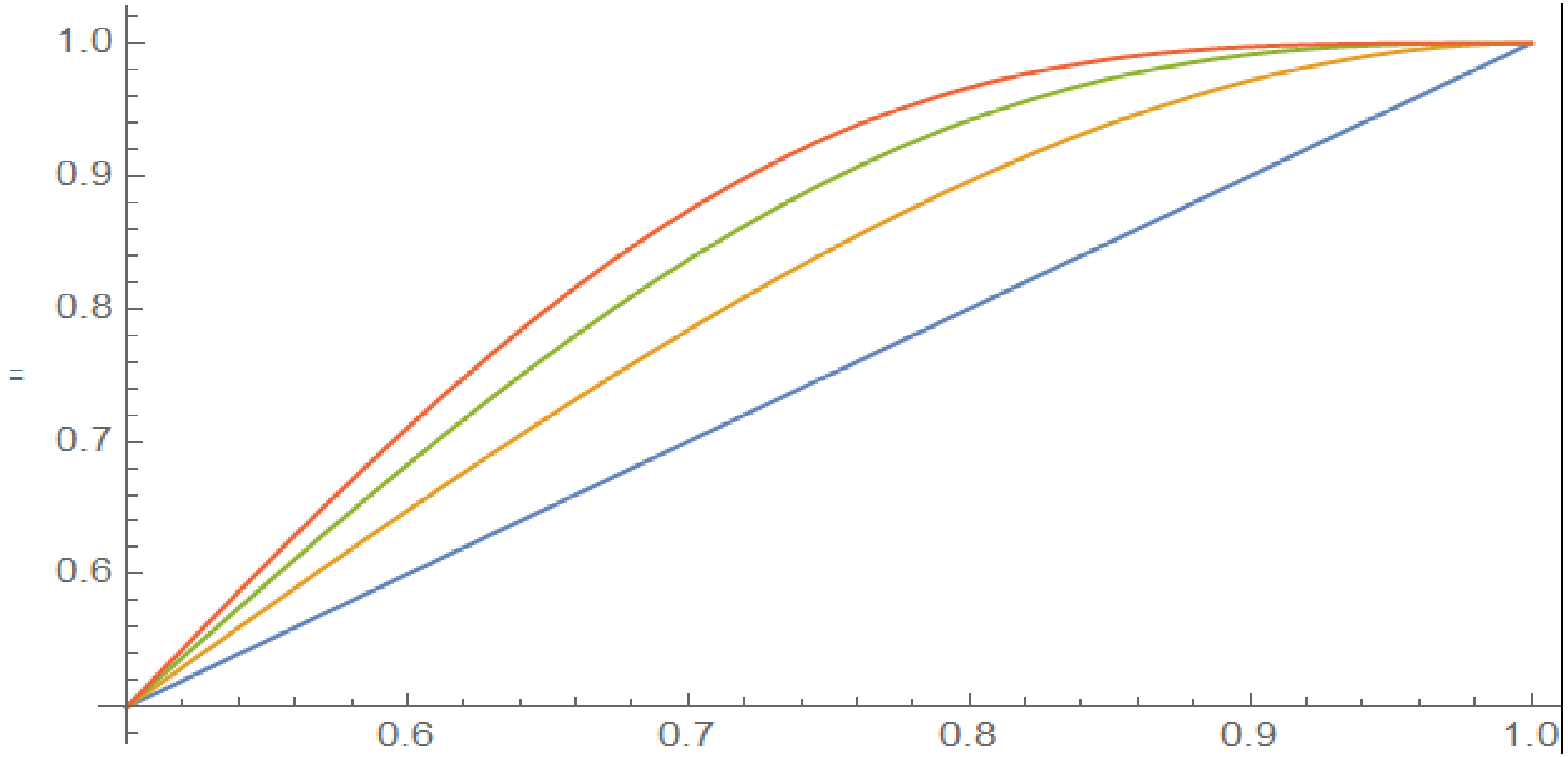
Russell looks better here … again, metrics matter!!



BUT Brady played *GAMES;* Russell *SERIES*

Horizontal axis: P(stronger team wins any one game).

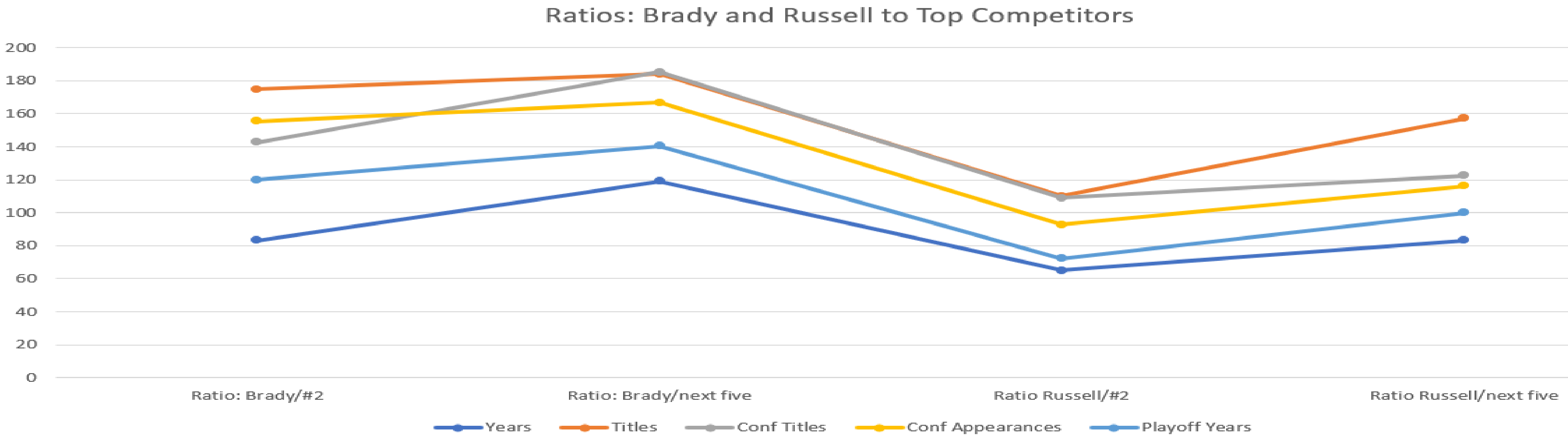
Vertical axis: P(stronger team wins series of 1,3,5,7 games) under independence.



BUT Brady played *GAMES;* Russell *SERIES*

Interpretation: Series help better teams avoid upsets. Also Brady had to win 3-4 games while Russell usually played two series. Does that make up the difference? IT MIGHT!

	Years	Titles	Championship	Conferences	Playoffs
Brady/#2*	83	175	143	156	120
Brady/next five	119	184	185	167	141
Russell/#2*	65	110	109	93	72
Russell/next five	83	157	122	116	100



Building Intuition: The $\log 5$ Method

Assume team A wins p percent of their games, and team B wins q percent of their games. Which formula do you think does a good job of predicting the probability that team A beats team B?

$$\frac{p+pq}{p+q+2pq}, \quad \frac{p+pq}{p+q-2pq}$$

$$\frac{p-pq}{p+q+2pq}, \quad \frac{p-pq}{p+q-2pq}$$

Building intuition: *A* wins *p* percent, *B* wins *q* percent

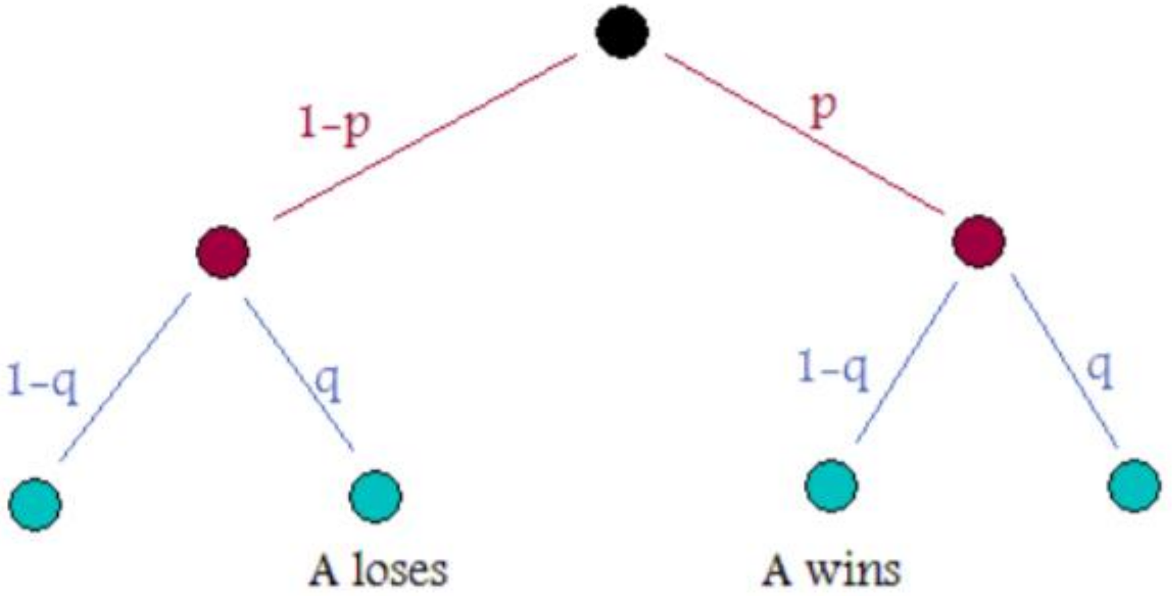
$$\frac{p+pq}{p+q+2pq}, \quad \frac{p+pq}{p+q-2pq}$$

$$\frac{p-pq}{p+q+2pq}, \quad \frac{p-pq}{p+q-2pq}$$

Consider special cases:

1. $\text{Prob}(A \text{ beats } B) + \text{Prob}(B \text{ beats } A) = 1$.
2. If $p = q$ then the probability A beats B is 50%.
3. If $p = 1$ and $q \neq 0, 1$ then A always beats B.
4. If $p = 0$ and $q \neq 0, 1$ then A always loses to B.
5. If $p > 1/2$ and $q < 1/2$ then $\text{Prob}(A \text{ beats } B) > p$.
6. If $q = 1/2$ prob A wins is p ($p = 1/2$ the prob B wins is q).

Building intuition: Sketch of proof: $\frac{p-pq}{p+q-2pq}$



- A beats B has probability $p(1-q)$.
- A and B do not have the same outcome has probability $p(1-q)+(1-p)q$.
- $\mathrm{Prob}(A \text{ beats } B) = \frac{p(1-q)}{p(1-q)+(1-p)q} = \frac{p-pq}{p+q-2pq}$.

The log 5 rule (due to Bill James)

Suppose team A wins games with probability p, and team B wins with probability q.

A good estimate of P(A wins a game vs. B) is given by

$$\frac{\boldsymbol{p(1-q)}}{\boldsymbol{p(1-q)+(1-p)q}} = \frac{\boldsymbol{p-pq}}{\boldsymbol{p+q-2pq}}$$

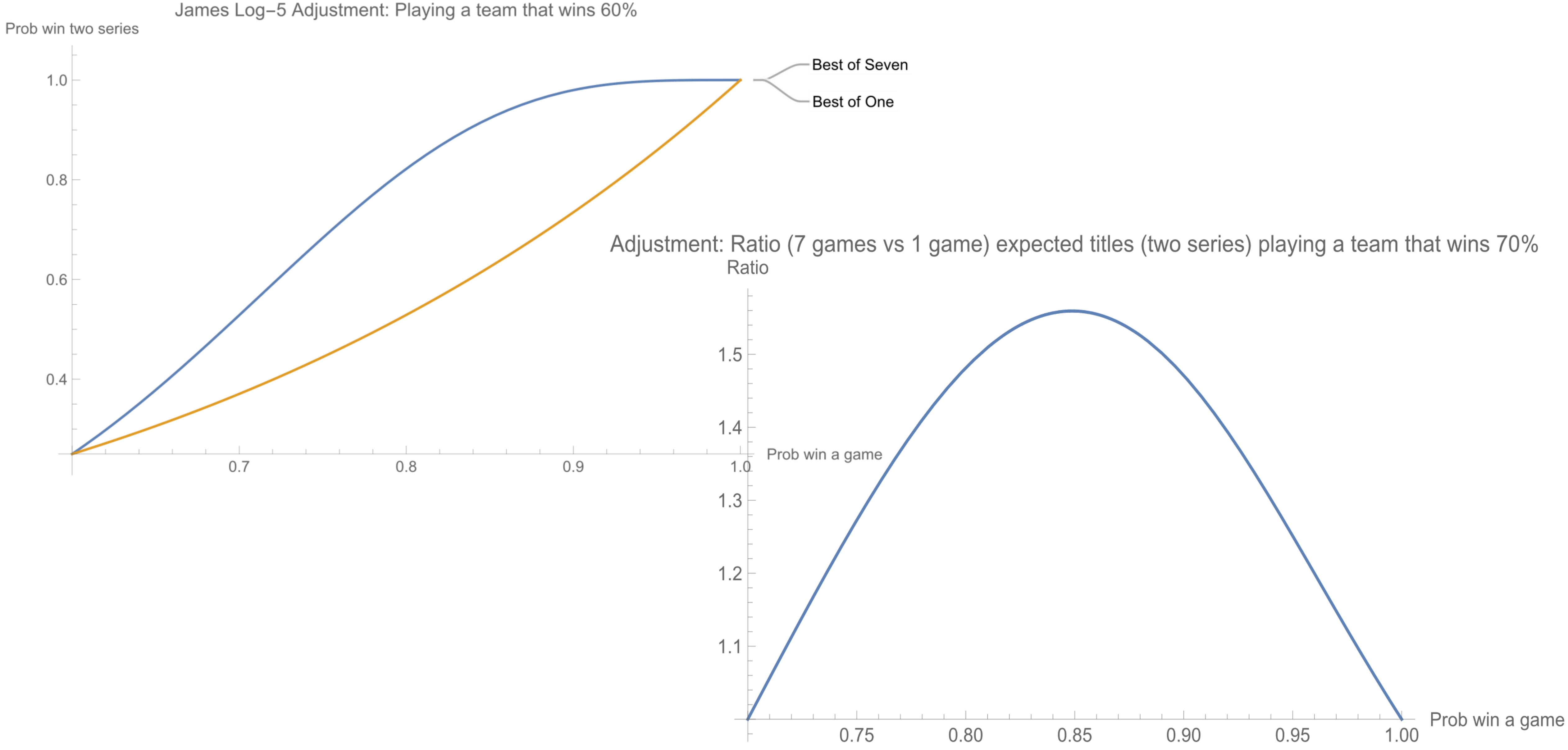
Example: In a playoff round we might have p = .8 and q = .6

So P(A wins a game vs. B) = (.8 - .48) / (1.4 - .96) ≈ .727.

(See SJM’s paper at https://web.williams.edu/Mathematics/sjmiller/public_html/399/handouts/Log5WonLoss_Paper.pdf)

A Log-5 Adjustment for series vs. games

Team A, P(win)=p, plays B with q=0.6. Maximum ratio is about 1.6. Two series.

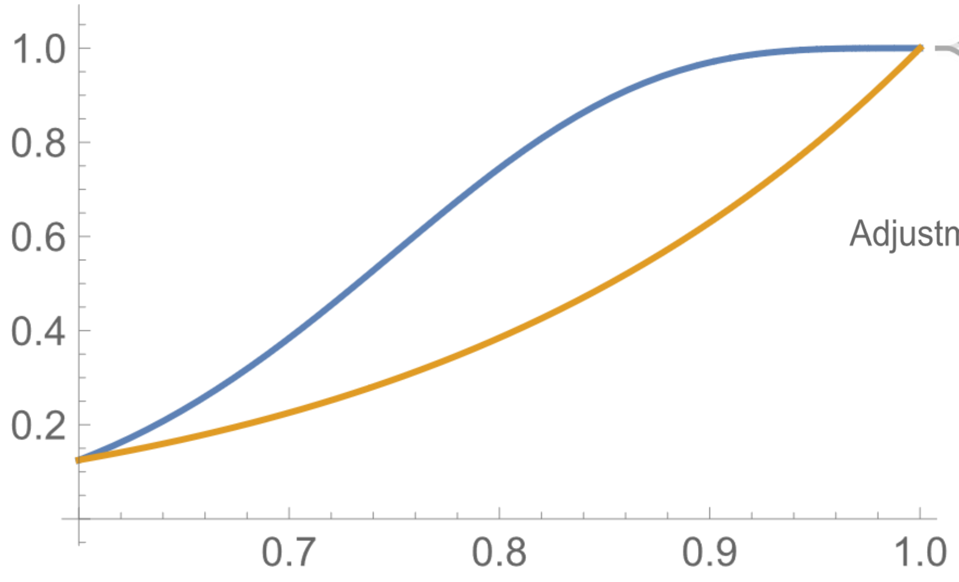


A Log-5 Adjustment for series vs. games

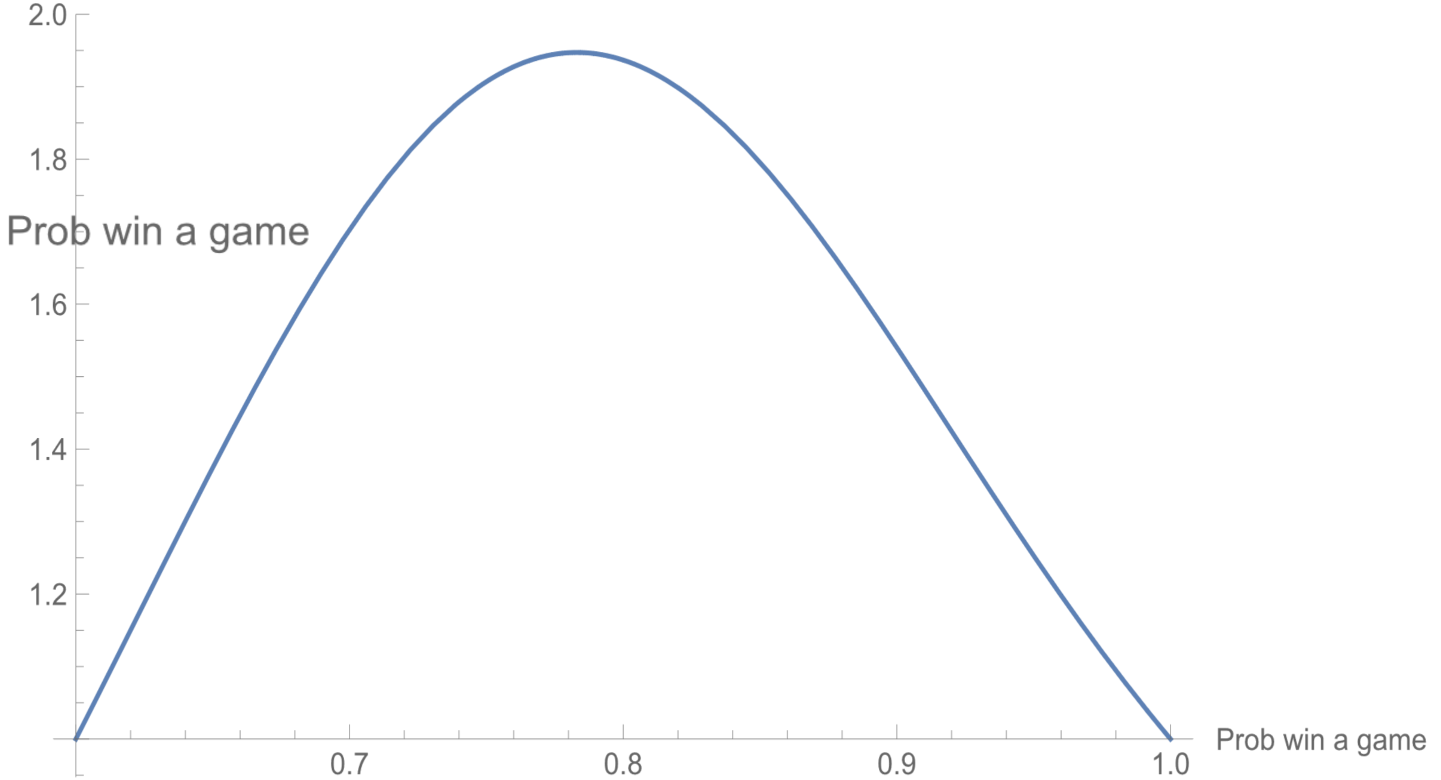
Team A, P(win)=p, plays B with q=0.6. Maximum ratio is about 1.9. Three series.

James Log−5 Adjustment: Playing a team that wins 60%

Prob win three series



Adjustment: Ratio (7 games vs 1 game) expected titles (three series) playing a team that wins 60%



So here’s an argument for Brady…

RUSSELL … 11 titles.

BUT NFL titles might be about 1.8 times more difficult because of the games/series issue!

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RUSSELL … 11 titles.

BUT NFL titles might be about 1.8 times more difficult because of the games/series issue!

So BRADY’s seven titles might be worth about:

$7 * 1.8 \ = \ \ 12.6$ RUSSELL titles!

-Some next steps …

We have begun considering (and will recruit students to help us consider):

-More sophisticated metrics with playoff wins models (Poisson vs. binomial);

-Championship round success (proponents of Michael “six for six” Jordan and Joe “four for four” Montana

-Similar argument for longevity vs. high peak. Is four title in eight year career more or less impressive than four in 20?

-Influence by sport (Russell one of five, on floor 80% of time; Brady one of eleven, of field 40% of time … but Brady key in all of those plays while Russell might go some time without a touch.)

-We’ll think of more!

One more vote …

Based on our discussion, please use the chat to vote again for BOAT:

BRADY RUSSELL RUTH OTHER

Did we change any minds?

(We did change the question!)

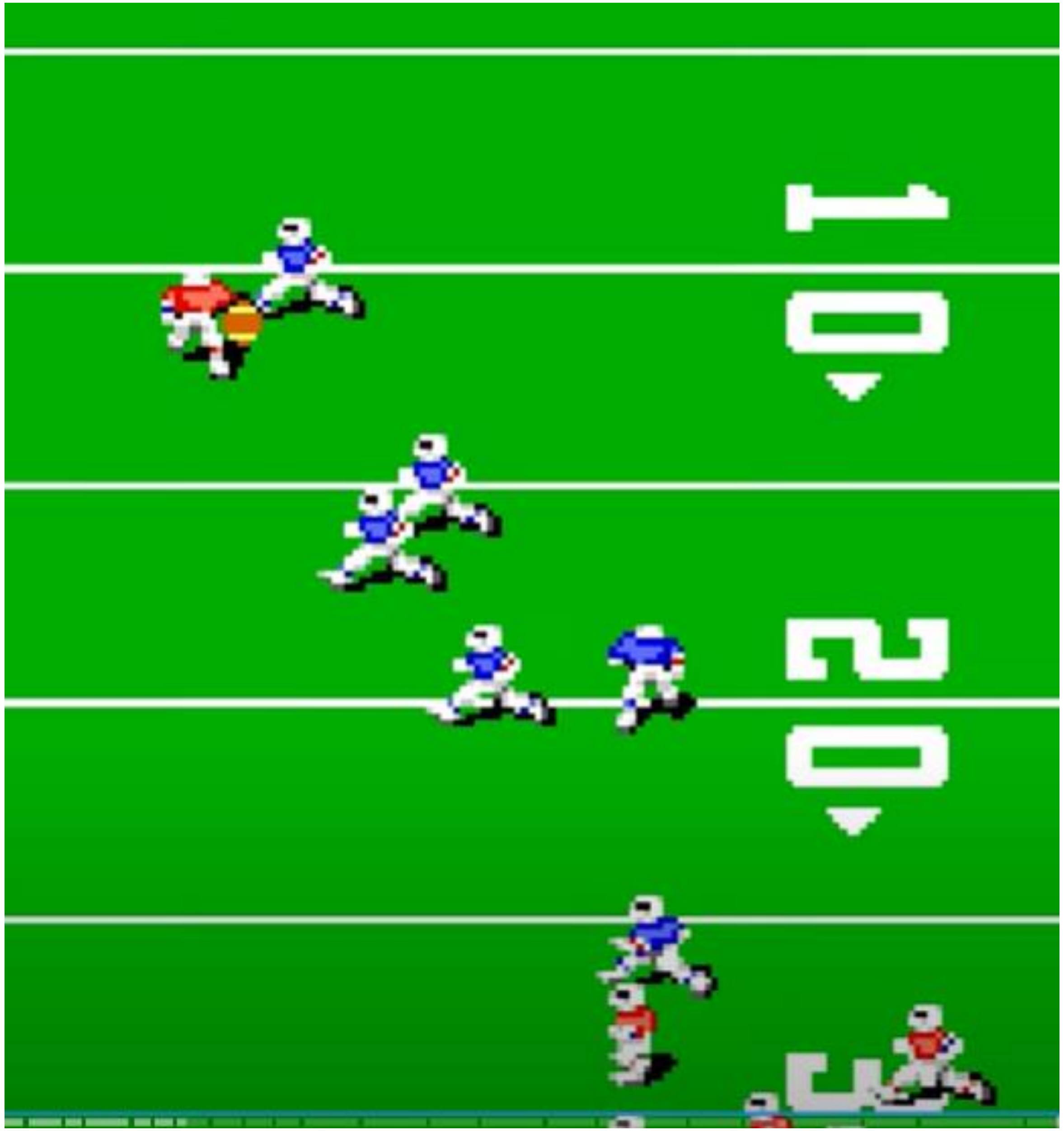
Math 344: Mathematics of Sports: Spring 2023:
Lecture 05: Markov Chains: https://youtu.be/Iah5_f4QoQc

Plan for the day.

- Review Linear Algebra.
- Study deterministic processes.
- Study random processes.

https://www.youtube.com/watch?v=b_gWfsPfokY

https://www.youtube.com/watch?v=bm0HNoCQGD8



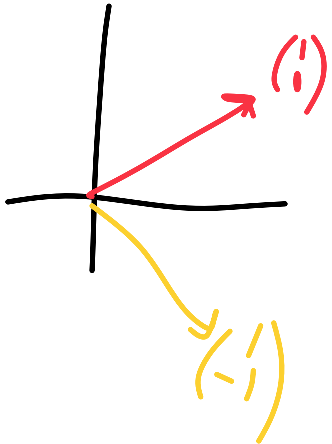
https://www.youtube.com/watch?v=b_gWfsPfokY

Review Linear Algebra: Eigenvalues, Eigenvectors.

Eigenvalue, Eigenvector

Say $\vec{v} \neq \vec{0}$ is an eigenvector of A with eigenvalue λ if $A\vec{v} = \lambda\vec{v}$.

Example:



$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\vec{v}_1$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\vec{v}_2$$

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ so $A^n \vec{v} = c_1 3^n \vec{v}_1 + c_2 (-1)^n \vec{v}_2$

Solving the Fibonacci Problem:

0, 1, 1, 2, 3, 5, 8, 13, ... $F_{n+1} = F_n + F_{n-1}$

$$\vec{v}_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

$$\vec{v}_{n+1} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_n + F_{n-1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v}_n$$

$$\vec{v}_{n+1} = A \vec{v}_n = A^2 \vec{v}_{n-1} = \cdots = A^n \vec{v}_1$$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solve $A\vec{v} = \lambda \vec{v}$

$$\text{Det}(A - \lambda I) = 0$$

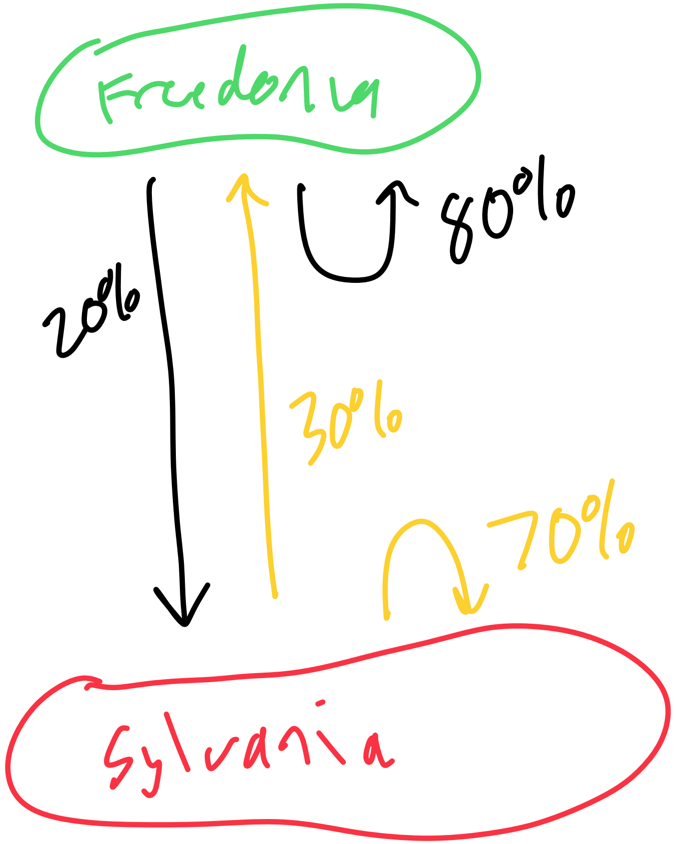
$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \text{or} \quad (1-\lambda)(-\lambda) - 1 = 0$$

$\downarrow$

Binet's Formula

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Freedonia / Sylvania:



Start with a in Freedonia, b in Sylvania

$\begin{pmatrix} f_n \\ s_n \end{pmatrix}$ population in each at time n

$$\begin{pmatrix} f_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} .8 f_n + .3 s_n \\ .2 f_n + .7 s_n \end{pmatrix} = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \end{pmatrix}$$

columns add to 1

transition matrix

entries non-neg, b/w 0 and 1

$A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$ $A^T = \begin{pmatrix} .8 & .2 \\ .3 & .7 \end{pmatrix}$

$\text{Det}(A - \lambda I) = 0$ same as $\text{Det}(A^T - \lambda I) = 0$

<u>Boola-Boola Thm</u>: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1

$\lambda_1(A) + \lambda_2(A) = Tr(A) = .8 + .7 = 1.5$

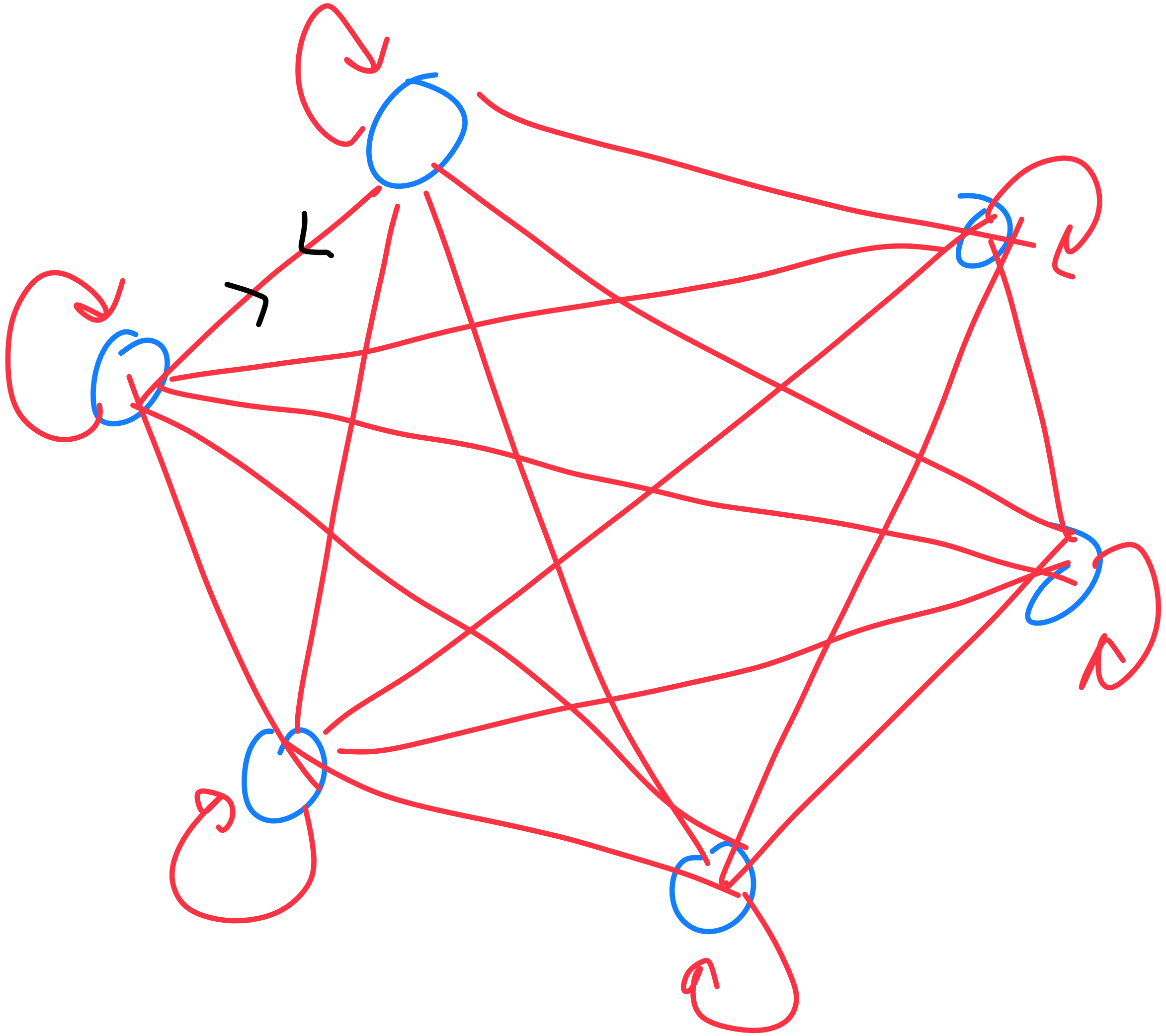
Since $\lambda_1(A) = 1$ must have $\lambda_2(A) = .5$

Find $\vec{v}_1, \vec{v}_2$, write $\begin{pmatrix} f_0 \\ s_0 \end{pmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

What is $\lim\limits_{n \to \infty} \begin{pmatrix} f_n \\ s_n \end{pmatrix}$?

Transition Matrices:

n states there are n possibilities for each



transition probabilities is n^2

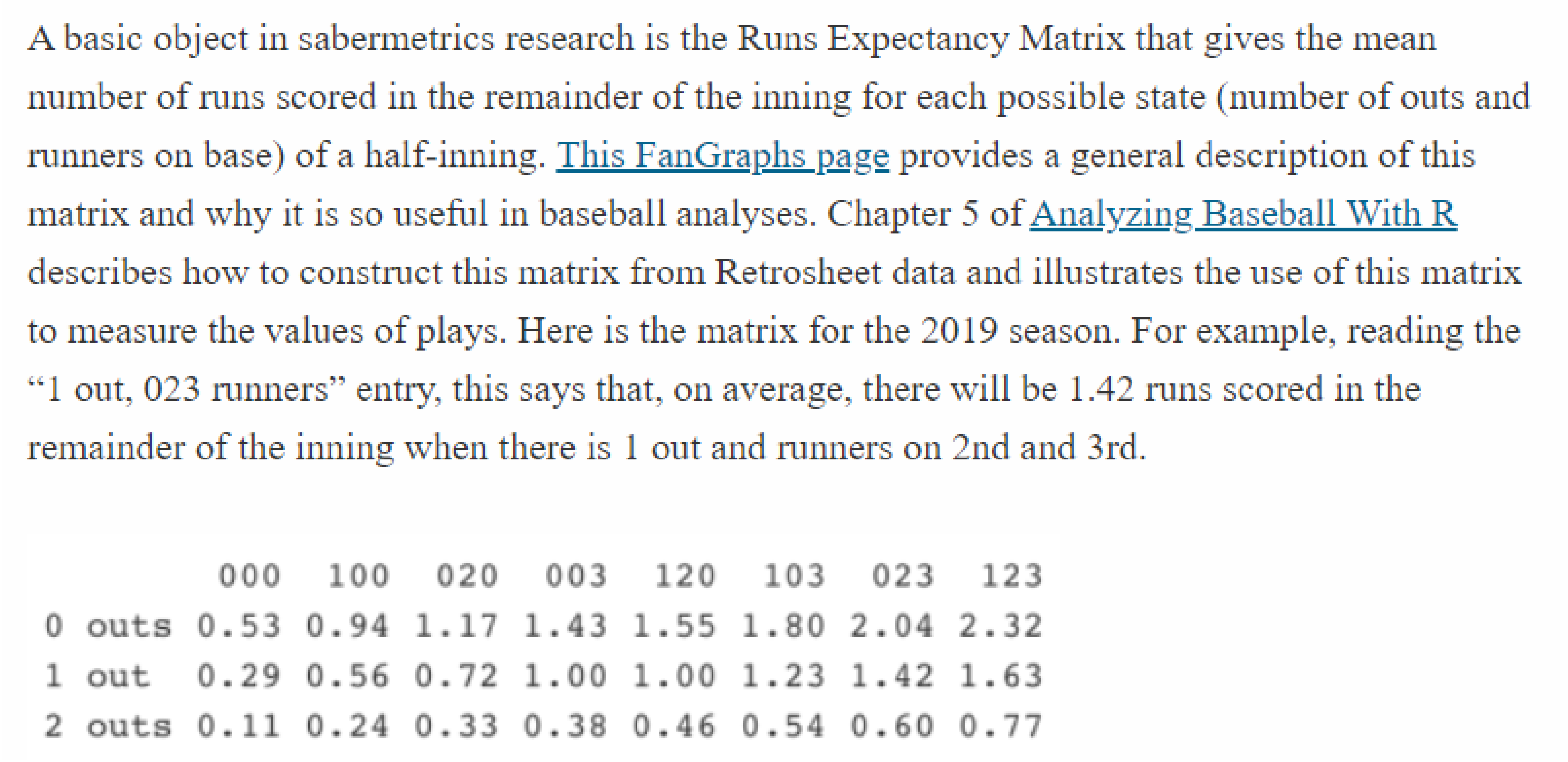
$$= 2\binom{n}{2} + n$$

$$= 2\frac{n(n-1)}{2} + n$$

$$= n^2 - n + n$$

$$= n^2$$ ✓

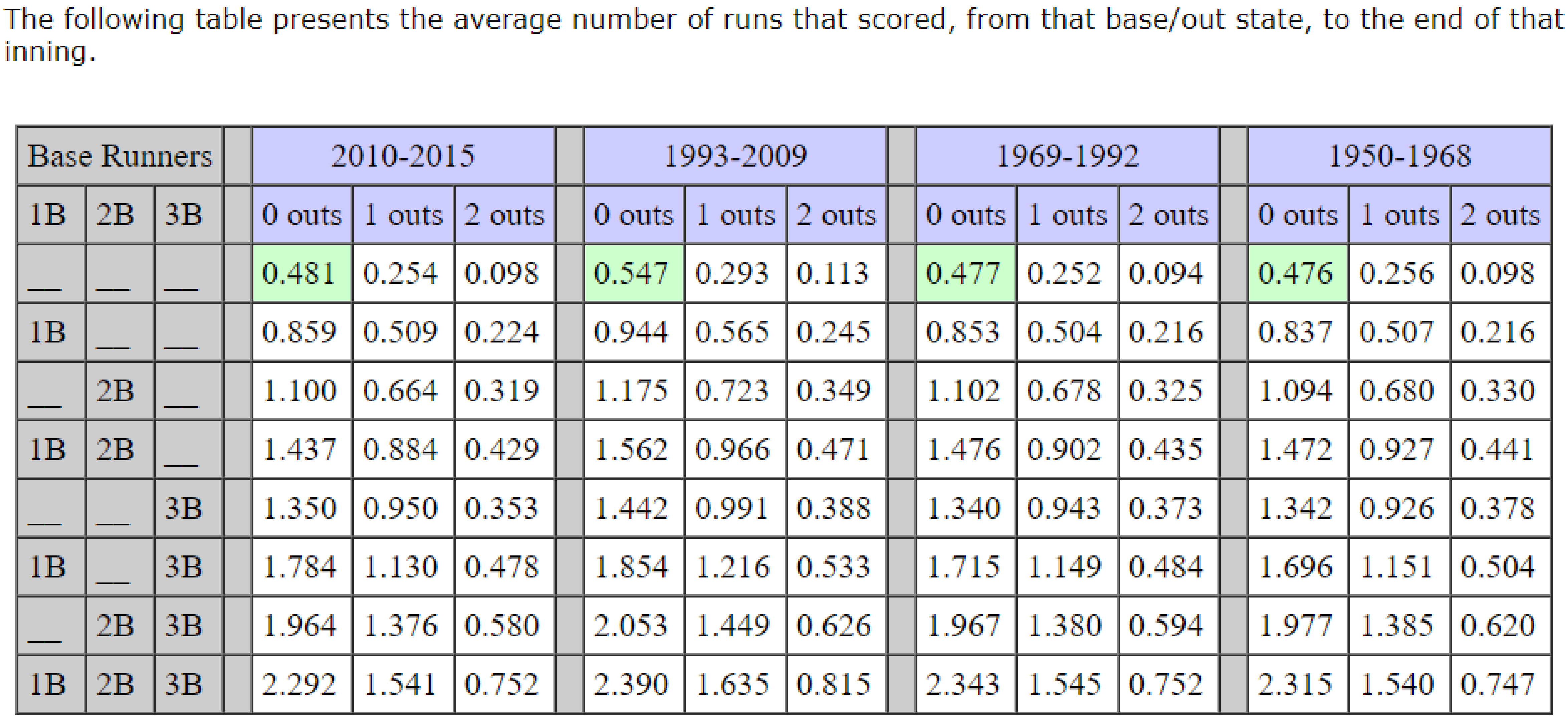
Run Expectancy Matrices:



A basic object in sabermetrics research is the Runs Expectancy Matrix that gives the mean number of runs scored in the remainder of the inning for each possible state (number of outs and runners on base) of a half-inning. This FanGraphs page provides a general description of this matrix and why it is so useful in baseball analyses. Chapter 5 of Analyzing Baseball With R describes how to construct this matrix from Retrosheet data and illustrates the use of this matrix to measure the values of plays. Here is the matrix for the 2019 season. For example, reading the "1 out, 023 runners" entry, this says that, on average, there will be 1.42 runs scored in the remainder of the inning when there is 1 out and runners on 2nd and 3rd.

```
         000  100  020  003  120  103  023  123
0 outs 0.53 0.94 1.17 1.43 1.55 1.80 2.04 2.32
1 out  0.29 0.56 0.72 1.00 1.00 1.23 1.42 1.63
2 outs 0.11 0.24 0.33 0.38 0.46 0.54 0.60 0.77
```

https://baseballwithr.wordpress.com/2020/12/21/summarizing-a-runs-expectancy-matrix/#:~:text=A%20basic%20object%20in%20sabermetrics%20research%20is%20the,why%20it%20is%20so%20useful%20in%20baseball%20analyses.

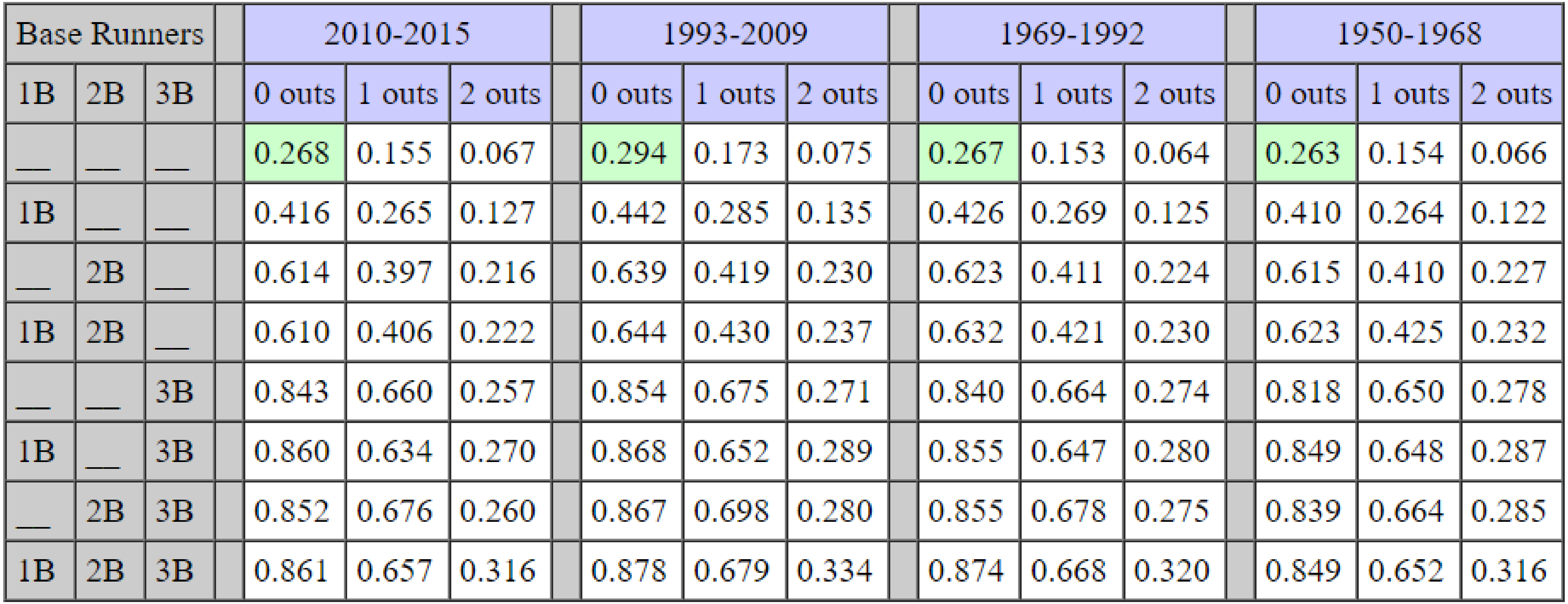


The following table presents the average number of runs that scored, from that base/out state, to the end of that inning.

Base Runners			2010-2015			1993-2009			1969-1992			1950-1968		
1B	2B	3B	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs
__	__	__	0.481	0.254	0.098	0.547	0.293	0.113	0.477	0.252	0.094	0.476	0.256	0.098
1B	__	__	0.859	0.509	0.224	0.944	0.565	0.245	0.853	0.504	0.216	0.837	0.507	0.216
__	2B	__	1.100	0.664	0.319	1.175	0.723	0.349	1.102	0.678	0.325	1.094	0.680	0.330
1B	2B	__	1.437	0.884	0.429	1.562	0.966	0.471	1.476	0.902	0.435	1.472	0.927	0.441
__	__	3B	1.350	0.950	0.353	1.442	0.991	0.388	1.340	0.943	0.373	1.342	0.926	0.378
1B	__	3B	1.784	1.130	0.478	1.854	1.216	0.533	1.715	1.149	0.484	1.696	1.151	0.504
__	2B	3B	1.964	1.376	0.580	2.053	1.449	0.626	1.967	1.380	0.594	1.977	1.385	0.620
1B	2B	3B	2.292	1.541	0.752	2.390	1.635	0.815	2.343	1.545	0.752	2.315	1.540	0.747

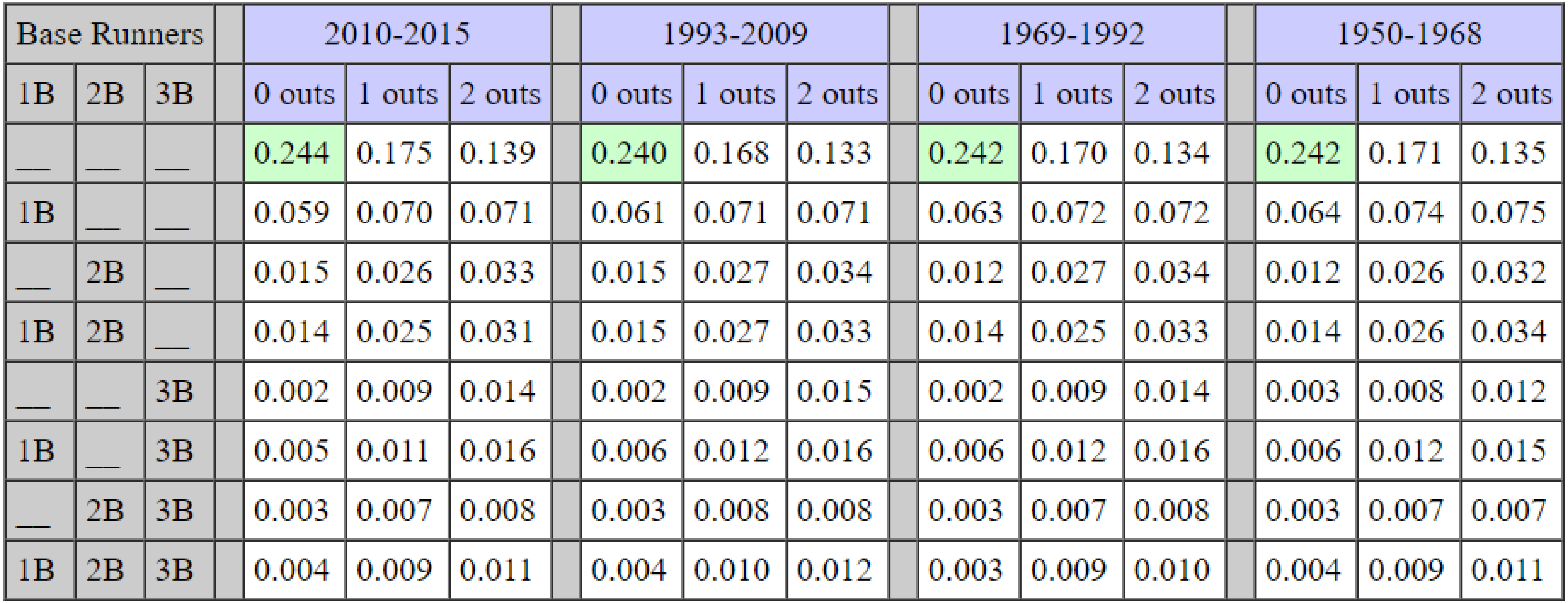
http://www.tangotiger.net/re24.html

The following table presents the chance that a run will score at some point in the inning, from each base/out state



Base Runners			2010-2015			1993-2009			1969-1992			1950-1968		
1B	2B	3B	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs
__	__	__	0.268	0.155	0.067	0.294	0.173	0.075	0.267	0.153	0.064	0.263	0.154	0.066
1B	__	__	0.416	0.265	0.127	0.442	0.285	0.135	0.426	0.269	0.125	0.410	0.264	0.122
__	2B	__	0.614	0.397	0.216	0.639	0.419	0.230	0.623	0.411	0.224	0.615	0.410	0.227
1B	2B	__	0.610	0.406	0.222	0.644	0.430	0.237	0.632	0.421	0.230	0.623	0.425	0.232
__	__	3B	0.843	0.660	0.257	0.854	0.675	0.271	0.840	0.664	0.274	0.818	0.650	0.278
1B	__	3B	0.860	0.634	0.270	0.868	0.652	0.289	0.855	0.647	0.280	0.849	0.648	0.287
__	2B	3B	0.852	0.676	0.260	0.867	0.698	0.280	0.855	0.678	0.275	0.839	0.664	0.285
1B	2B	3B	0.861	0.657	0.316	0.878	0.679	0.334	0.874	0.668	0.320	0.849	0.652	0.316

The following table presents the frequency of plate appearances that started in each base/out state.



Base Runners			2010-2015			1993-2009			1969-1992			1950-1968		
1B	2B	3B	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs
__	__	__	0.244	0.175	0.139	0.240	0.168	0.133	0.242	0.170	0.134	0.242	0.171	0.135
1B	__	__	0.059	0.070	0.071	0.061	0.071	0.071	0.063	0.072	0.072	0.064	0.074	0.075
__	2B	__	0.015	0.026	0.033	0.015	0.027	0.034	0.012	0.027	0.034	0.012	0.026	0.032
1B	2B	__	0.014	0.025	0.031	0.015	0.027	0.033	0.014	0.025	0.033	0.014	0.026	0.034
__	__	3B	0.002	0.009	0.014	0.002	0.009	0.015	0.002	0.009	0.014	0.003	0.008	0.012
1B	__	3B	0.005	0.011	0.016	0.006	0.012	0.016	0.006	0.012	0.016	0.006	0.012	0.015
__	2B	3B	0.003	0.007	0.008	0.003	0.008	0.008	0.003	0.007	0.008	0.003	0.007	0.007
1B	2B	3B	0.004	0.009	0.011	0.004	0.010	0.012	0.003	0.009	0.010	0.004	0.009	0.011

Simple Baseball

outs

0 outs

1 out ends inning

(maybe 2 outs ends innings)

Math 344: Mathematics of Sports: Spring 2023:
Lecture 06: Markov Chains II: https://youtu.be/46yCrsD1nBM

Plan for the day.

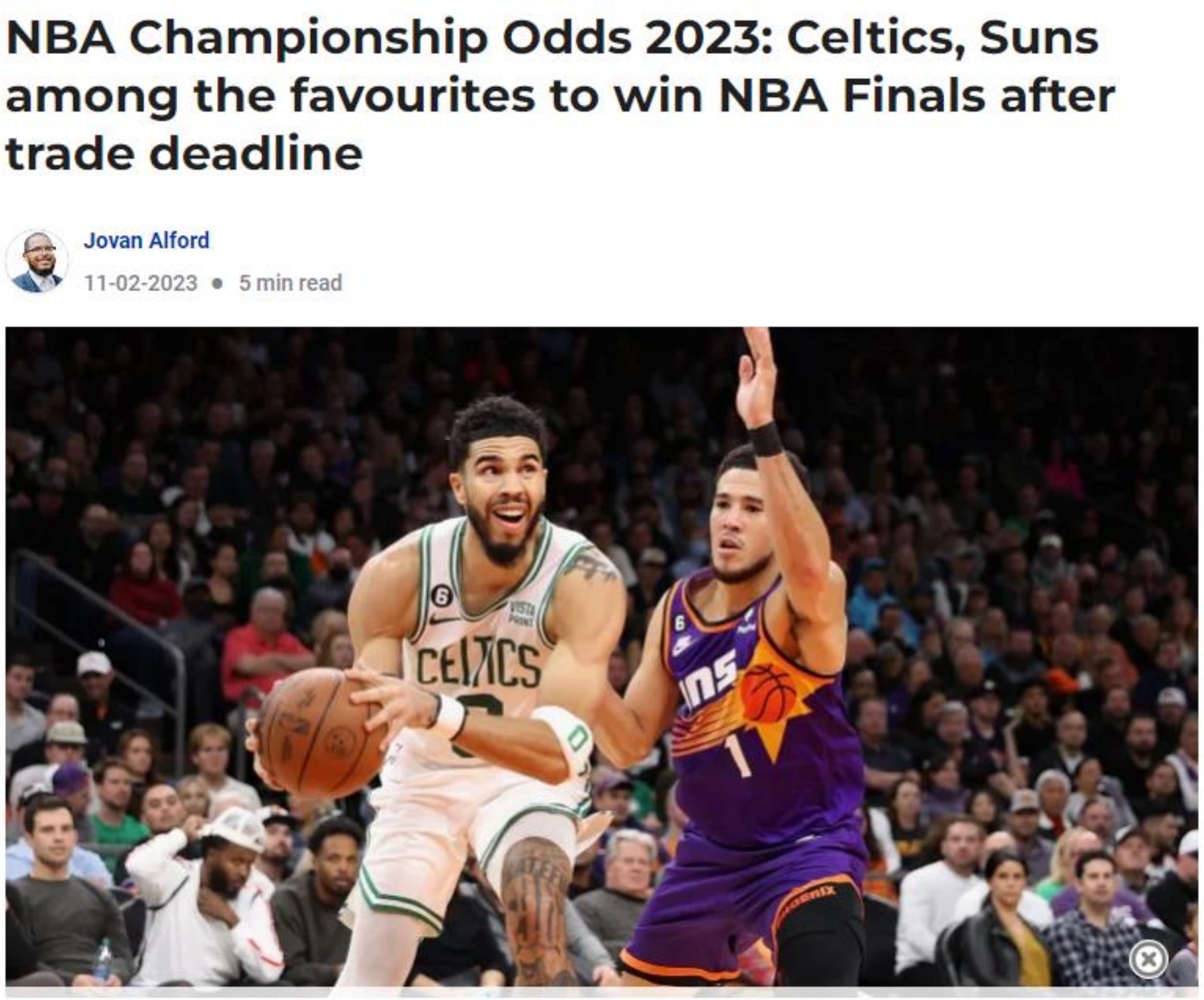
- Discuss misleading statistics
- Discuss the Freedonia / Sylvania Problem.
- Markov Model for Baseball.
- Discuss presentations.

ClutchPoints Betting
@CPBetting

Lakers Championship odds last week:
+5000

Lakers odds after dumping Westbrook, acquiring D'Angelo Russell, Malik Beasley and Mo Bamba:
+5000 🤷‍♂️

It's not getting easier for Lakers fans 😂



NBA Championship Odds 2023: Celtics, Suns among the favourites to win NBA Finals after trade deadline

Jovan Alford
11-02-2023 • 5 min read

https://www.sportingnews.com/au/nba/news/nba-championship-odds-2023-celtics-suns-favourites-trade-deadline/nmeqbsckgszzd5gyb9uat3ms

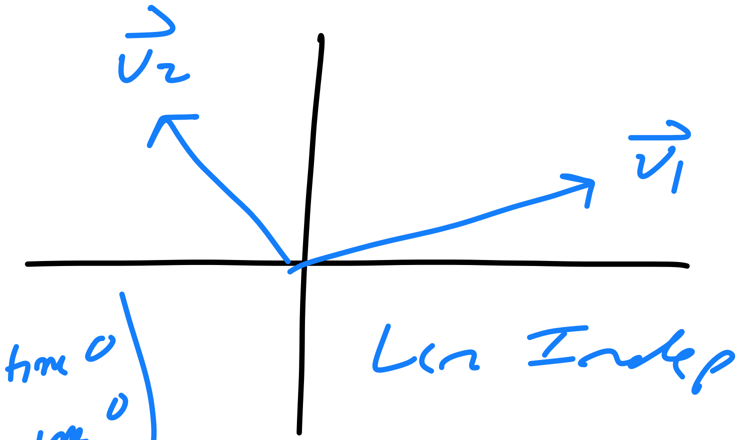
NBA Championship Odds 2023

Odds courtesy of BetMGM

Team	Odds
Celtics	$4.00
Suns	$5.50
Bucks	$5.50
Nuggets	$9.00
Clippers	$13.00
Sixers	$15.00
Warriors	$15.00
Mavericks	$17.00
Grizzlies	$21.00
Cavaliers	$26.00
Lakers	$34.00
Pelicans	$51.00
Heat	$51.00
Kings	$67.00

```
In[5]:= A = {{.8, .3}, {.2, .7}};
        MatrixForm[A]

Out[6]//MatrixForm=
        ( 0.8  0.3 )
        ( 0.2  0.7 )

In[10]:= Eigenvalues[A]
         Eigenvectors[A]

Out[10]= {1., 0.5}

Out[11]= {{0.83205, 0.5547}, {-0.707107, 0.707107}}
```

$\vec{V}_1$ $\vec{V}_2$



Lin Indep

$$\vec{v} = \begin{pmatrix} \text{freedonia at time } 0 \\ \text{sylvania at time } 0 \end{pmatrix}$$

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$A^n\vec{v} = c_1 1^n \vec{v}_1 + c_2\left(\tfrac{1}{2}\right)^n \vec{v}_2$$

as $n \to \infty$

$$A^n\vec{v} \to c_1\vec{v}_1$$

```
In[19]:= A[x_, y_] := {{.8 - x, .3 + y}, {.2 + x, .7 - y}};
         MatrixForm[A[x, y]]
```

Out[20]//MatrixForm=

$$\begin{pmatrix} 0.8 - x & 0.3 + y \\ 0.2 + x & 0.7 - y \end{pmatrix}$$

$$\vec{V} = c_1(x,y)\vec{v}_1(x,y) + c_2(x,y)\vec{v}_2(x,y)$$

```
In[34]:= Simplify[Eigenvalues[A[x, y]]]
         Eigenvectors[A[x, y]]
```

Out[34]=

$$\left\{0.75 - 0.5\,x - 0.5\,y - 0.5\sqrt{0.25 + 1.\,x + x^2 + 1.\,y + 2\,x\,y + y^2},\; 0.75 - 0.5\,x - 0.5\,y + 0.5\sqrt{0.25 + 1.\,x + x^2 + 1.\,y + 2\,x\,y + y^2}\right\}$$

Out[35]=

$$\left\{\left\{-\frac{0.5\left(-0.1 + 1.\,x - 1.\,y - 1.\,\sqrt{0.25 + 1.\,x + 1.\,x^2 + 1.\,y + 2.\,x\,y + 1.\,y^2}\right)}{0.2 + x}, 1.\right\}, \left\{-\frac{0.5\left(-0.1 + 1.\,x - 1.\,y + 1.\,\sqrt{0.25 + 1.\,x + 1.\,x^2 + 1.\,y + 2.\,x\,y + 1.\,y^2}\right)}{0.2 + x}, 1.\right\}\right\}$$

smallest

In[36]:= ~~largest~~[x_, y_] :=

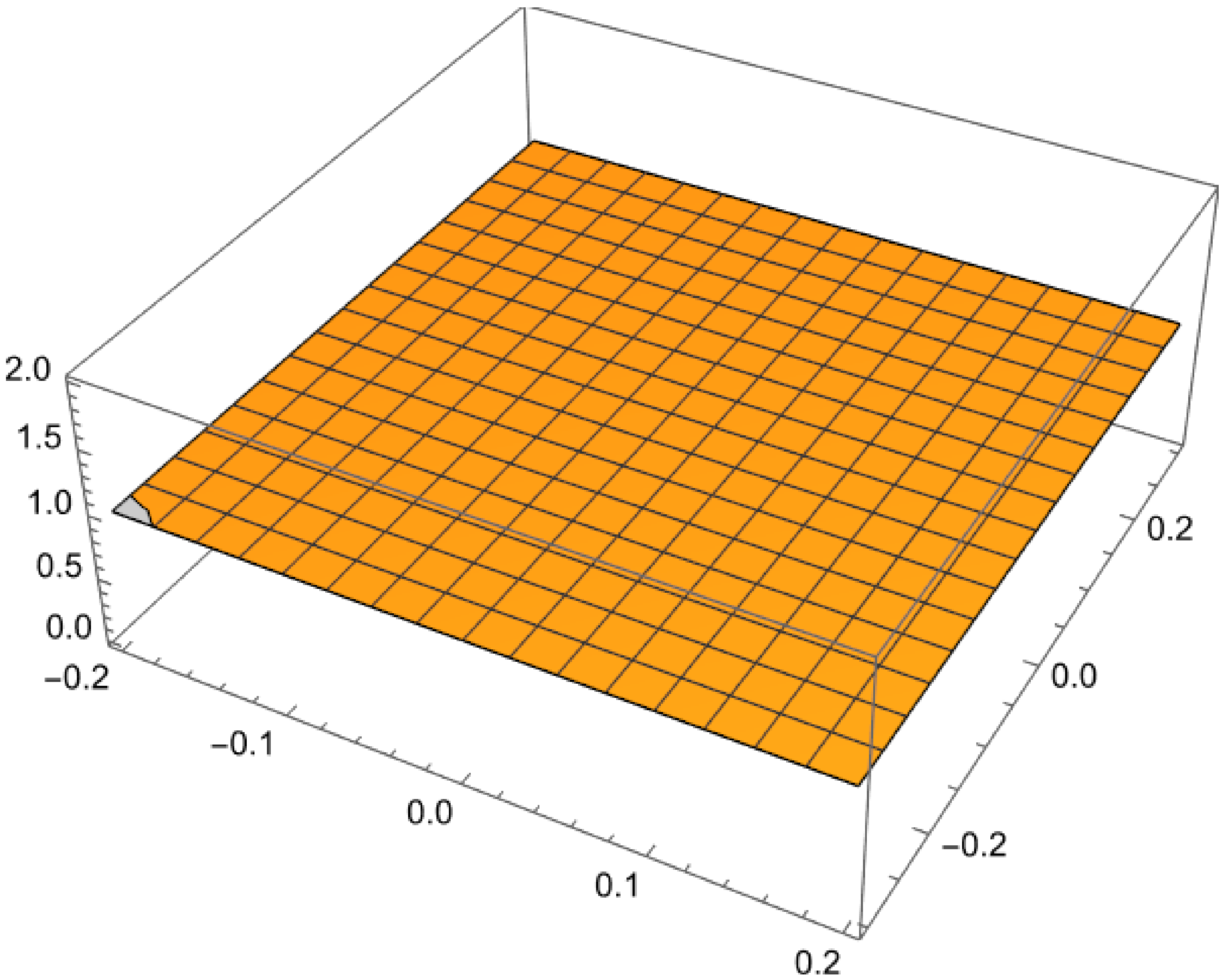
$$0.75` - 0.5`\,x - 0.5`\,y - 0.5`\sqrt{0.25000000000000002` + 1.`\,x + x^2 + 1.`\,y + 2\,x\,y + y^2};$$

largest ~~smallest[~~x_, y_] :=

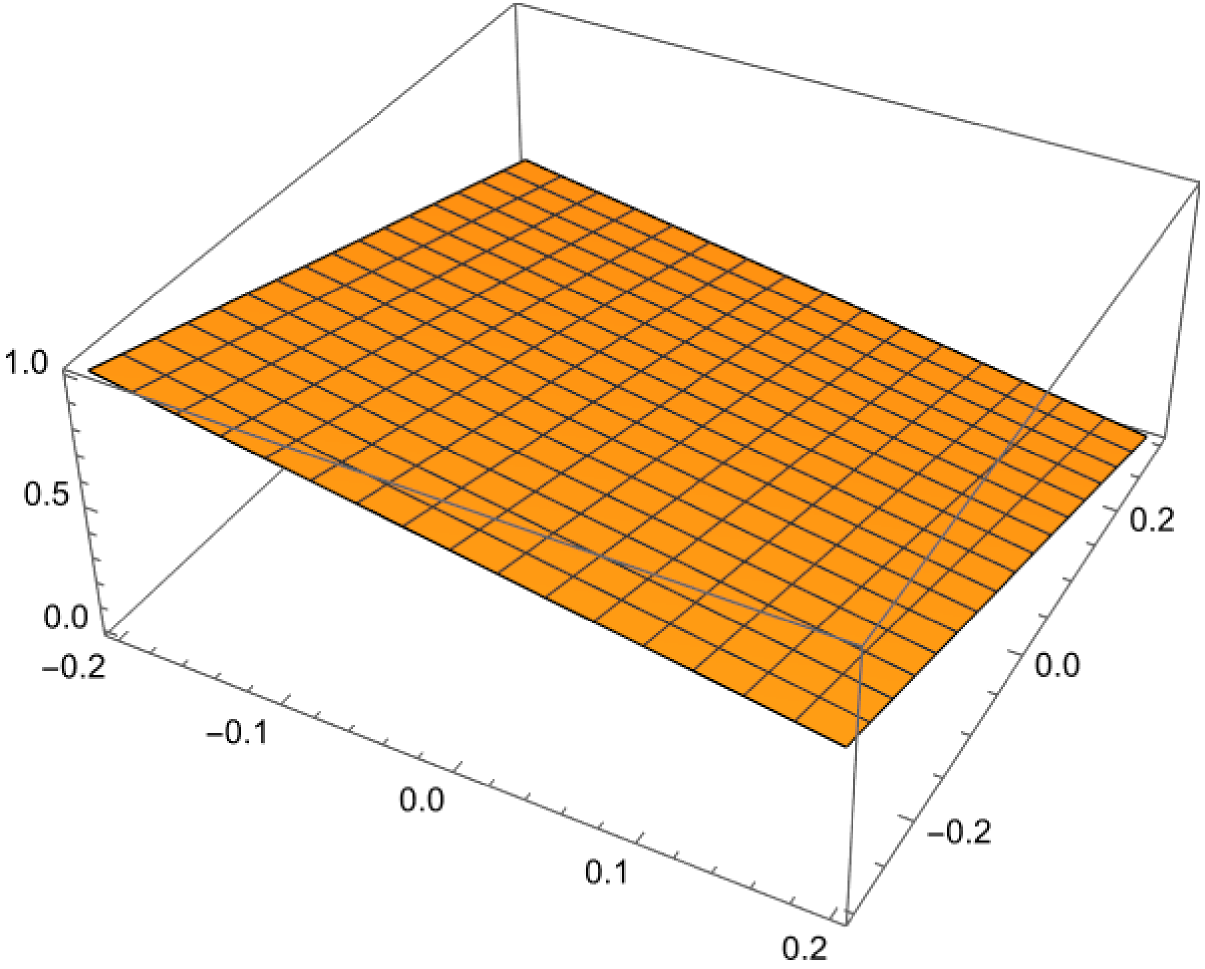
$$0.75` - 0.5`\,x - 0.5`\,y + 0.5`\sqrt{0.25000000000000002` + 1.`\,x + x^2 + 1.`\,y + 2\,x\,y + y^2};$$

```
In[29]:= Plot3D[largest[x, y], {x, -.2, .2}, {y, -.3, .3}]
```

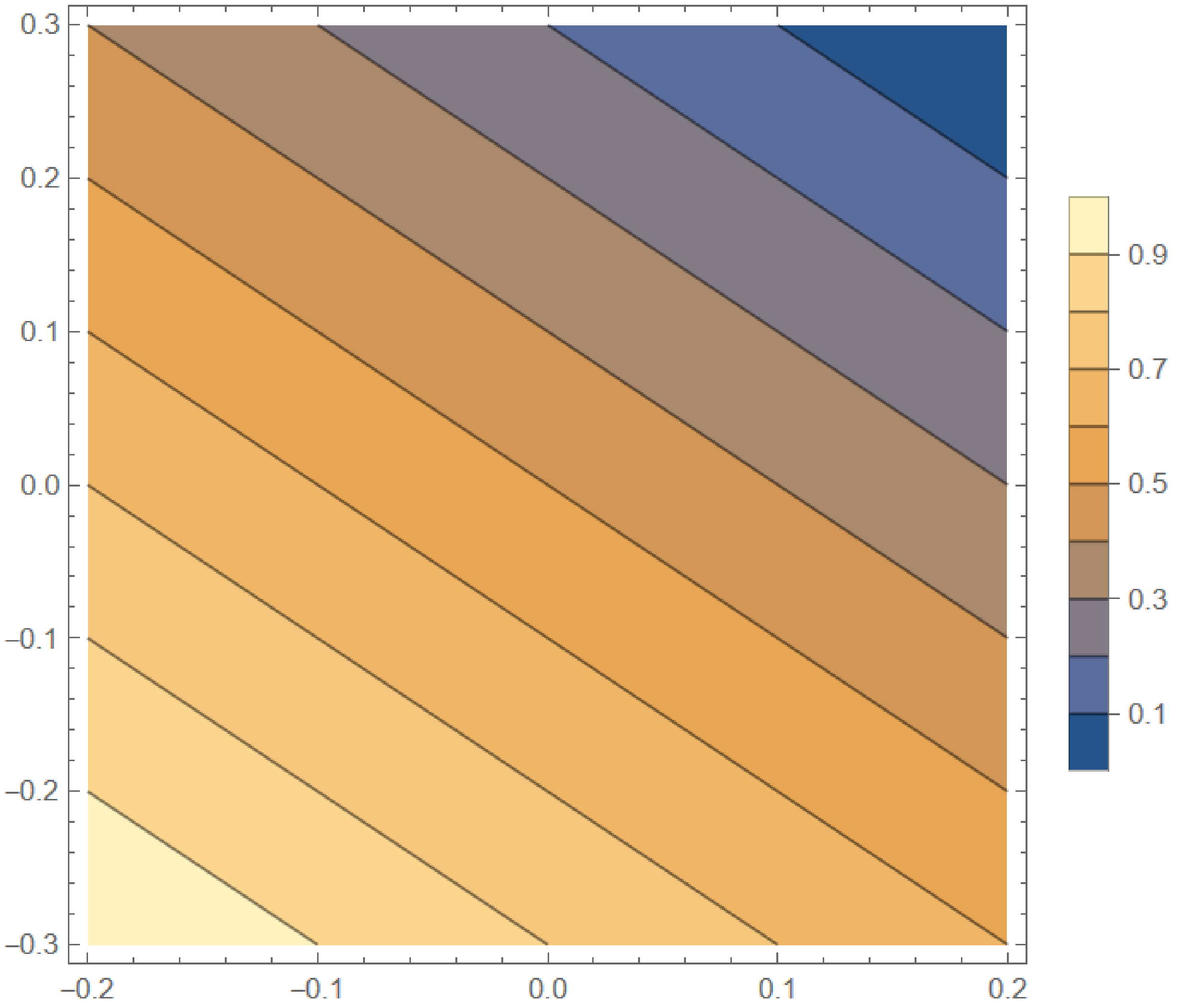
Out[29]=

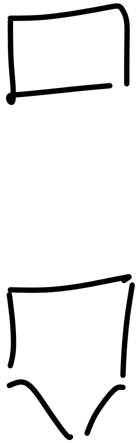


```
Plot3D[smallest[x, y], {x, -.2, .2}, {y, -.3, .3}]
```



```
ContourPlot[smallest[x, y] 1.0, {x, -.2, .2}, {y, -.3, .3},
 PlotLegends → Automatic]
```





outs

0, 1

KNOWN

	0	1
0 outs	r_{00}	r_{01}
1 out	r_{10}	r_{11}

YES

14 unknowns?

Underdetermined system

BUT

P_{11} +1

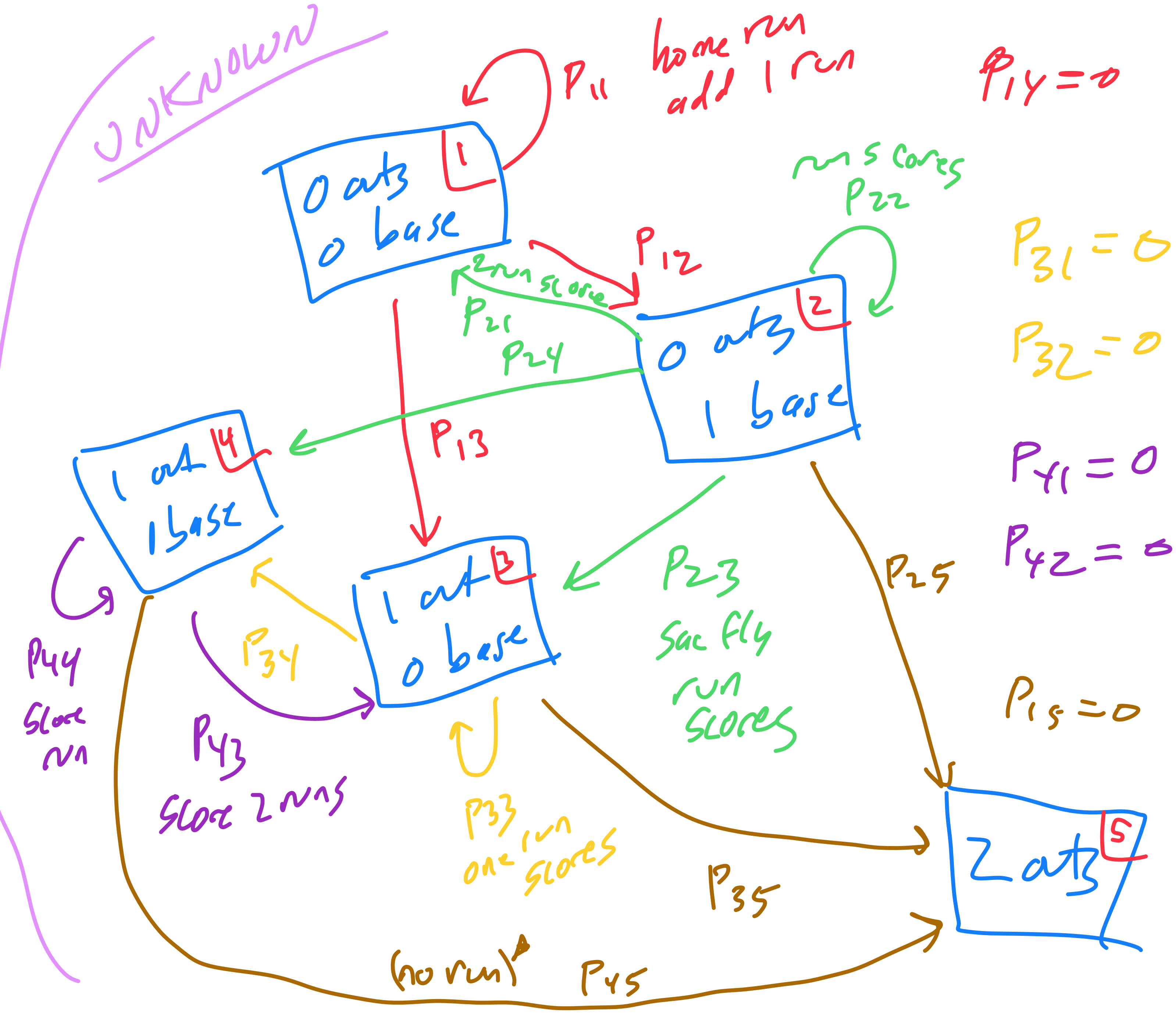
P_{22} +1

P_{21} +2

P_{33} +1

P_{34} +2

P_{44} +1



$P_{14} = 0$

$P_{31} = 0$

$P_{32} = 0$

$P_{41} = 0$

$P_{42} = 0$

$P_{15} = 0$

BUT

① More eq: 5 more!

$p_{11} + p_{12} + p_{13} + p_{14} + p_{15} = 1$

$\vdots \qquad \vdots$

$p_{51} + \cdots + p_{55} = 0$

9 Eqs

14 unknowns

underdetermined

BUT

② each $p_{ij} \in [0,1]$

Math 344: Mathematics of Sports: Spring 2023:
Lecture 07:
Pythagorean Won-Loss Formula: https://www.youtube.com/watch?v=iJyoyUV2JWQ

Plan for the day.

- Video: https://www.youtube.com/watch?v=iJyoyUV2JWQ
- Slides: https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/talks/PythagWLTalk_WilliamsRecruit2018.pdf

Papers here:

- A derivation of James' Pythagorean projection, By The Numbers -- The Newsletter of the SABR Statistical Analysis Committee (**16** (February 2006), no. 1, 17--22). pdf (expanded version: pdf). Chance Magazine **(20** (Winter 2007), no. 1, 40-48**).**
- First Order Approximations of the Pythagorean Won-Loss Formula for Predicting MLB Teams Winning Percentages (with Kevin Dayaratna), By The Numbers -- The Newsletter of the SABR Statistical Analysis Committee, pdf (expanded version with appendix proving main result with just one variable calculus: pdf); entire issue of By The Numbers here **(**22 **(2012),** no 1, 15--19)**.**
- The Pythagorean Won-Loss Formula and Hockey: A Statistical Justification for Using the Classic Baseball Formula as an Evaluative Tool in Hockey (with Kevin Dayaratna), The Hockey Research Journal: A Publication of the Society for International Hockey Research. ((2012/2013), pages 193--209) pdf
- Pythagoras at the Bat (with Taylor Corcoran, Jen Gossels, Victor Luo and Jaclyn Porfilio), book chapter in Social Networks and the Economics of Sports (edited by Panos M. Pardalos and Victor Zamaraev, Springer-Verlag, 2014, pages 89--114). Pdf
- Relieving and Readjusting Pythagoras (with Victor Luo), By The Numbers -- The Newsletter of the SABR Statistical Analysis Committee. **(25** (2015), no. 1, 5-14**)** pdf (older, expanded version: pdf)

Math 344: Mathematics of Sports: Spring 2023:
Lecture 08: Modeling Baseball Games: https://youtu.be/pPzUc28zz6E

Plan for the day.

- Markov Model for Baseball.
- Discuss presentations.

Baseball

Average is $\dfrac{\#\text{ hits}}{\#\text{ at bats}}$

On Base Percentage is $\dfrac{\#\text{ hits} + BB + \text{hit by pitch} + \cdots}{\#\text{ Plate appearances}}$

Slugging is $\dfrac{1\,\#\,1B + 2\,\#\,2B + 3\,\#\,3B + 4\,\#\,HR}{\#\text{ at bats}}$

On-Base Plus Slugging $= OBP + SLG$

The following table presents the average number of runs that scored, from that base/out state, to the end of that inning.

Base Runners			2010-2015			1993-2009			1969-1992			1950-1968		
1B	2B	3B	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs	0 outs	1 outs	2 outs
__	__	__	0.481	0.254	0.098	0.547	0.293	0.113	0.477	0.252	0.094	0.476	0.256	0.098
1B	__	__	0.859	0.509	0.224	0.944	0.565	0.245	0.853	0.504	0.216	0.837	0.507	0.216
__	2B	__	1.100	0.664	0.319	1.175	0.723	0.349	1.102	0.678	0.325	1.094	0.680	0.330
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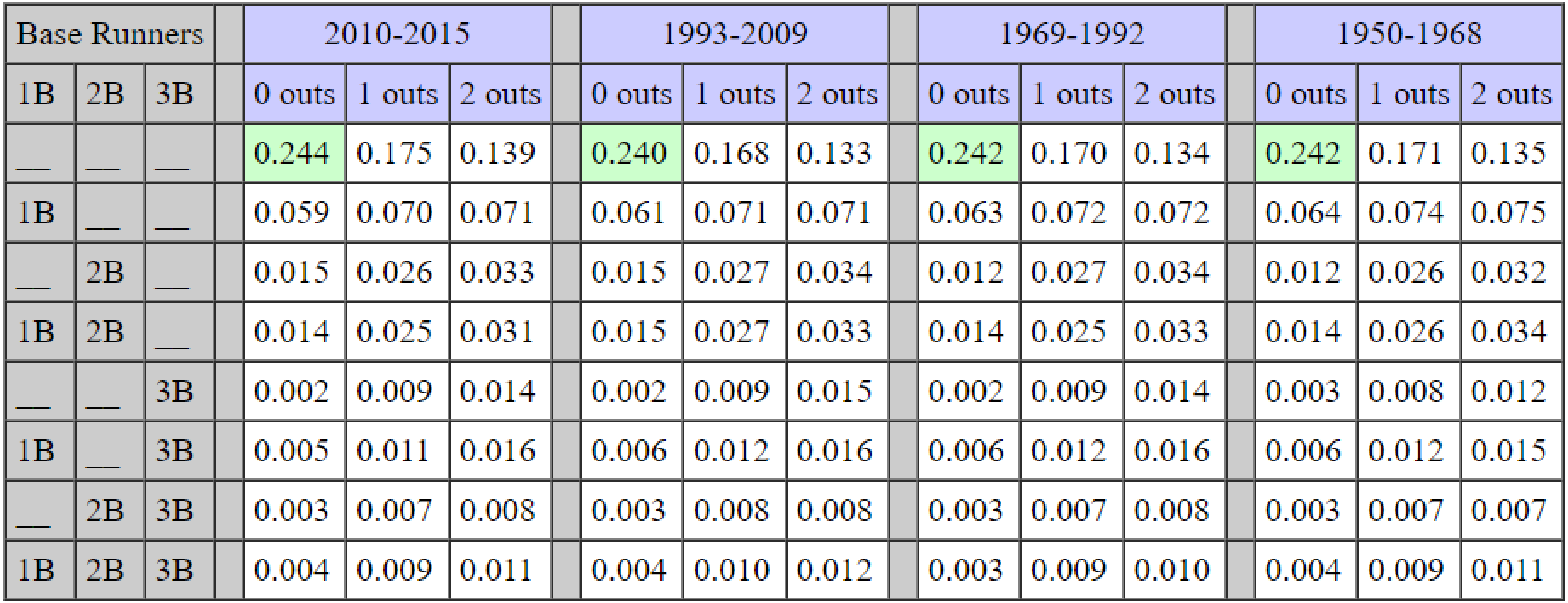
1B	__	3B	1.784	1.130	0.478	1.854	1.216	0.533	1.715	1.149	0.484	1.696	1.151	0.504
__	2B	3B	1.964	1.376	0.580	2.053	1.449	0.626	1.967	1.380	0.594	1.977	1.385	0.620
1B	2B	3B	2.292	1.541	0.752	2.390	1.635	0.815	2.343	1.545	0.752	2.315	1.540	0.747

The following table presents the frequency of plate appearances that started in each base/out state.



Base Runners			2010-2015			1993-2009			1969-1992			1950-1968		
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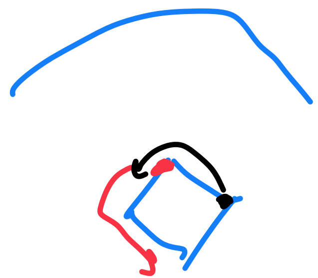
__	2B	__	0.015	0.026	0.033	0.015	0.027	0.034	0.012	0.027	0.034	0.012	0.026	0.032
1B	2B	__	0.014	0.025	0.031	0.015	0.027	0.033	0.014	0.025	0.033	0.014	0.026	0.034
__	__	3B	0.002	0.009	0.014	0.002	0.009	0.015	0.002	0.009	0.014	0.003	0.008	0.012
1B	__	3B	0.005	0.011	0.016	0.006	0.012	0.016	0.006	0.012	0.016	0.006	0.012	0.015
__	2B	3B	0.003	0.007	0.008	0.003	0.008	0.008	0.003	0.007	0.008	0.003	0.007	0.007
1B	2B	3B	0.004	0.009	0.011	0.004	0.010	0.012	0.003	0.009	0.010	0.004	0.009	0.011

all runners same speed

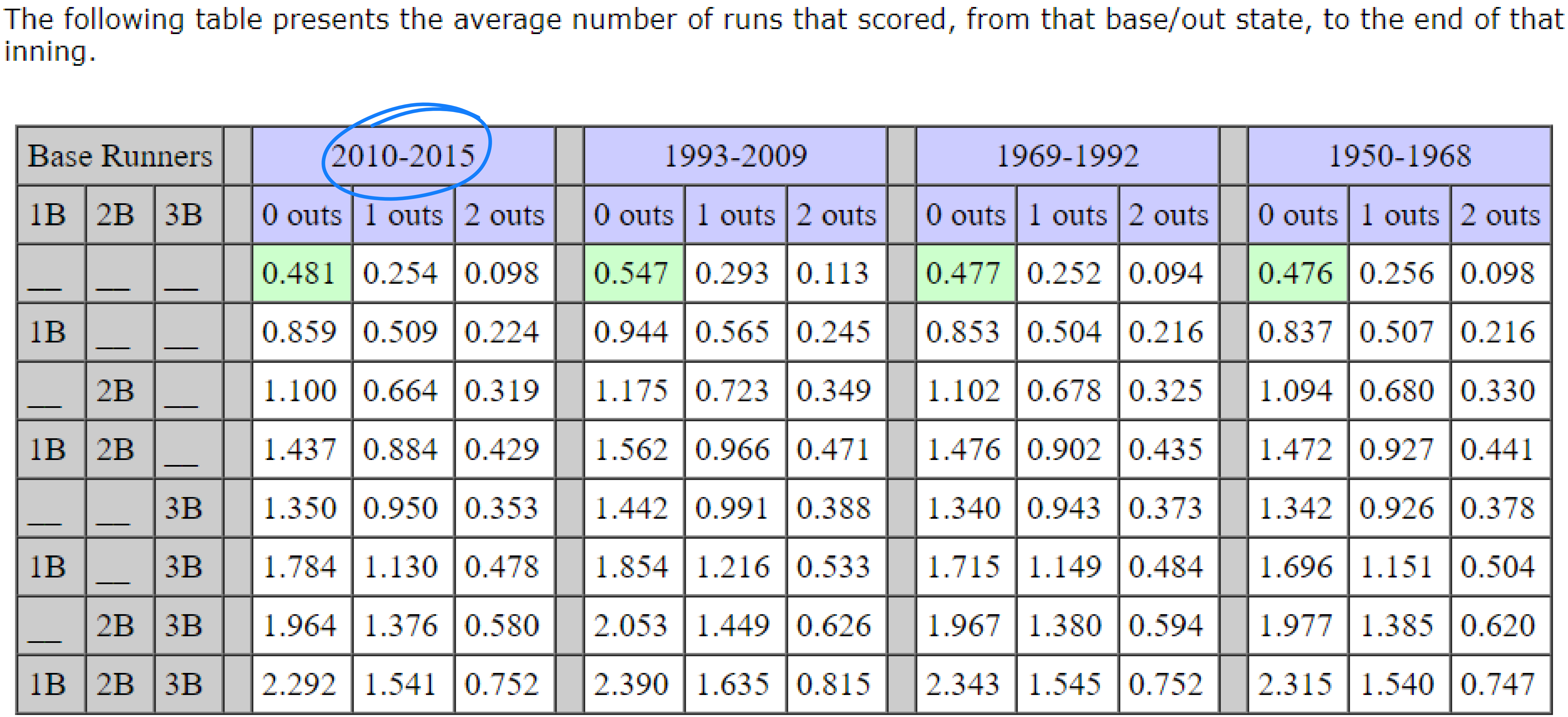
all runners advance same # of bases in simple model
with prob b_1 of 1 base, b_2 of 2 bases, . . .



Eventually do all, do first with fixed $b_1, b_2, \ldots$

Then allow to depend on start/end base

Easiest is 3B vs HR : clear bases



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__	2B	__	1.100	0.664	0.319	1.175	0.723	0.349	1.102	0.678	0.325	1.094	0.680	0.330
1B	2B	__	1.437	0.884	0.429	1.562	0.966	0.471	1.476	0.902	0.435	1.472	0.927	0.441
__	__	3B	1.350	0.950	0.353	1.442	0.991	0.388	1.340	0.943	0.373	1.342	0.926	0.378
1B	__	3B	1.784	1.130	0.478	1.854	1.216	0.533	1.715	1.149	0.484	1.696	1.151	0.504
__	2B	3B	1.964	1.376	0.580	2.053	1.449	0.626	1.967	1.380	0.594	1.977	1.385	0.620
1B	2B	3B	2.292	1.541	0.752	2.390	1.635	0.815	2.343	1.545	0.752	2.315	1.540	0.747

Triple vs HR: difference end with runner on 3rd

no outs: .85 vs 1   if 1 out: .70 vs 1   if 2 outs: .25 vs 1

Triple vs HR: difference end with runner on 3rd

no outs: .85 vs 1   if 1 out: .70 vs 1   if 2 outs: .25 vs 1

extra base is worth 4 times more, but what did you bring in already?

Best Case Triple

Bases loaded, no outs:

Triple: 3.85 vs 4

← ratio $\frac{3.85}{4}$

Worse Case Triple

Bases empty, 2 outs:

Triple: .25 vs 1 ← ratio of 1/4

Slugging

Triple is $\frac{3}{4}$ HR

between reasonable

average

$\frac{4.85/2}{4} = \frac{2.425}{4}$

Math 344: Mathematics of Sports: Spring 2023:
Lecture 09: Modeling Baseball Games: https://youtu.be/FwaqGavXMDI

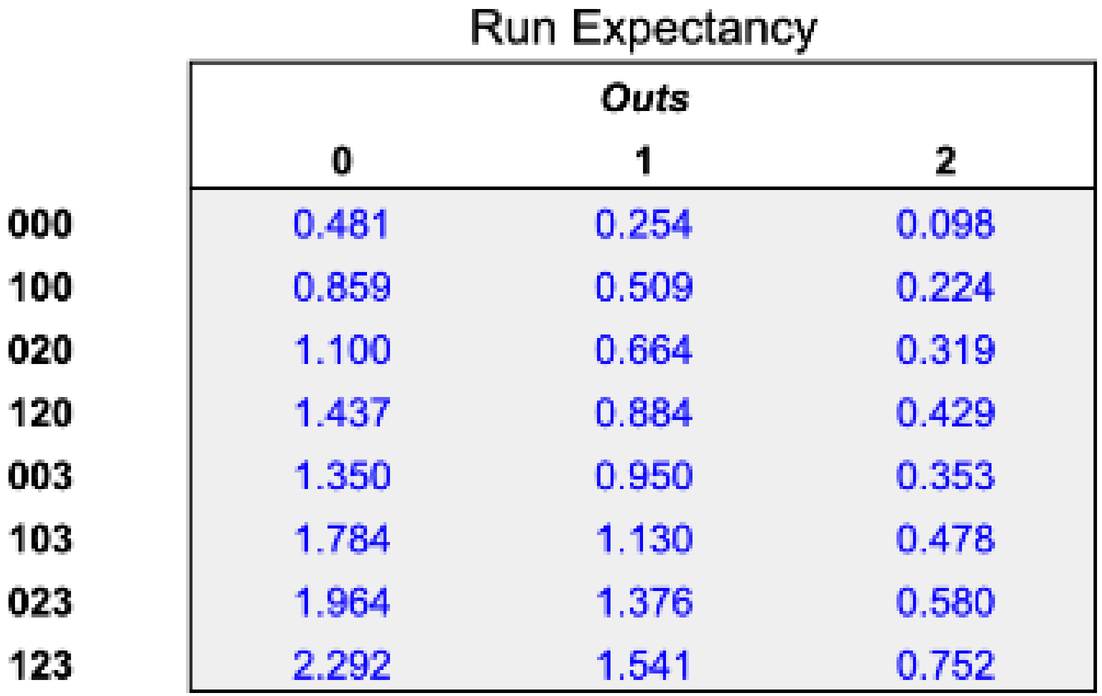
Plan for the day.

- Markov Model for Baseball.
- Relative value of different hits.

Calculations from Alex Cardonick and Jacob Eckerle

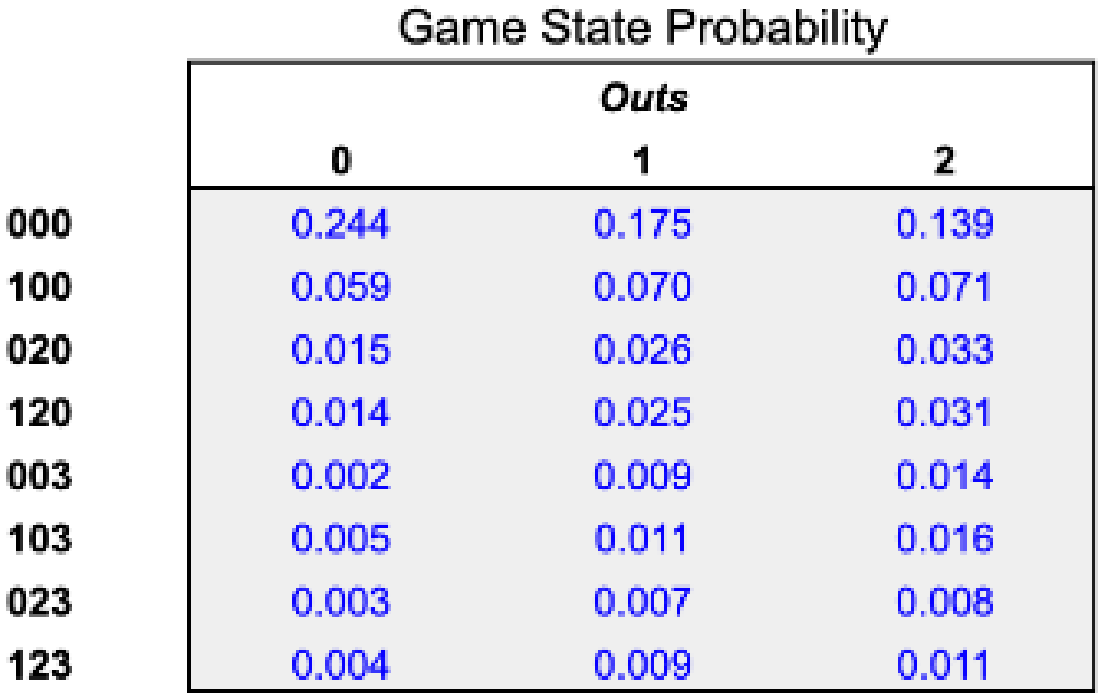
Game State

2010-2015



Run Expectancy

	Outs		
	0	1	2
000	0.481	0.254	0.098
100	0.859	0.509	0.224
020	1.100	0.664	0.319
120	1.437	0.884	0.429
003	1.350	0.950	0.353
103	1.784	1.130	0.478
023	1.964	1.376	0.580
123	2.292	1.541	0.752



Game State Probability

	Outs		
	0	1	2
000	0.244	0.175	0.139
100	0.059	0.070	0.071
020	0.015	0.026	0.033
120	0.014	0.025	0.031
003	0.002	0.009	0.014
103	0.005	0.011	0.016
023	0.003	0.007	0.008
123	0.004	0.009	0.011

See also: Twinkie Town Analytics Fundamentals: Using Linear Weights to Accurately Measure Run Production

Part 4: Run Value Stats: wRC, wRAA, Batting Runs, and wRC+

https://www.twinkietown.com/2020/6/15/21283205/twinkie-town-analytics-fundamentals-come-learn-baseball-with-john-linear-weights-run-value-stats

	Runs Scored
	Triple
000	0
100	1
020	1
120	2
003	1
103	2
023	2
123	3

	Runs Scored
	HR
000	1
100	2
020	2
120	3
003	2
103	3
023	3
123	4

Triple Gain

	Outs		
	0	**1**	**2**
000	0.869	0.696	0.255
100	1.491	1.441	1.129
020	1.250	1.286	1.034
120	1.913	2.066	1.924
003	1.000	1.000	1.000
103	1.566	1.820	1.875
023	1.386	1.574	1.773
123	2.058	2.409	2.601

HR Gain

	Outs		
	0	**1**	**2**
000	1.000	1.000	1.000
100	1.622	1.745	1.874
020	1.381	1.590	1.779
120	2.044	2.370	2.669
003	1.131	1.304	1.745
103	1.697	2.124	2.620
023	1.517	1.878	2.518
123	2.189	2.713	3.346

Triple Gain - Normalized

	Outs		
	0	**1**	**2**
000	0.212	0.122	0.035
100	0.088	0.101	0.080
020	0.019	0.033	0.034
120	0.027	0.052	0.060
003	0.002	0.009	0.014
103	0.008	0.020	0.030
023	0.004	0.011	0.014
123	0.008	0.022	0.029

Expected Triple Gain **1.033**

HR Gain - Normalized

	Outs		
	0	**1**	**2**
000	0.244	0.175	0.139
100	0.096	0.122	0.133
020	0.021	0.041	0.059
120	0.029	0.059	0.083
003	0.002	0.012	0.024
103	0.008	0.023	0.042
023	0.005	0.013	0.020
123	0.009	0.024	0.037

Expected HR Gain **1.420**

Model Triple/HR Value

0.728

SLG Triple/HR Value

0.750

Model HR/Triple Value

1.374

SLG HR/Triple Value

1.333

% Difference

3.08%

Question: If two players have the same
- batting average,
- on-base percentage, and
- slugging percentage,

are they equally valuable?

No OPS = on-base plus slugging

Why: it's implied!

Smartass: Player one: 1 HR, rest walks
Player two: all HRs

Normal Players?

Same # of at-bats and plate appearances (never walked, never HBP, ...)
(or done equally)

hit similarly independent of game state

Compare 2B + 3B vs HR and Single

↳ Same batting average

↳ same on-base

↳ same slugging

Compare n_3 triples vs n_4 HRs and n_1 singles

$3n_3 = 1 \cdot n_1 + 4n_4$ slugging equal

$n_3 = n_1 + n_4$ batting ave equal

$\Longrightarrow 2n_3 = 3n_4$

$n_1 = n_3 - n_4$

Given $n_4 \rightarrow n_3 \rightarrow n_1$

Take $n_4 = 2 \rightarrow n_3 = 3 \rightarrow n_1 = 1$

3 triples vs 2 HRs and a single

Triple Gain

	Outs		
	0	**1**	**2**
000	0.869	0.696	0.255
100	1.491	1.441	1.129
020	1.250	1.286	1.034
120	1.913	2.066	1.924
003	1.000	1.000	1.000
103	1.566	1.820	1.875
023	1.386	1.574	1.773
123	2.058	2.409	2.601

HR Gain

	Outs		
	0	**1**	**2**
000	1.000	1.000	1.000
100	1.622	1.745	1.874
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120	2.044	2.370	2.669
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103	1.697	2.124	2.620
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Triple Gain - Normalized

	Outs		
	0	**1**	**2**
000	0.212	0.122	0.035
100	0.088	0.101	0.080
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120	0.027	0.052	0.060
003	0.002	0.009	0.014
103	0.008	0.020	0.030
023	0.004	0.011	0.014
123	0.008	0.022	0.029

3 triples ≈ 3.1

Expected Triple Gain 1.033

HR Gain - Normalized

	Outs		
	0	**1**	**2**
000	0.244	0.175	0.139
100	0.096	0.122	0.133
020	0.021	0.041	0.059
120	0.029	0.059	0.083
003	0.002	0.012	0.024
103	0.008	0.023	0.042
023	0.005	0.013	0.020
123	0.009	0.024	0.037

2 HRs ≈ 2.84

is a single worth at least .267?

Expected HR Gain 1.420

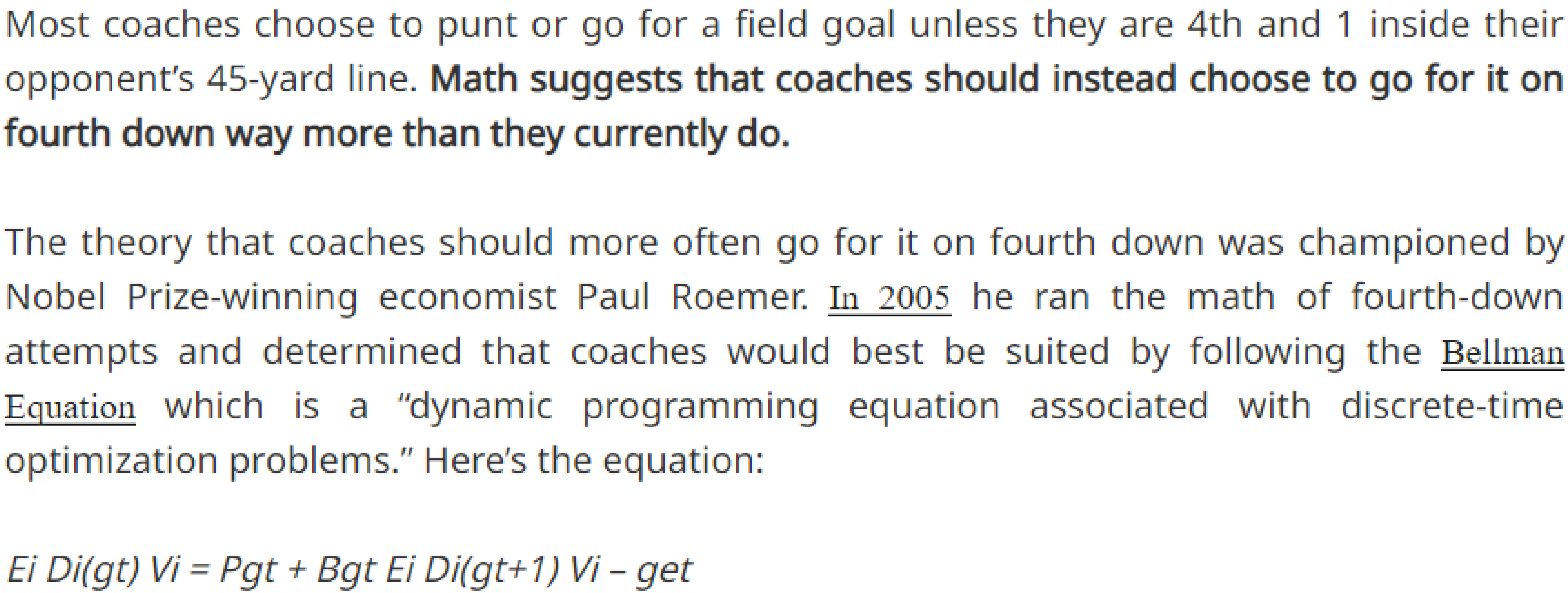
Math 344: Mathematics of Sports: Spring 2023:
Lecture 10: https://youtu.be/deWwJzqK1EQ

Plan for the day.

- Consequences of Position Evaluation.
  (1) Going on fourth down in football.
  (2) Doubling in backgammon.

- Constraint Optimization.

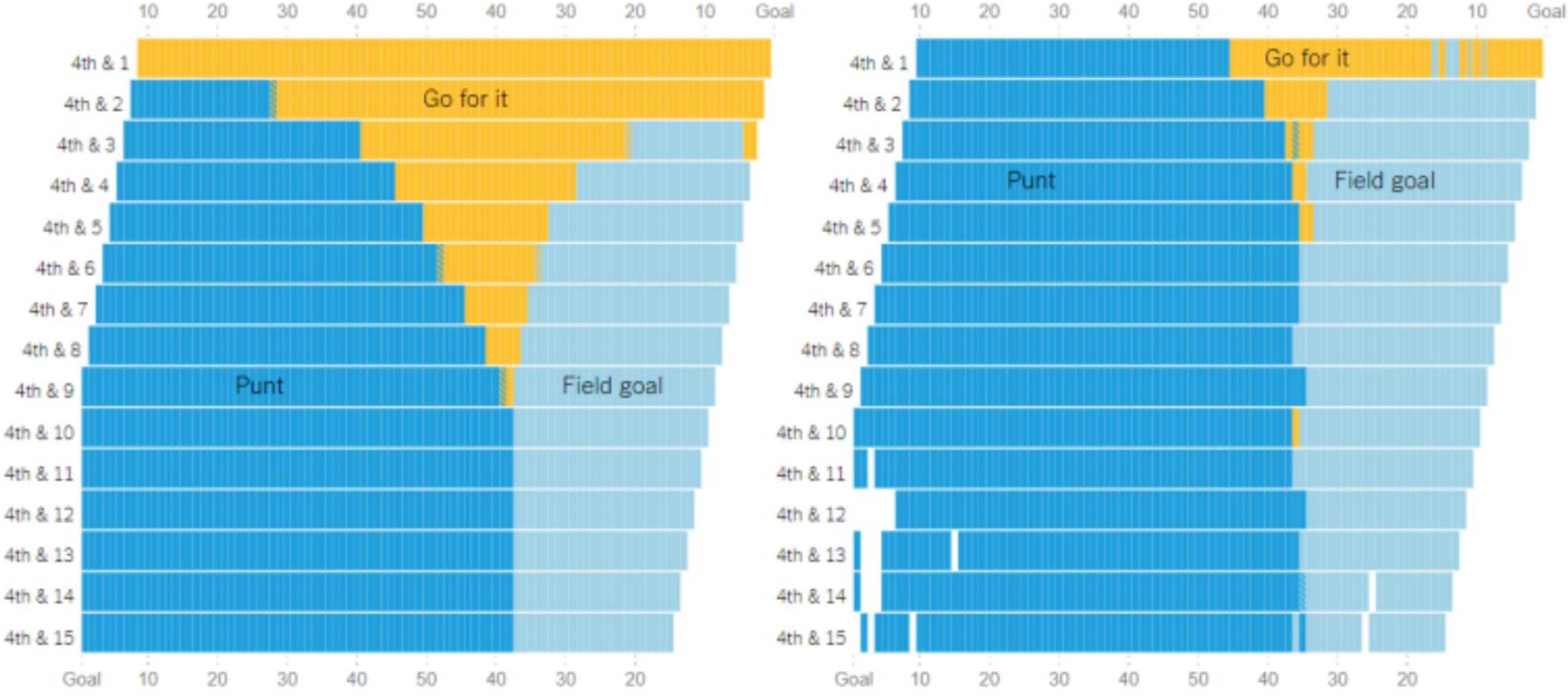
https://www.theifod.com/math-says-go-for-it-on-fourth-down/



Most coaches choose to punt or go for a field goal unless they are 4th and 1 inside their opponent's 45-yard line. **Math suggests that coaches should instead choose to go for it on fourth down way more than they currently do.**

The theory that coaches should more often go for it on fourth down was championed by Nobel Prize-winning economist Paul Roemer. In 2005 he ran the math of fourth-down attempts and determined that coaches would best be suited by following the Bellman Equation which is a "dynamic programming equation associated with discrete-time optimization problems." Here's the equation:

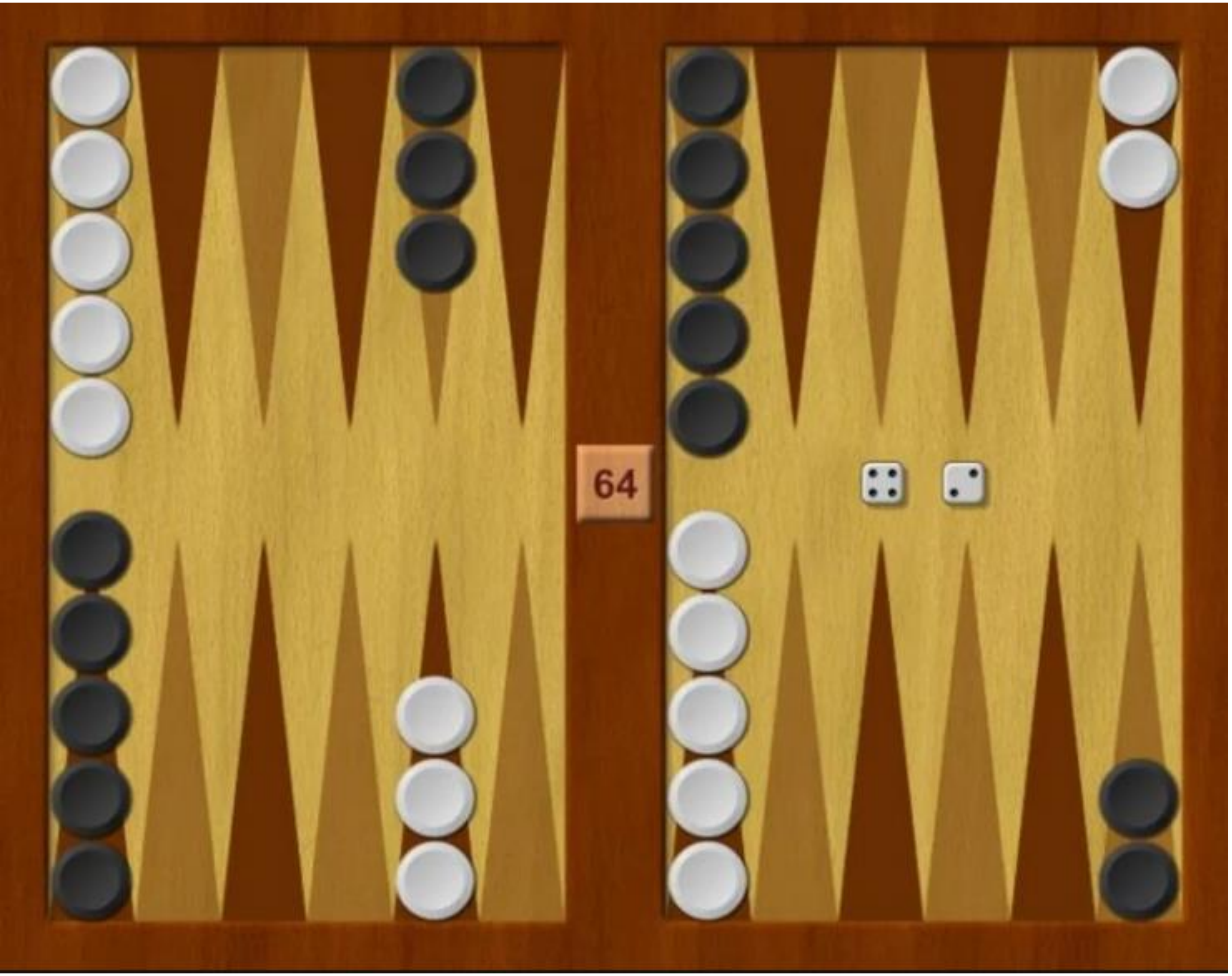
$$Ei\ Di(gt)\ Vi = Pgt + Bgt\ Ei\ Di(gt+1)\ Vi - get$$



WHAT NYT 4TH DOWN BOT RECOMMENDS ON 4TH DOWN

WHAT N.F.L. COACHES DO MOST OFTEN

Backgammon: https://en.wikipedia.org/wiki/Backgammon



https://www.backgammon-rules.com/backgammon-rules/

https://youtu.be/tBz2GIxfYXA

From Wikipedia: https://en.wikipedia.org/wiki/Backgammon

The most recent major development in backgammon was the addition of the doubling cube. Doubles had originally been recorded by placing "common parlour matches" on the bar in the centre of the board.[29] A doubling cube was first introduced in the 1920s in New York City among members of gaming clubs in the Lower East Side.[30] The cube required players not only to select the best move in a given position, but also to estimate the probability of winning from that position, transforming backgammon into the expected value-driven game played in the 20th and 21st centuries.

Given number S (maybe 100)

write S as a sum of positive integers $a_1, \dots, a_n$ st

$$a_1 * a_2 * \cdots * a_n \text{ is as large as possible}$$

I.E., maximize $a_1 \cdots a_n$ given $a_1 + \cdots + a_n = S$ and each $a_k > 0$ integer

Why POSITIVE?

- if an $a_k = 0$ then product is 0
- if negative allowed: $-mS - mS + (2m+1)S$

↳ product is as large as wish

Constraints on the pos integers a_k

$a_1 + \cdots + a_n = S$, each $a_k > 0$ int, $\max(a_1 \cdots a_n)$

- If $a_n = 1$ replace a_{n-1} with $1 + a_{n-1}$, ↑ product, sum same

- If had 6, replace with $4+2$
  7, replace with $5+2$
  8, replace with $6+2$ . . . .
  5, replace with $3+2$
  4, replace with $2+2$

$\left.\begin{matrix}6\\7\\8\\5\end{matrix}\right\}$ improves

$\left.4\right\}$ indifferent

each $a_k \in \{2, 3\}$

Solve $2x = 3y$

Try $x = 3$ get $y = 2$

So $\underbrace{2+2+2}_{\text{product}} = \underbrace{3+3}_{\text{product}}$

$8 < 9$

All 3's and either 0, 1 or 2 two's

Generalize: Each $a_k \geq 0$ is real number

$0 \leq a_k \leq S'$ and $a_1 + \cdots + a_n = S'$, maximize $a_1 \cdots a_n$

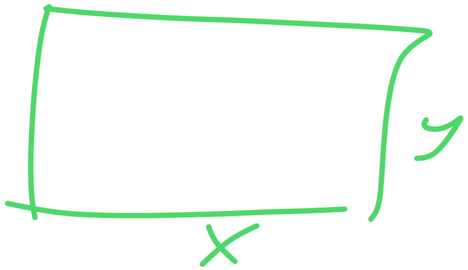
wlog $1 \leq a_k \leq 4$

For each n find best, then find best n.

Case 1: $n=1$: easy! $a_1 = S'$

Case 2: $n=2$: $x+y=S'$ max(xy) = max$(x*(S'-x))$

Fixed semi-perim S'



max area xy

$f(x) = x(S'-x) = S'x - x^2$

$f(0) = f(S') = 0$

$f'(x) = S' - 2x$

$f'(x) = 0 \rightarrow \boxed{x = S'/2}$

$f''(x) = -2$ so max

General n

$a_1^* + a_2^* + a_3^* + \cdots + a_n^* = S$ has largest product

Claim: each is S/n

Proof: If not all equal, make larger by replacing a_i and a_j with $\frac{a_i+a_j}{2}$ and $\frac{a_i+a_j}{2}$

For each n, best is $a_k = S/n$ giving $f(n) = \left(\frac{S}{n}\right)^n$

Find n that is best

$$f(n) = (S'/n)^n$$

Extend: $f(x) = (S'/x)^x = e^{x\ln(S'/x)}$

endpoints: $x=1$, $x=S'$: not best

$\ln(\frac{S'}{x})$

$= \ln S' - \ln x$

$$f'(x) = f(x)\left[\ln(\tfrac{S'}{x}) - x\tfrac{1}{x}\right]$$

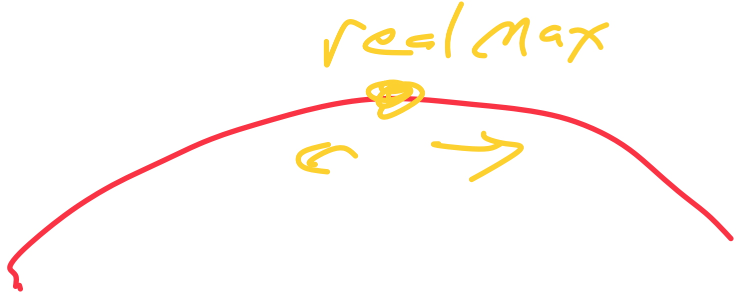
so $f'(x)=0$ means $\ln(S'/x) - 1 = 0$

so $\ln(S'/x) = 1$

so $S'/x = e$ or $x = S'/e$

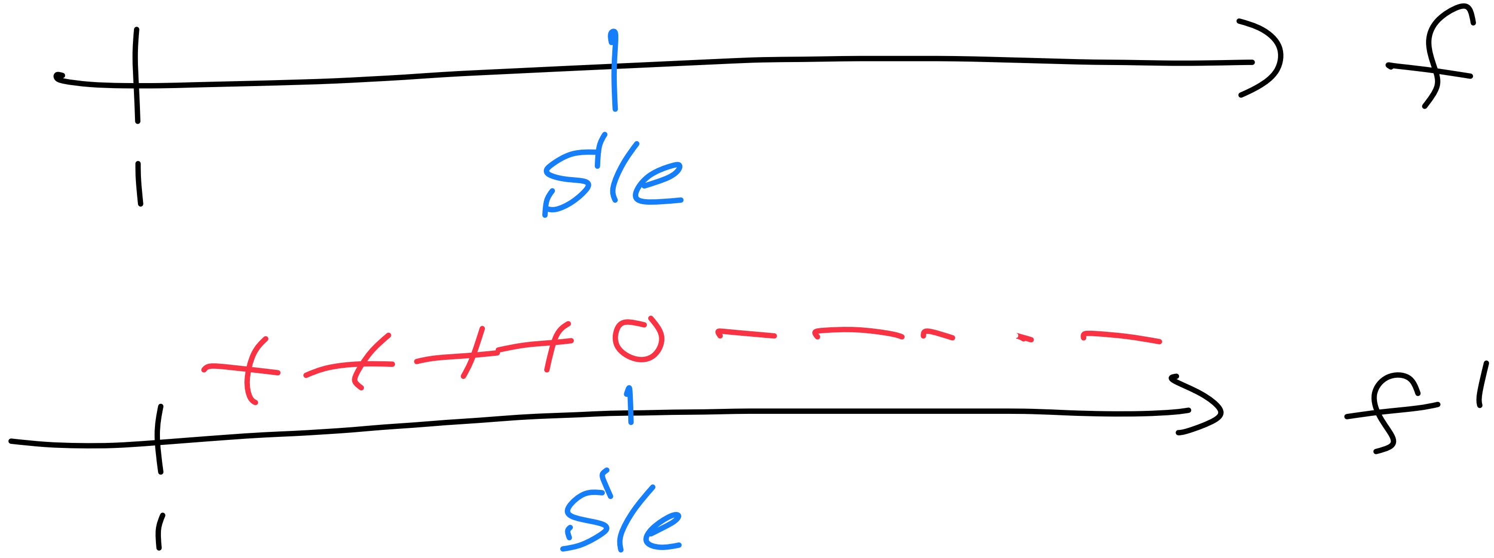
each piece of size $\frac{S}{S/e} = e$

$$f'(x) = f(x) \ln\left(\frac{s'}{xe}\right)$$



integer max is either

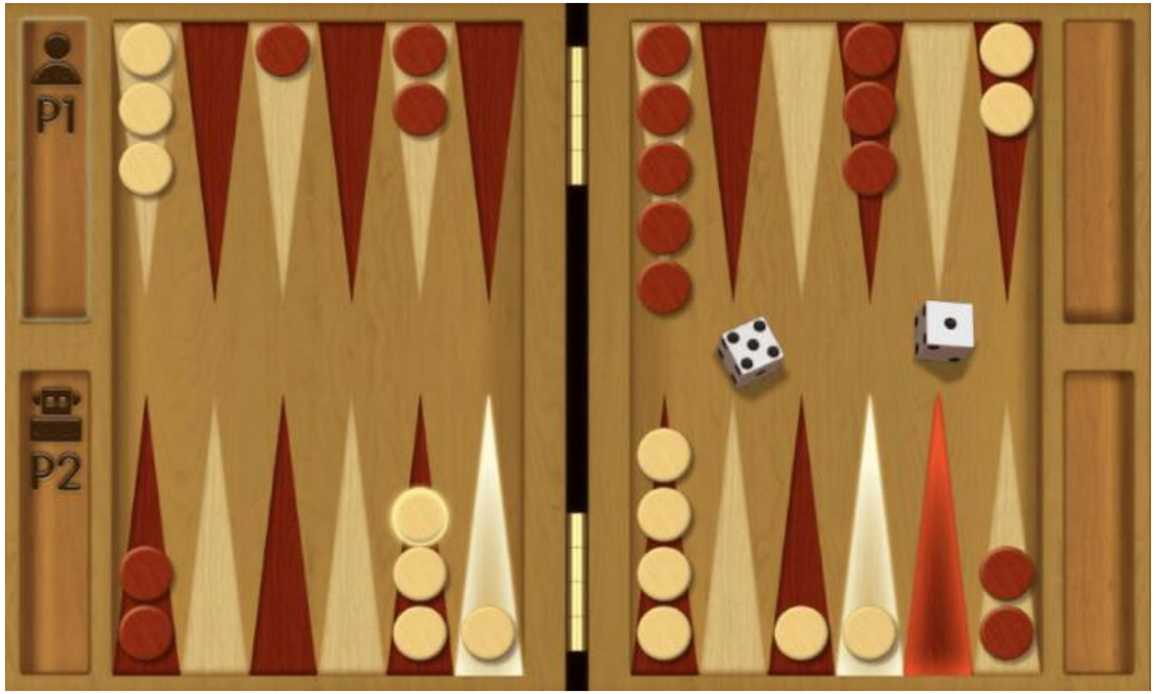
$\lfloor s'/e \rfloor$ or $\lfloor s'/e \rfloor + 1$



Math 344: Mathematics of Sports: Spring 2023:
Lecture 11: Backgammon: Doubling I: https://youtu.be/1-whvEQxaO4

Plan for the day.

- Backgammon: When to double, when to accept?



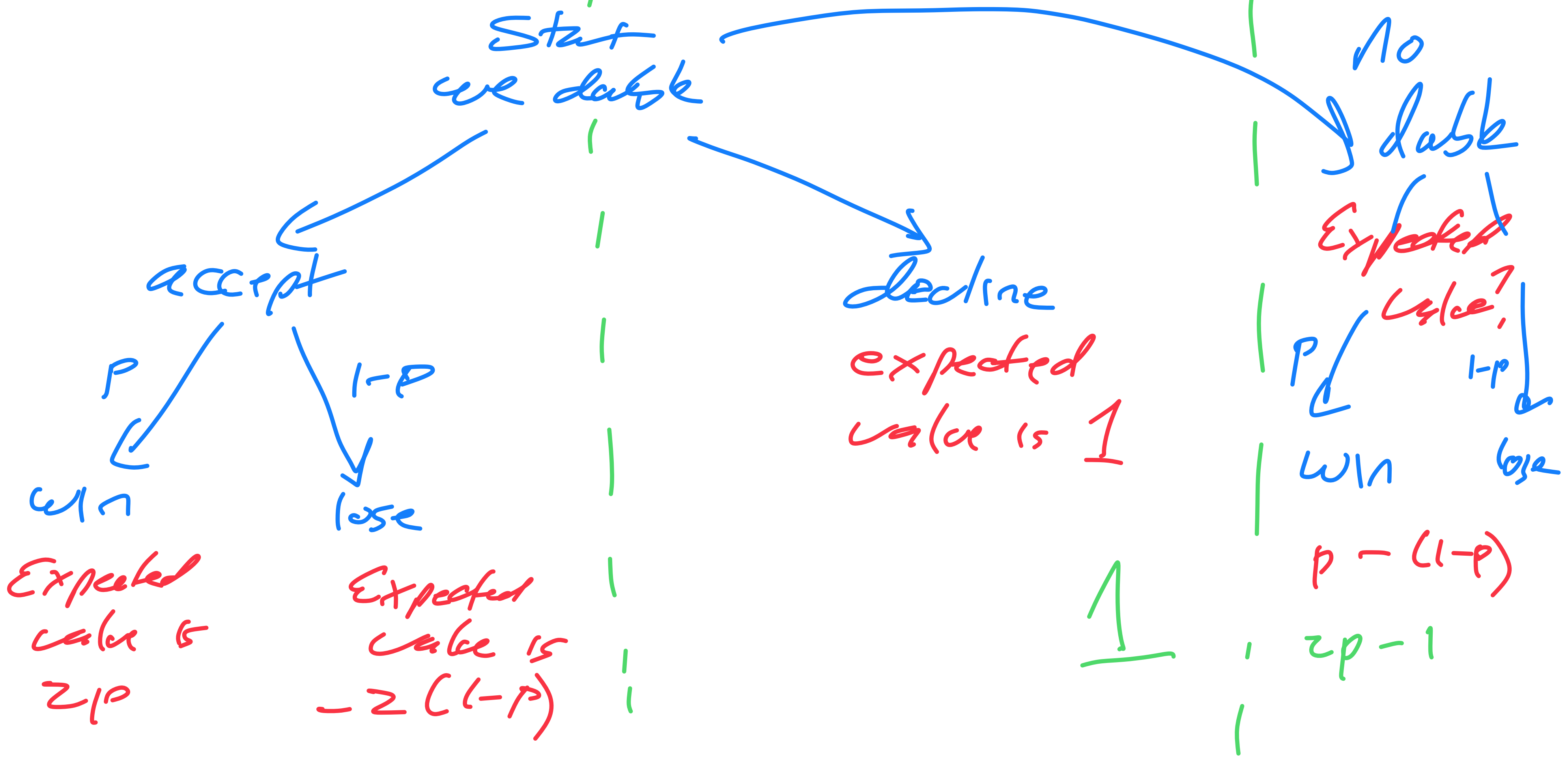
https://image.winudf.com/v2/image/Y29tLmZlbmcuYmFja2dhbW1vbl9zY3JlZW5fM18xNTMxMzMyMDAzXzAzOQ/screen-3.jpg?fakeurl=1&type=.jpg

https://d3d71ba2asa5oz.cloudfront.net/12015220/images/08807_kop.jpg

Only one Double

Double when have prob p of winning

Expected Values



Start
we double

accept

p win

Expected value is $2p$

$1-p$ lose

Expected value is $-2(1-p)$

decline

expected value is 1

1

No double

Expected value?

p win

$1-p$ lose

$p-(1-p)$

$2p-1$

if $p > 1/2$ then $4p-2 > 2p-1$

$4p-2$
$= 2(2p-1)$

	double accepted	double rejected	no double
Expected Value	$2(2p-1)$	1	$2p-1$

Double if $2(2p-1) > 2p-1$ i.e., if $p > 1/2$

↳ if $p=1$ doesn't matter

Note if $p<1$ then $2p-1 < 1$

	double accepted	double rejected	~~no double~~
Expected Value	$2(2p-1)$	1	~~$2p-1$~~

Double if $p > 1/2$

Accepting the double vs rejecting!

our vantage if accept or reject

p = prob that player one wins

$$-2(2p-1) \quad vs \quad -1$$

$$2-4p \quad vs \quad -1$$

$$3 \quad vs \quad 4p$$

$$\frac{3}{4} \quad vs \quad p$$

observe if $\frac{3}{4} > p > \frac{1}{2}$

then $-2(2p-1) > -1$

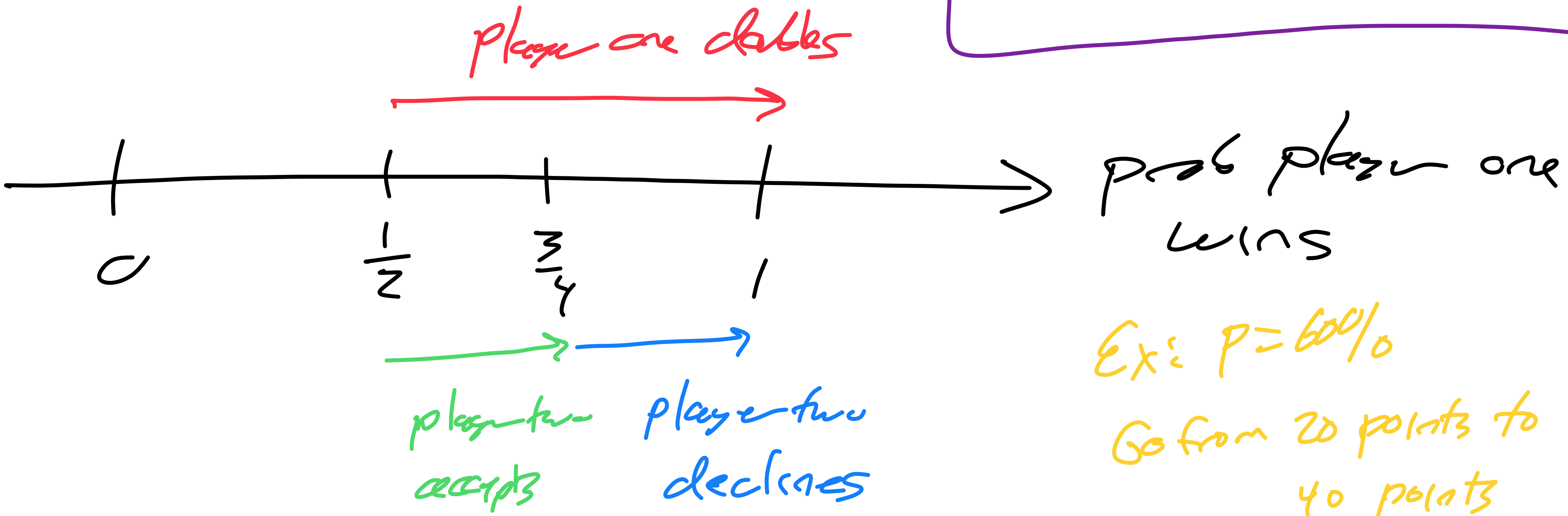
accept if $\frac{1}{2} < p < 3/4$

---

if $\frac{3}{4} < p$ then it is worse, decline if $p > 3/4$

Indifferent if $p = 3/4$

Investigate if a forfeit (decline) is now worth $f \in (0,1)$, not $f = 1$



Ex: $p = 60\%$

Go from 20 points to 40 points

$\frac{1}{2} < p < 3/4$

by doubling player one goes from expected earnings of $p - (1-p) = 2p-1$ to expected $2(2p-1)$, so we gain $2p-1$

Allow at most two doubles, Player One is leading and has prob p of winning.

Say Player One has a 60% chance of winning

$T = P_1$

$H = P_2$

Imagine we have 100 tokens

↳ 60 are H and 40 are T

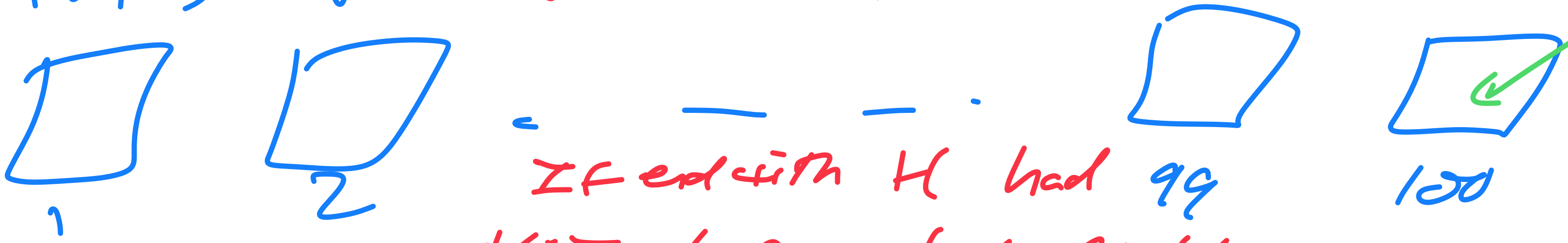
Look at all strings of 60 H's and 40 T's

Have a lot of strings..... Maybe $H_1, \dots, H_{60}$ and $T_1, \dots, T_{40}$

Have 100! strings

If end with a T then had 60 H before we got 40 T.

H or T



Prob(H) = 60%

Prob(T) = 40%

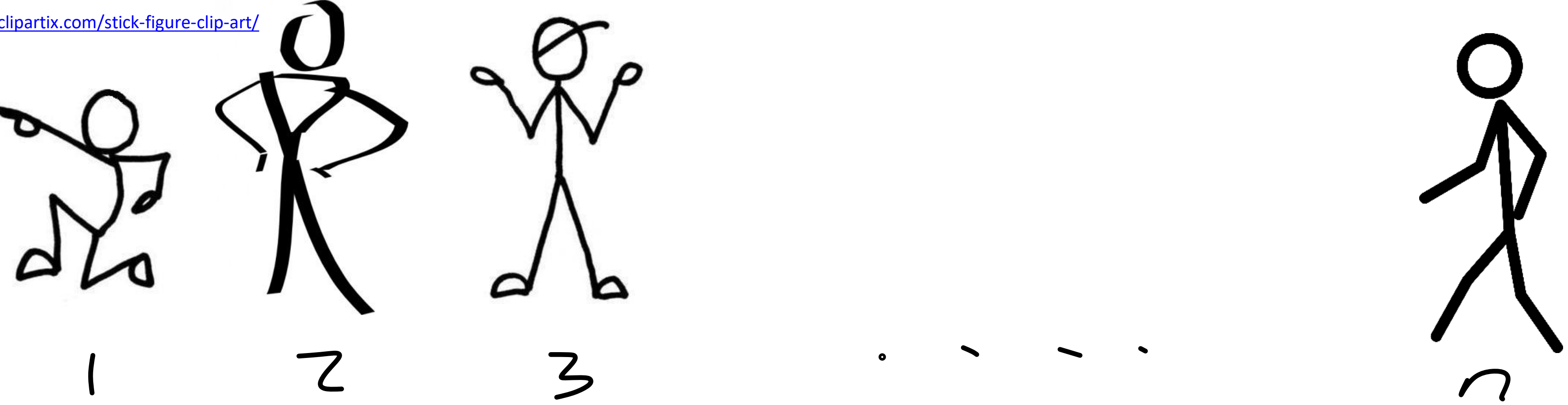
If end with H had 40 T before hit 60 H

Math 344: Mathematics of Sports: Spring 2023:
Lecture 12: Marriage/Secretary Problem: https://youtu.be/f0NwSmtuRlc

Plan for the day.

- State the Marriage/Secretary Problem.
- Discuss Strategies.
- Analyze and find the Optimal.
- Extract lessons for Backgammon Doubling.

https://www.youtube.com/watch?v=dafvzF66vzY https://clipartix.com/stick-figure-clip-art/



See one at a time, when see either offer or decline

↳ if offer → accepted

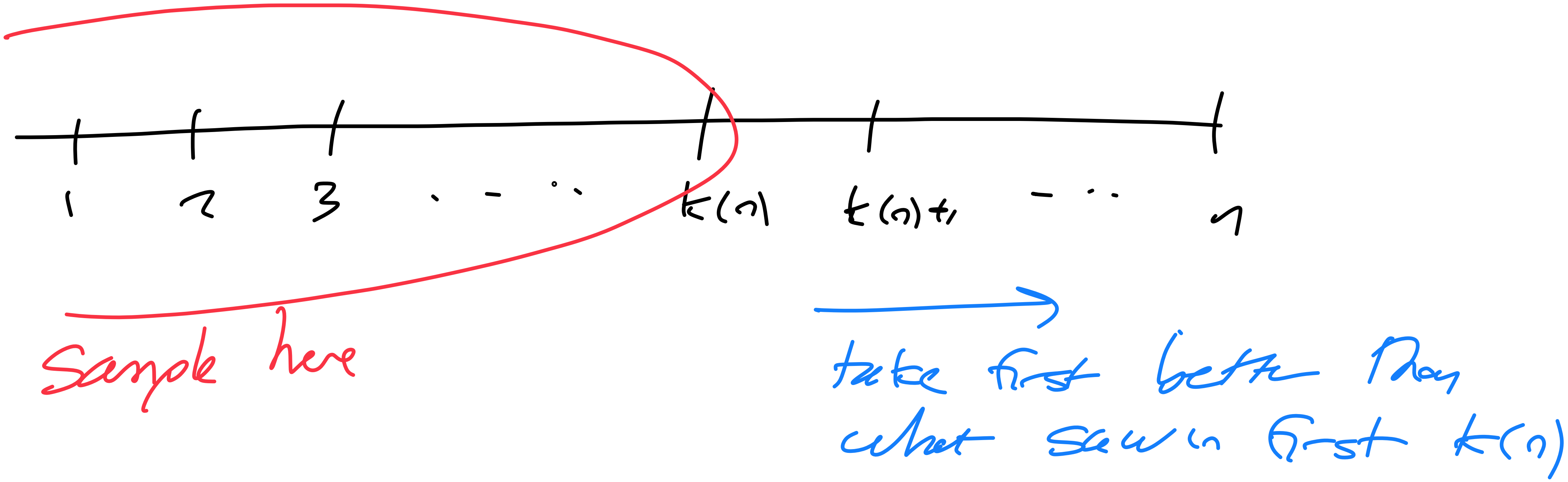
if decline → killed (unavailable)

Goal: Hire the best

Strategy 1: hire first: Prob(best) = $1/n$

if hire k^{th}: Prob(best) = $1/n$

Strategy Build Intuition: Look at first $k = k(n)$, choose first one better than best seen.



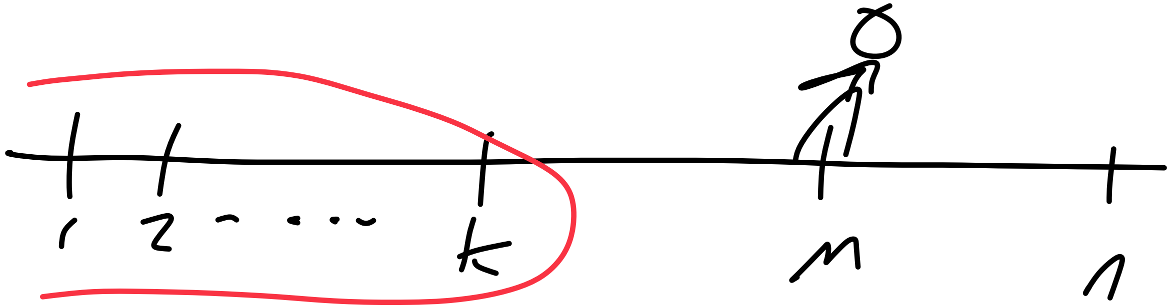
- Will lose if best is in first k(n) happens with probability $\frac{k(n)}{n}$ ← want k(n) small to ↓ this prob
- If 2nd best is in first k(n) and best is not, win.
- Smaller k(n) means more likely to find someone better than first k(n) but not best

← want k(n) ↗

Assume best is at m

If $m \leq k$ Then lose

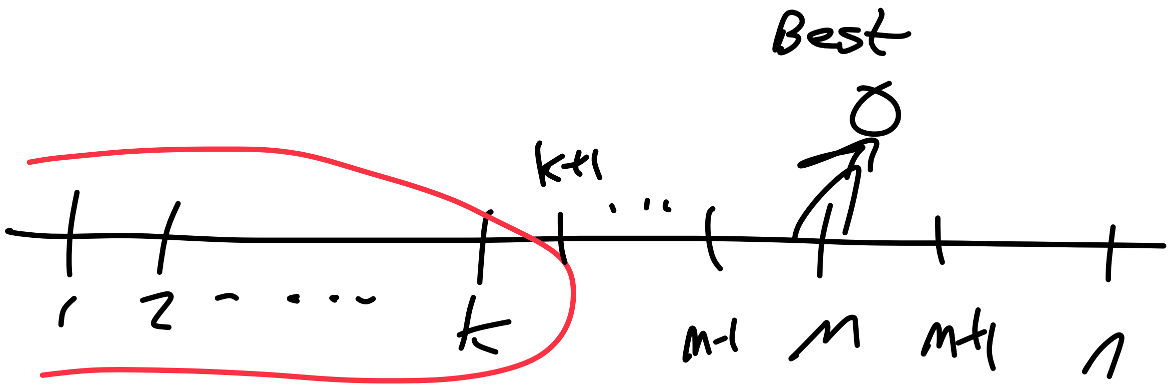
Wlog, assume $k < m \leq n$



$$\text{Prob}(\text{best is at } m) = \frac{1}{n}$$

What is Prob(win | best is at m)?

$$\text{Prob}(\text{win}) = \sum_{m=k+1}^{n} \underbrace{\text{Prob}(\text{win} \mid \text{best at } m)}_{??} \underbrace{\text{Prob}(\text{best at } m)}_{1/n}$$



What is Prob(win | best at m)

Happens if the best of the first m-1 is in the first k

Prob best of first m-1 in first k is $\frac{k}{m-1}$

best of the first m-1 : k slots of m-1 work

$$\text{Prob}(\text{win}) = \sum_{m=k+1}^{n} \underbrace{\text{Prob}(\text{win} \mid \text{best at } m)}_{\frac{k}{m-1}} \underbrace{\text{Prob}(\text{best at } m)}_{1/n}$$

$$= \sum_{m=k+1}^{n} \frac{k}{m-1} \frac{1}{n}$$

$$= \frac{k}{n} \sum_{m=k+1}^{n} \frac{1}{m-1} = \frac{k}{n} \sum_{m=k}^{n-1} \frac{1}{m}$$

$$= \frac{k}{n} \left(\sum_{m=1}^{n-1} \frac{1}{m} - \sum_{m=1}^{k-1} \frac{1}{m} \right)$$

use $\sum_{m=1}^{x} \frac{1}{m} \simeq \log x \pm \text{Const}$

error of size $1/x$

$$= \frac{k}{n} \left[\log(n-1) - \log(k-1) \right] = \frac{k}{n} \log\left(\frac{n-1}{k-1} \right)$$

Maximize $\frac{k}{n} \log\left(\frac{n-1}{k-1}\right)$ ← makes k small

makes k large

"k = 2" $\rightarrow \frac{2}{n} \log(n-1)$ ≈ on the order of $\frac{\log n}{n}$

$k = n \rightarrow 0$

Study $\frac{k}{n} \log\left(\frac{n}{k}\right) \longrightarrow f(x) = \frac{\log x}{x}$ $x = n/k$

Critical Points: $f'(x) = 0$ or $\frac{\frac{1}{x}x - \log x}{x^2} = 0$

So $1 - \log x = 0 \rightarrow x = e$ or $\frac{n}{k} = e$ or

$k = n/e$

Look at first n/e or about 37%, take the first that is better, win with probability

$$\frac{k}{n}\log\left(\frac{n-1}{k-1}\right) \approx \frac{k}{n}\log\left(\frac{n}{k}\right) \quad \text{with } \frac{n}{k}=e$$

$$= \frac{1}{e}\log(e)$$

$$= \frac{1}{e} \approx 37\%$$

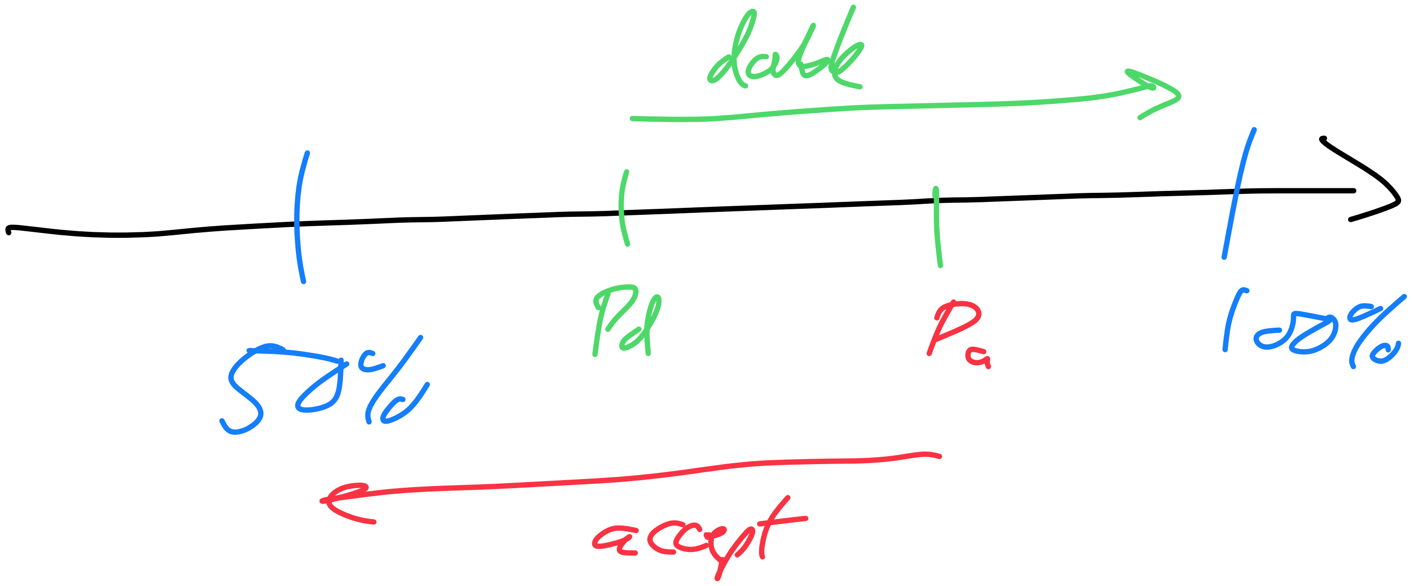
If choose someone better than 1st k but worse than best, at least better than 37% of the people

If have to take the last person could be BAD!

Backgammon

Double if prob of winning is $\geq P_d$,

accept if prob of winning is $\leq P_a$



Marriage/ Secretary Problem

```
secretaryproblem[k_, numpeople_, numdo_] := Module[{},
   (* numdo is the number of times we play, numpeople is number of people, k is where we stop looking to build intuition *)
   success = 0;
   people = {};
   If[k == numpeople, Print["YOU ARE AN IDIOT."]];
   For[p = 1, p ≤ numpeople, p++, people = AppendTo[people, p]];  (* makes list of people *)
   For[n = 1, n ≤ numdo, n++,
    {
     (* the higher the score, the better the person *)
     ordergroup = RandomSample[people, numpeople];
     bestofk = 0;
     For[j = 1, j ≤ k, j++,
      If[ordergroup⟦j⟧ > bestofk, bestofk = ordergroup⟦j⟧]]; (* end of j loop *)
     For[j = k + 1, j ≤ numpeople, j++,
      {
       If[ordergroup⟦j⟧ > bestofk,
         {
          If[ordergroup⟦j⟧ == numpeople, success = success + 1];
          j = numpeople + 100; (* break us out of the j loop *)
         }]; (* end of If statement where found someone better than first k *)
      }]; (* end of j loop *)
    }]; (* end of n loop *)
   Print["If k =approx= numpeople/e expect ", 100. / E];
   Print["We won ", 100. success / numdo, "%."];
  ];
(* end of module *)
```

```
Timing[secretaryproblem[Floor[1000 / E], 1000, 10 000]]
If k =approx= numpeople/e expect 36.7879
We won 50.81%.
{10.1713, Null}
```

Math 344: Mathematics of Sports: Spring 2023:
Lecture 13: Coding

Plan for the day.

- Discuss coding problems.
- Interplay of theory and experiment.
- Try to get a feel for the answer….

Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get. NOTE: $nCr = \binom{n}{r}$

What is the probability you are dealt at least 7 cards in a suit?

It is Prob(exactly one 7 card suit) + … + Prob(exactly one 13 card suit).

$$4C1 * 13C7 * 39C6 \ + \ 4C1 * 13C8 * 39C5 + \cdots + 4C1 * 13C13 * 39C0$$

Can write compactly as $\sum_{k=7}^{13} \binom{4}{1}\binom{13}{k}\binom{39}{13-k}$ = 25,604,567,408.

There are $52C13$ = 635,013,559,600 hands.

Probability at least 7 in a suit is $\frac{25{,}604{,}567{,}408}{635{,}013{,}559{,}600}$ or about .04 (thus 4%).

Low probability, but happens enough that need to be prepared for it!

Trump Splits II: The Bad 5-0 Split

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

What if you and your partner have 8 trump; what are the odds the remaining 5 are all in the same hand?

One solution: There are $2C1 * 5C5 * 21C8 * 13C13 = 406{,}980$.

Thus probability is $406{,}980 / 104{,}006{,}000$ = 9/230 or about .039 (or 3.9%).

Could we say the answer is 2 * $(1/2)^5$ as there are two players who could get all 5, and each card has a 50-50 chance? Note this equals 1/16 or 6.25%. Why is this wrong?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Darth Vader Problem

Only the Emperor is less forgiving than Darth Vader; one mistake and you are dead! No one seems to fail him twice….

If your probability of failing him on a task is p, how many tasks till you die?

The Darth Vader Problem

If your probability of failing him on a task is p,

how many tasks till you die?

Could be unlucky and fail at the first task and die.

Could be very lucky and never fail and live a long, long time….

- What is the probability your first failure is on your first task?
- What is the probability your first failure is on your second task?
- What is the probability your first failure is on your third task?
- What is the probability your first failure is on your n^{th} task?

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP

The Darth Vader Problem

If your probability of failing him on a task is p,

how many tasks till you die?

Could be unlucky and fail at the first task and die.

Could be very lucky and never fail and live a long, long time….

- What is the probability your first failure is on your first task? p
- What is the probability your first failure is on your second task? $(1-p)\ p$
- What is the probability your first failure is on your third task? $(1-p)^2\ p$
- What is the probability your first failure is on your nth task? $(1-p)^{n-1}\ p$

The Darth Vader Problem

If your probability of failing him on a task is p,

how many tasks till you die?

- What is the probability your first failure is on your first task? p
- What is the probability your first failure is on your second task? $(1-p)\ p$
- What is the probability your first failure is on your third task? $(1-p)^2\ p$
- What is the probability your first failure is on your nth task? $(1-p)^{n-1}\ p$

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

Here it is : $1 * Prob(first\ fail\ at\ 1) + 2 * Prob(first\ fail\ at\ 2) + \cdots$

The Darth Vader Problem

If your probability of failing him on a task is p,

how many tasks till you die?

- What is the probability your first failure is on your first task? p
- What is the probability your first failure is on your second task? $(1-p)\ p$
- What is the probability your first failure is on your third task? $(1-p)^2 p$
- What is the probability your first failure is on your nth task? $(1-p)^{n-1} p$

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

Here it is : $1 * p + 2 * (1 - p)p + 3 * (1 - p)2\, p + \cdots \ + n * (1\text{-}p)^{n-1} p + \cdots$

The Darth Vader Problem: LOWER BOUND

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots)$$

Note $p(1 + (1 - p) + (1 - p)2 \ + (1 - p)3 + \cdots) \leq S(p)$

Using the Geometric Series formula with r = 1-p we get $p\frac{1}{1-(1-p)} \leq S(p)$

Gives the useless lower bound of S(p) is at least 1.

The Darth Vader Problem: UPPER BOUND

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots)$$

Note $p(1 + 2(1 - p) + 22(1 - p)2 \ + 23(1 - p)3 + \cdots) \geq S(p)$

If (1-p) < ??? then we can use the geometric series with ratio r = ???.

The Darth Vader Problem: UPPER BOUND

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)2 \ + 4(1-p)3 + \cdots)$$

Note $p(1 + 2(1-p) + 22(1-p)2 \ + 23(1-p)3 + \cdots) \geq S(p)$

If (1-p) < ½ then 2(1-p) < 1 so can use the Geometric Series formula and get $p \frac{1}{1-2(1-p)} \geq S(p)$

For example, if p = ¾ gives an upper bound of 3/2 or 1.5.

The Darth Vader Problem

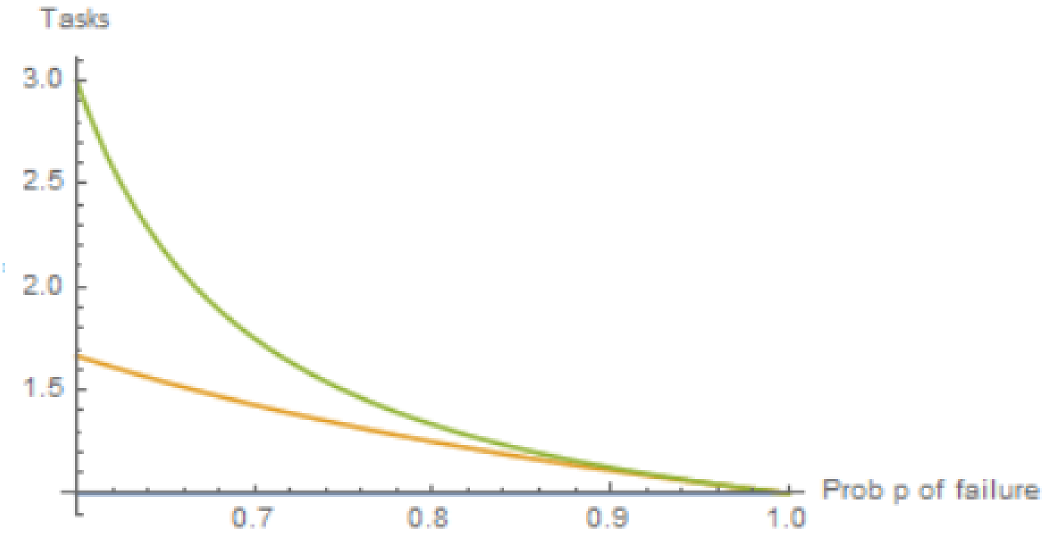
If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)2 \ + 4(1-p)3 + \cdots)$$

Bounds: $If\ (1-p) < \frac{1}{2}\ so\ p > \frac{1}{2}$

Then $1 \ \le S(p) \le \frac{p}{1-2(1-p)}$.



The Darth Vader Problem

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots)$$

Using Calculus one can show S(p) = 1/p; is this formula reasonable?

Look at extreme cases: what happens as p goes to 0 or 1?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Darth Vader Problem

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots)$$

Using Calculus one can show S(p) = 1/p; is this formula reasonable?

Look at extreme cases: what happens as p goes to 0 or infinity?

- As p goes to 1 you are a complete failure, and only do one tasks.
- As p goes to 0 you never fail, and tasks goes to infinity!

The Darth Vader Problem

Probability of failing a task is p, how many tasks till you die?

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots)$$

Let $q = 1\text{-}p$. Note this is $p(1 + 2q + 3q^2 + 4q^3 + \ \cdots)$.

We can rewrite: It is

$p(1 + q + q^2 + q^3 + \ \cdots) + p(q + q^2 + q^3 + \ \cdots) + p\ (q^2 + q^3 + q^4 + \ \cdots) + \cdots$

Each is a geometric series with ratios ???

The Darth Vader Problem

Probability of failing a task is p, how many tasks till you die?

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots)$$

Let $q = 1\text{-}p$. Note this is $p(1 + 2q + 3q^2 + 4q^3 + \cdots)$.

We can rewrite: It is

$p(1 + q + q^2 + q^3 + \cdots) + p(q + q^2 + q^3 + \cdots) + p\,(q^2 + q^3 + q^4 + \cdots) + \cdots$

Each is a geometric series with ratios q, q, q, … but different starting terms.

$S(p) = p\,(1 + q + q^2 + \cdots) + pq\,(1 + q + q^2 + \cdots) \ + pq^2\,(1 + q + q^2 + \cdots) + \cdots$

$S(p) = (p + pq + pq2 + pq3 + \cdots)\,\frac{1}{1-q}$ =

The Darth Vader Problem

Probability of failing a task is p, how many tasks till you die?

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots)$$

Let q = 1-p. Note this is p(1 + 2q + $3q^2$ + $4q^3$ + ⋯).

We can rewrite: It is

p(1 + q + q^2 + q^3 + ⋯) + p(q + q^2 + q^3 + ⋯) + p (q^2 + q^3 + q^4 + ⋯) + ⋯

Each is a geometric series with ratios q, q, q, … but different starting terms.

S(p) = p (1 + q + q^2 + ⋯) + pq (1 + q + q^2 + ⋯) + pq^2 (1 + q + q^2 + ⋯) + ⋯

S(p) = $(p + pq + pq2 + pq3 + \cdots)\,\frac{1}{1-q}$ = $p\ (1 + q + q2 + q3 + \cdots)\,\frac{1}{1-q}$ = $p\,\frac{1}{1-q}\,\frac{1}{1-q}$

Thus S(p) = 1/p as claimed! And without calculus!

Die Another Game

https://youtu.be/tBz2GIxfYXA?t=2

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

The Darth Vader Problem: Review

Probability of failing a task is p, how many tasks till you die?

Answer: Expect 1/p.

Equivalently, if the probability of a success is p, the number of tasks or tries you need before the first success is 1/p.

The Sixes Game

Probability of failing a task is p, how many tasks till you die?

Answer: Expect 1/p.

Equivalently, if the probability of a success is p, the number of tasks or tries you need before the first success is 1/p.

We can use this to study a new game!

The sixes game: you roll a fair die until you get a 6. How many rolls do you expect before this happens?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Sixes Game

Probability of failing a task is p, how many tasks till you die?

Answer: Expect 1/p

Equivalently, if the probability of a success is p, the number of tasks or tries you need before the first success is 1/p.

We can use this to study a new game!

The sixes game: you roll a fair die until you get a 6. How many rolls do you expect before this happens?

Answer: As the probability of rolling a 6 is $p = 1/6$ (all six outcomes are equally likely) we expect it will take 6 rolls.

The Double Sixes Game

You have two fair die.

On each turn you can roll one or both of the die.

The goal is to have both show a 6.

Thus once one of the die lands on a 6 you can stop rolling it.

Questions:

- How many rolls do you expect before you have double sixes?
- What is the probability you win on your first turn? On your second? On your nth?

Can we use the Darth Vader Theorem here? Why or why not?

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP

The Double Sixes Game

You have two fair die.

On each turn you can roll one or both of the die.

The goal is to have both show a 6.

Thus once one of the die lands on a 6 you can stop rolling it.

Questions:

- How many rolls do you expect before you have double sixes?
- What is the probability you win on your first turn? On your second? On your nth?

Can we use the Darth Vader Theorem here? Why or why not?

Hard to use: the difficulty is that our probability of a success is NOT constant; it depends on whether or not we rolled a 6 earlier…. Need a new method.

The Double Sixes Game

You have two fair die.

On each turn you can roll one or both of the die.

The goal is to have both show a 6.

Thus once one of the die lands on a 6 you can stop rolling it.

We will first find the probability of winning after a given number of rolls.

It is easy to find the probability of winning on the first roll: It is 1/36.

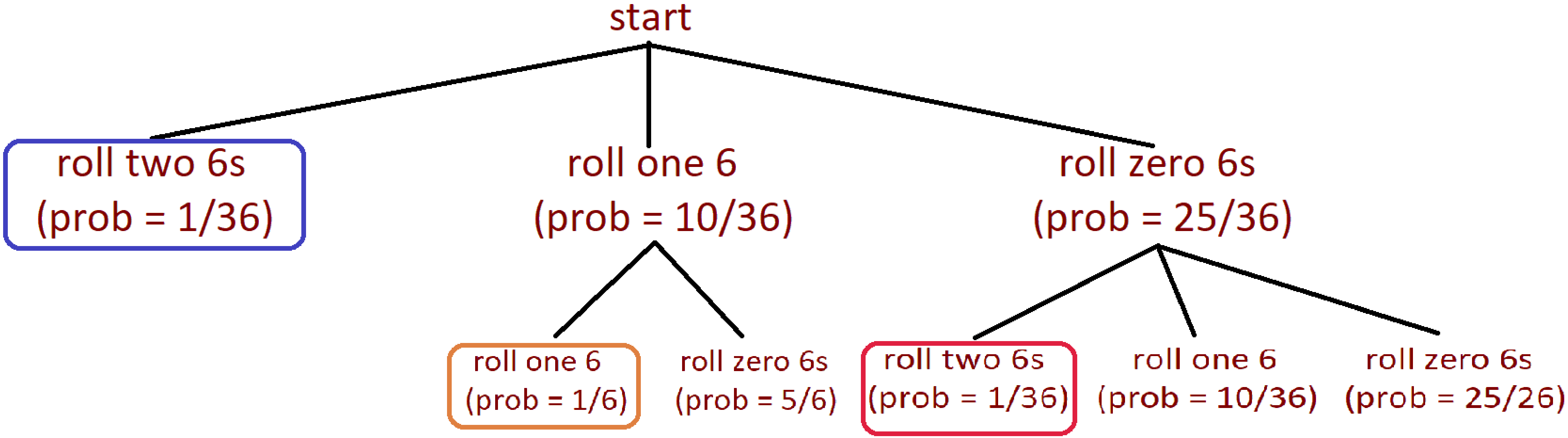
What is the probability you win on the second roll? It is

$10/36 * 1/6 + 25/36 * 1/36$. But why???

The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.



start

roll two 6s (prob = 1/36)

roll one 6 (prob = 10/36)

roll zero 6s (prob = 25/36)

roll one 6 (prob = 1/6)

roll zero 6s (prob = 5/6)

roll two 6s (prob = 1/36)

roll one 6 (prob = 10/36)

roll zero 6s (prob = 25/26)

Prob(win first roll) = 1/36. Prob(win second roll) = 10/36*1/6 + 25/36*1/36 = 85/1296

Great Probability Results

We can continue the analysis, but there are more and more branches as we go down.

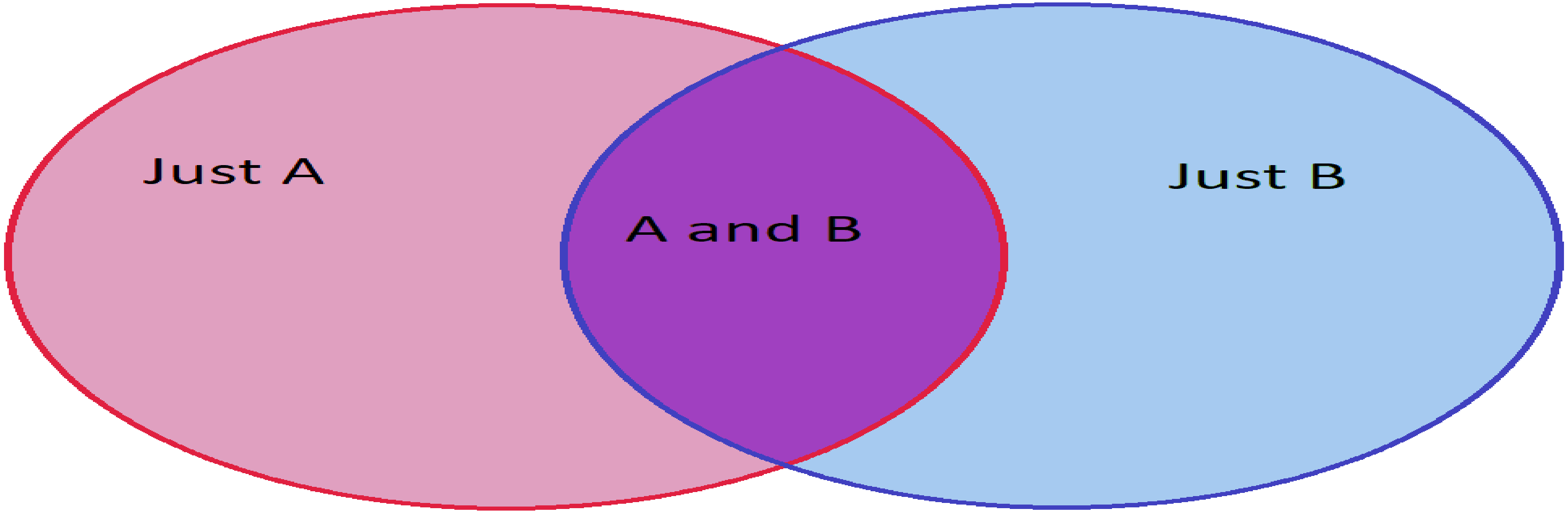
We introduce a WONDERFUL idea in probability:

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

Great Probability Results

We introduce another WONDERFUL idea in probability:

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).



The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

Want both to show a 6. Once one of the die lands on a 6 you can stop rolling it.

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

What is the probability we win by the n^{th} turn?

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is $(5/6)^n + (5/6)^n - (25/36)^n$.

Where did this come from? It is the probability the first die is never a 6 PLUS the probability the second is never a six, MINUS the probability neither die is ever a 6 (we must subtract as we we DOUBLE COUNTED that that probability).

The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

What is the probability we win **BY** the nth turn? $1 - 2*(5/6)^n + (25/36)^n$.

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is $(5/6)^n + (5/6)^n - (25/36)^n$.

So…, what is the probability we win **ON** the nth turn?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

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The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

What is the probability we win BY the nth turn? $1 - 2*(5/6)^n + (25/36)^n$.

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is $(5/6)^n + (5/6)^n - (25/36)^n$.

So…, what is the probability we win ON the nth turn?

It is the probability we win BY the nth turn MINUS the probability we win BY the (n-1)st turn! $(1 - 2*(5/6)^n + (25/36)^n) - (1 - 2*(5/6)^{n-1} + (25/36)^{n-1})$

The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

What is the probability we win BY the nth turn? $1 - 2*(5/6)^n + (25/36)^n$.

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is $(5/6)^n + (5/6)^n - (25/36)^n$.

So…, what is the probability we win ON the nth turn?

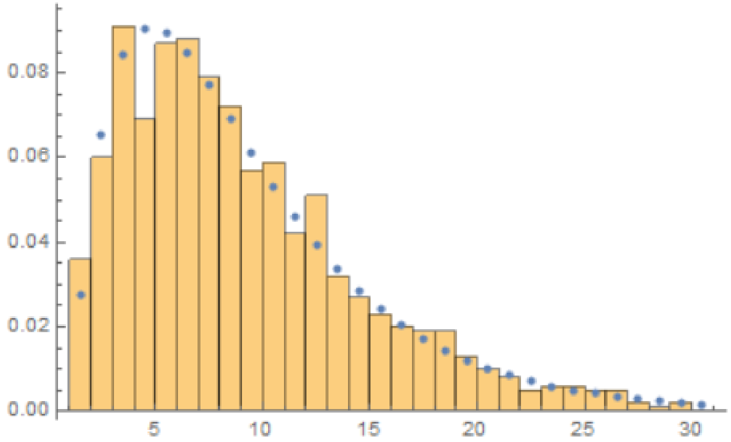
It is the probability we win BY the nth turn MINUS the probability we win BY the (n-1)st turn! $(2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1}$.

The Double Sixes Game

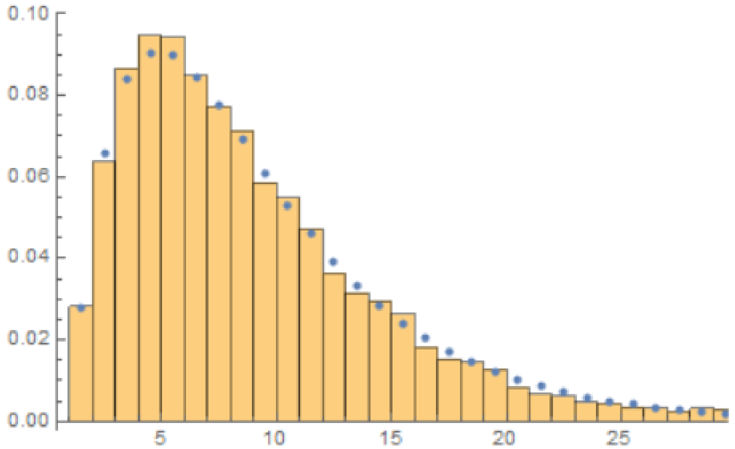
You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.

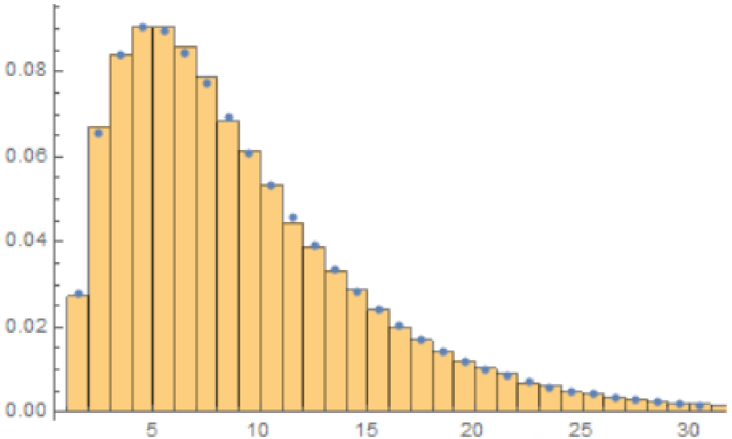
Probability win on nth turn: $(2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1}$.



100 trials



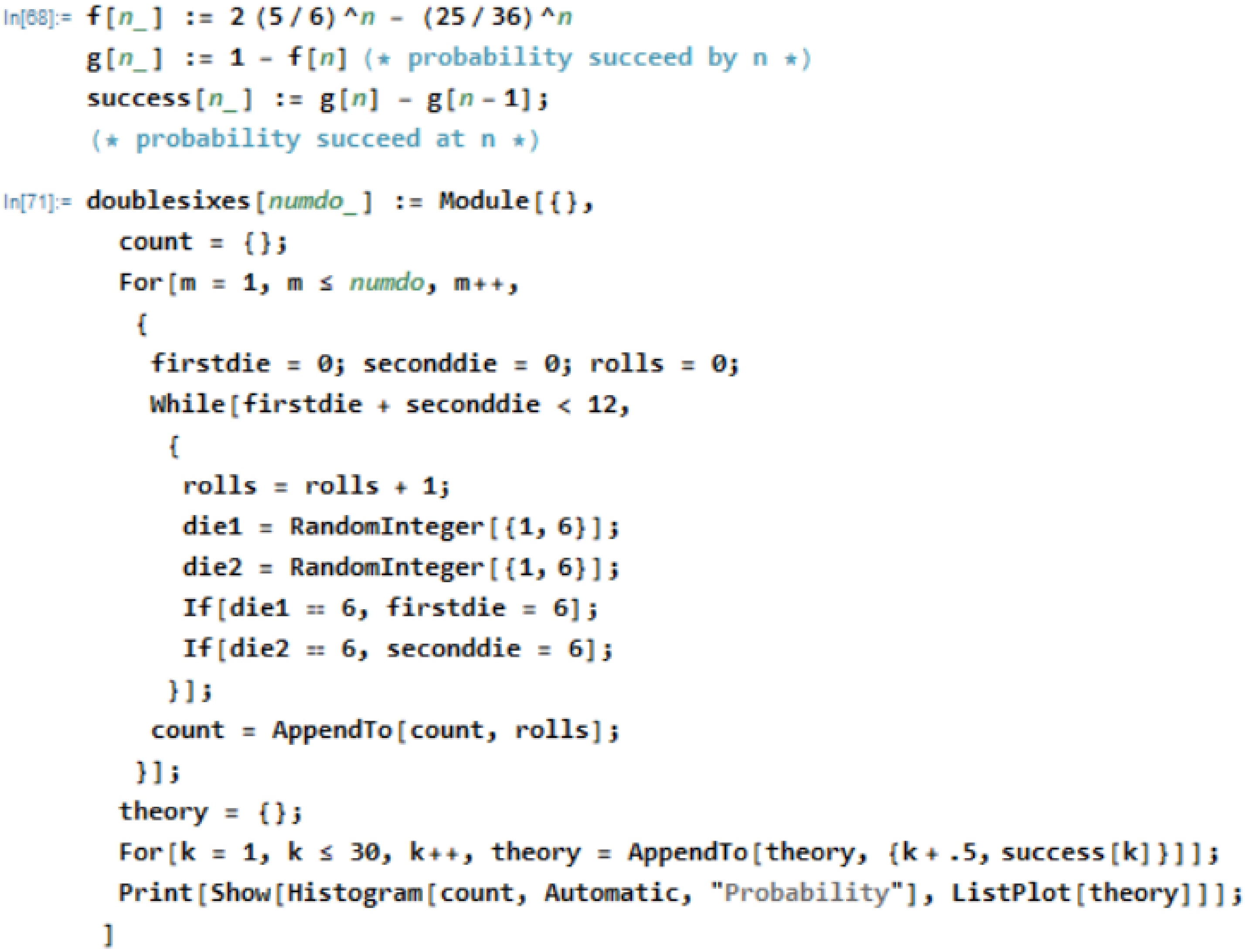
10,000 trials



100,000 trials

The Double Sixes Game: Code

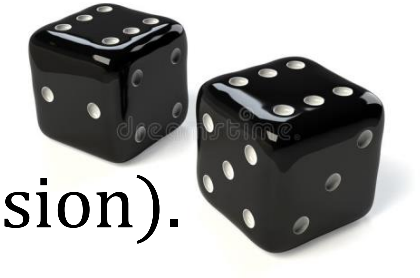
Mathematica code to simulate



```
In[68]:= f[n_] := 2 (5 / 6) ^n - (25 / 36) ^n
         g[n_] := 1 - f[n] (* probability succeed by n *)
         success[n_] := g[n] - g[n - 1];
         (* probability succeed at n *)

In[71]:= doublesixes[numdo_] := Module[{},
           count = {};
           For[m = 1, m ≤ numdo, m++,
            {
             firstdie = 0; seconddie = 0; rolls = 0;
             While[firstdie + seconddie < 12,
              {
               rolls = rolls + 1;
               die1 = RandomInteger[{1, 6}];
               die2 = RandomInteger[{1, 6}];
               If[die1 == 6, firstdie = 6];
               If[die2 == 6, seconddie = 6];
              }];
             count = AppendTo[count, rolls];
            }];
           theory = {};
           For[k = 1, k ≤ 30, k++, theory = AppendTo[theory, {k + .5, success[k]}]];
           Print[Show[Histogram[count, Automatic, "Probability"], ListPlot[theory]]];
          ]
```

The Double Sixes Game: Expected Value



Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2\ + 4(1 - p)3 + \cdots) = 1$/p.

The Double Sixes Game: Expected Value

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The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
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sum of n * Prob(takes exactly n rolls), n from 1 to infinity.

As Prob(takes exactly n rolls) = $(2/6)(5/6)^{n-1} – (11/36)(25/36)^{n-1}$.

Notation: $\sum_{n=1}^{\infty} an \quad means \ \ a1 + a2 + a3 + \cdots$ (using a Greek Sigma for Sum)

We have $\sum_{n=1}^{\infty} n((2/6)(5/6)n-1 - (11/36)(25/36)n-1)$.

First term: $\frac{2}{6}\left(1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + 4\left(\frac{5}{6}\right)^3 + \cdots\right)$

The Double Sixes Game: Expected Value

Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \; + 4(1 - p)3 + \cdots) = 1/\text{p}$.

Notation: $\sum_{n=1}^{\infty} an$ $\;$ means $\;a1 + a2 + a3 + \cdots$ (using a Greek Sigma for Sum)

We have $\sum_{n=1}^{\infty} n((2/6)(5/6)n{-}1 - (11/36)(25/36)n{-}1)$.

Equals $\frac{2}{6}\sum_{n=1}^{\infty} n(5/6)n{-}1$ - $\frac{11}{36}\sum_{n=1}^{\infty} n(25/36)n{-}1$.

Each looks a lot like the Darth Vader Theorem – need to adjust a bit. What should p be for the first? For the second?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Double Sixes Game: Expected Value

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We have $\sum_{n=1}^{\infty} n((2/6)(5/6)n{-}1 - (11/36)(25/36)n{-}1)$.

Equals $\frac{2}{6}\sum_{n=1}^{\infty} n(5/6)n{-}1$ - $\frac{11}{36}\sum_{n=1}^{\infty} n(25/36)n{-}1$.

Each looks a lot like the Darth Vader Theorem – need to adjust a bit. What should be for the first?

p = 1/6 (want 1-p = 5/6)

For the second? p = 11/36 (want 1-p = 25/36)

The Double Sixes Game: Expected Value

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The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots) = 1/\text{p}$.

We have $\sum_{n=1}^{\infty} n((2/6)(5/6)n{-}1 - (11/36)(25/36)n{-}1)$.

Equals $\frac{2}{6}\sum_{n=1}^{\infty} n(5/6)n{-}1$ - $\frac{11}{36}\sum_{n=1}^{\infty} n(25/36)n{-}1$.

Equals $2 * \frac{1}{6}\sum_{n=1}^{\infty} n(1 - 1/6)n{-}1$ - $\frac{11}{36}\sum_{n=1}^{\infty} n(1 - 11/36)n{-}1$.

What is the first term? What is second?

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP

The Double Sixes Game: Expected Value

Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore: $S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots) = 1/\mathrm{p}$.

We have $\sum_{n=1} n((2/6)(5/6)n{-}1 - (11/36)(25/36)n{-}1)$.

Equals $\frac{2}{6}\sum_{n=1}^{\infty} n(5/6)n{-}1$ - $\frac{11}{36}\sum_{n=1}^{\infty} n(25/36)n{-}1$.

Equals 2 $* \frac{1}{6}\sum_{n=1}^{\infty} n(1 - 1/6)n{-}1$ - $\frac{11}{36}\sum_{n=1}^{\infty} n(1 - 11/36)n{-}1$.

What is the first term? $2 * \frac{1}{1/6}$ What is second? $\frac{1}{11/36}$. Answer is ….

The Double Sixes Game: Expected Value

Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore: $S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots) = 1/\text{p}$.

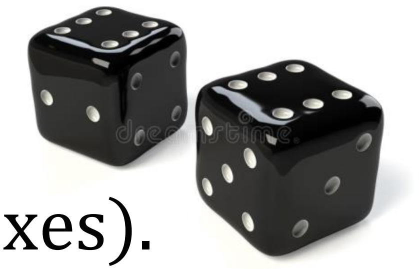
We have $\sum_{n=1} n((2/6)(5/6)n-1 - (11/36)(25/36)n-1)$.

Equals 2 $* \frac{1}{6}\sum_{n=1}^{\infty} n(1 - 1/6)n-1$ - $\frac{25}{11}\,\frac{11}{36}\sum_{n=1}^{\infty} n(1 - 11/36)n-1$.

What is the first term? $2 * \frac{1}{1/6}$ What is second? $\frac{1}{11/36}$.

Answer is $2 * 6 \ - \frac{36}{11} \ = \frac{96}{11}$ (or about 8.7 rolls until you get both sixes).

The Double Sixes Game: Expected Value



Answer is $2 * 6 - \frac{36}{11} = \frac{96}{11}$ (or about 8.7 rolls until you get both sixes).

Is this answer reasonable? Are you surprised by it? What tests can you do to see if it makes sense? What lower or upper bounds can you find?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Double Sixes Game: Expected Value

Answer is $2 * 6 - \frac{36}{11} = \frac{96}{11}$ (or about 8.7 rolls until you get both sixes).

Is this answer reasonable? Are you surprised by it? What tests can you do to see if it makes sense?

In the six game (roll one die, stop when you get a 6) we saw the expected number of rolls is 6; as we now need TWO 6s, reasonable that it takes LONGER, and 6 is a LOWER BOUND.

If we played the six game twice (roll the first die until we get a 6, then start rolling the second die till we get a 6) expect to need 12 rolls. Thus 12 should be an UPPER BOUND. (Actually, can improve to 11 as an upper bound….)

Review: Big Takeaways

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)2 \ + 4(1 - p)3 + \cdots) = 1/$p.

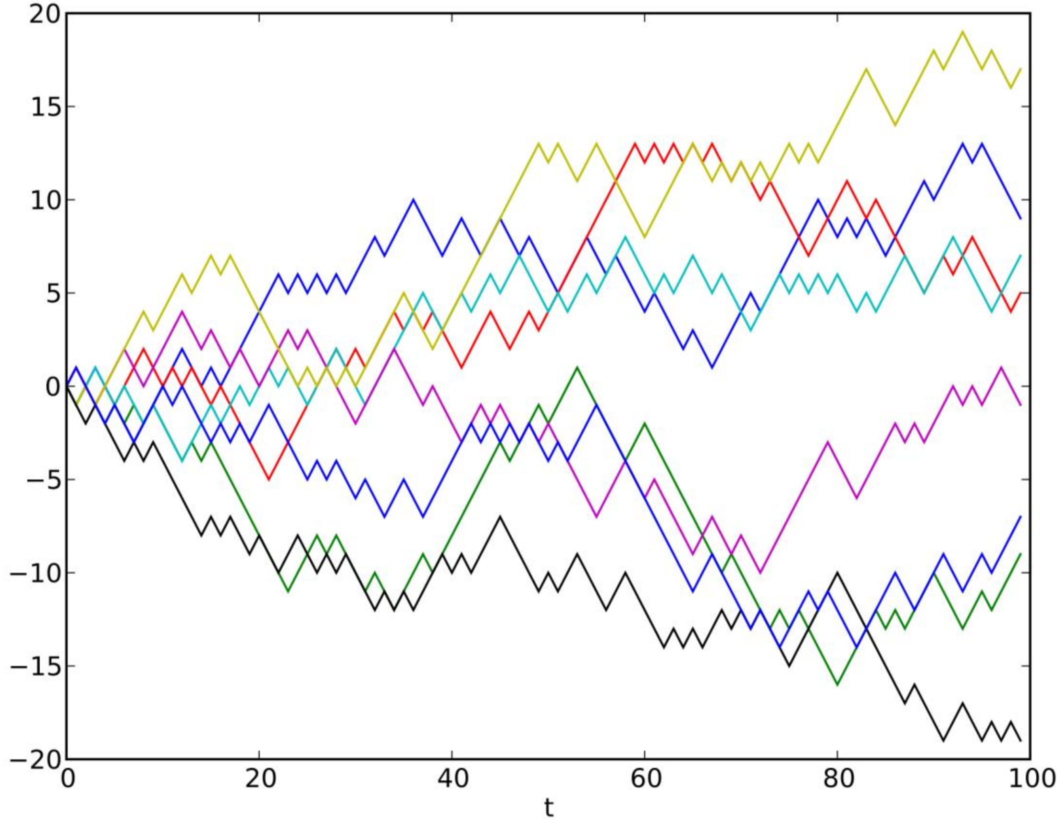
The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen:
Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

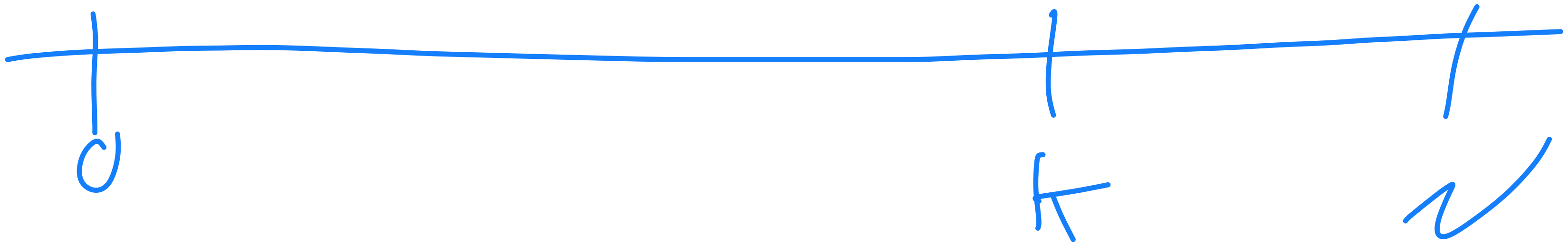
The Power of Algebra: Sometimes have to do a bit of algebraic manipulations to make what you have look like something you know.

Math 344
Lecture 14
Coding and Backgammon

https://youtu.be/00to2-PsHa0



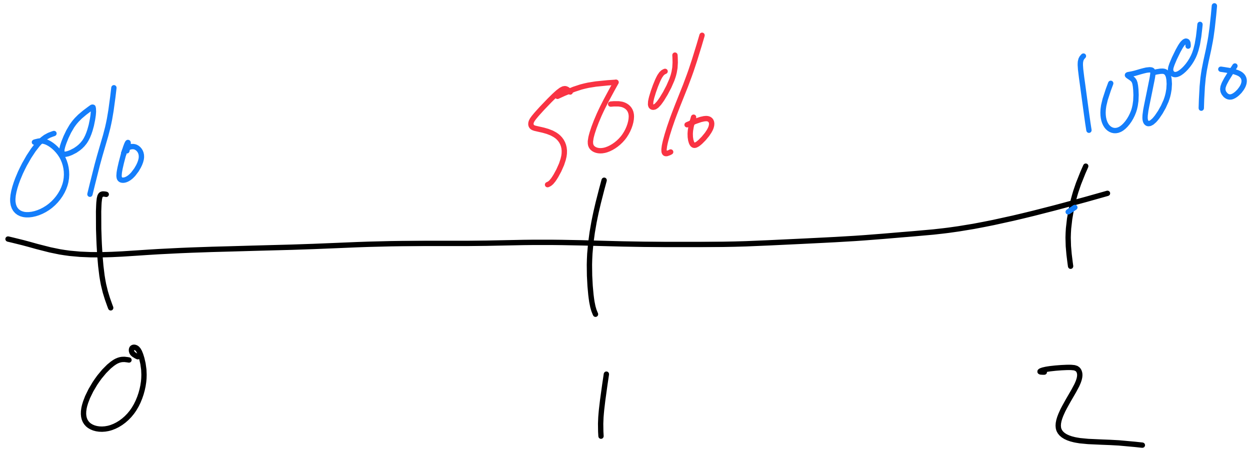
https://r.search.yahoo.com/_ylt=AwrEsrnEzghkreADXwSjzbkF;_ylu=c2VjA2ZwLWF0dHJpYgRzbGsDcnVybA--/RV=2/RE=1678327620/RO=11/RU=http%3a%2f%2fwww.alanzucconi.com%2f2015%2f09%2f09%2funderstanding-the-gaussian-distribution%2f/RK=2/RS=3z8IH4vbOMbhm1FX_Q79ohTfDCk-



Given at K and each turn go up 1 with prob 1/2 or down 1 with prob 1/2, what is the prob hit N before 0.

Conjecture: $\frac{K}{N}$

Easy N?, $N = 2$ ($N = 1$ already over)

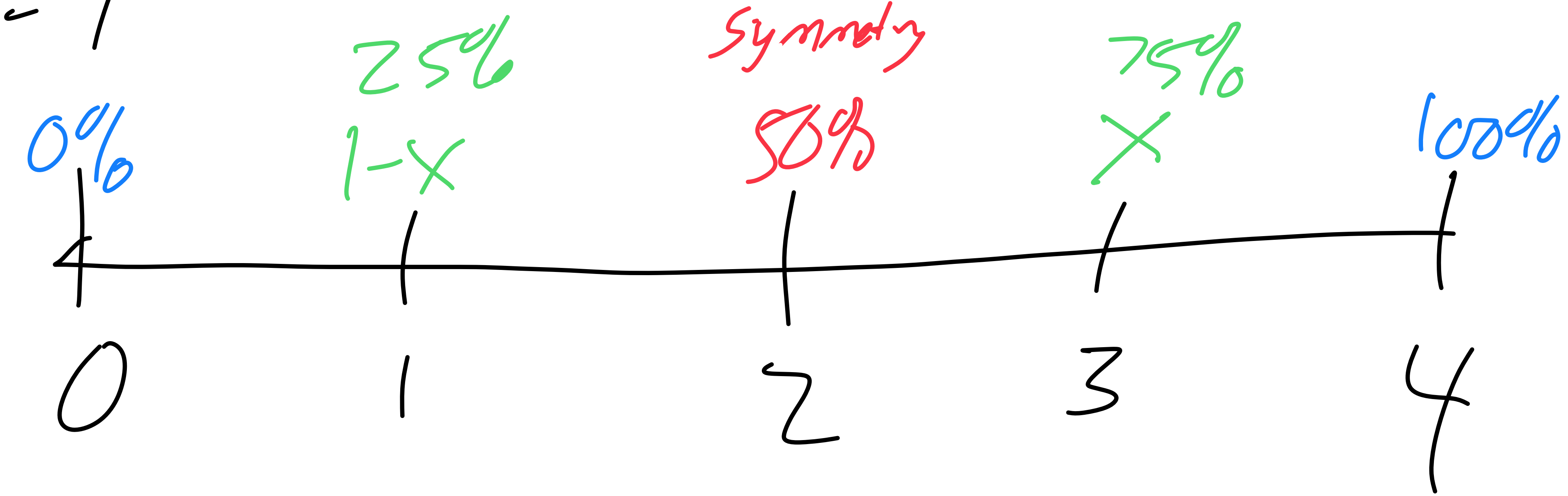


Game ends in one toss: win with prob $1/2$, lose with $1/2$

OR: Symmetry!

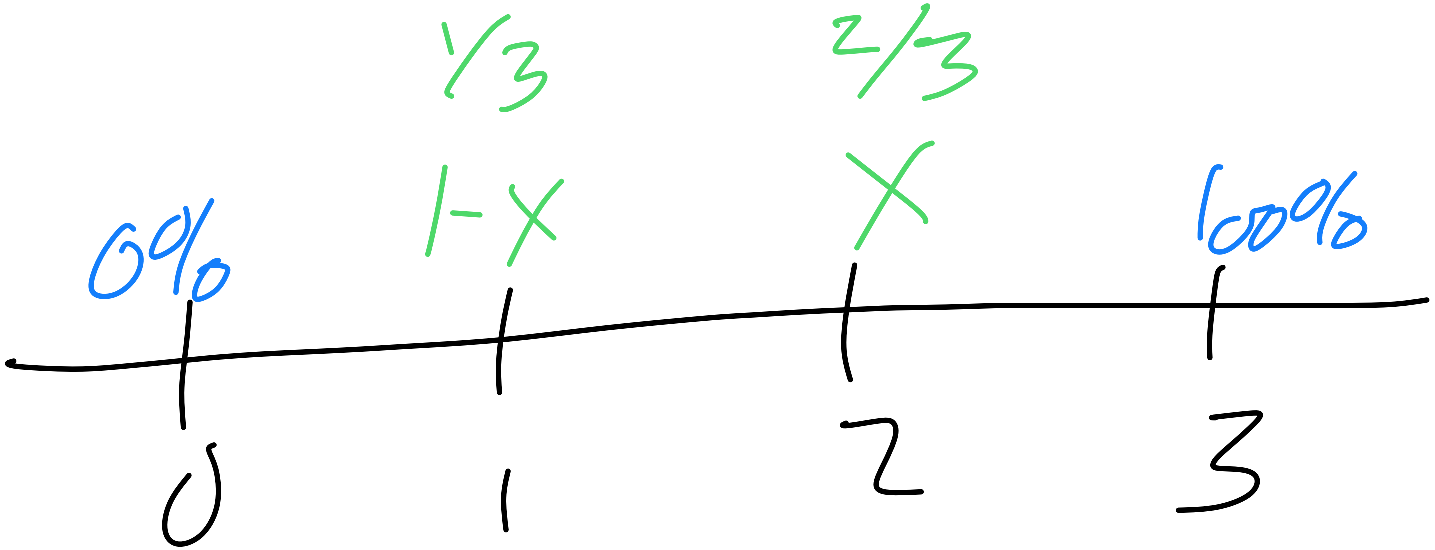
Did 1, 2, next is... 3 or 4

$N = 4$



$$x = \underset{(\text{at } 4)}{\frac{1}{2}\, 1} + \underset{(\text{at } 2)}{\frac{1}{2}\, \frac{1}{2}} = \frac{3}{4} = 75\%$$

N=3



$$X = \frac{1}{2} 1 + \frac{1}{2}(1-X)$$

(at 3) (at 1)

$$X = 1 - \frac{1}{2}X \rightarrow \frac{3}{2}X = 1 \rightarrow X = 2/3$$

$N = 5$ or $N = 8$

$$x = \underset{\substack{(a+8)\\ \text{win}}}{\frac{1}{2}\,1} + \underset{(a+6)}{\frac{1}{2}\,\frac{6}{8}} = \frac{8+6}{16} = \frac{7}{8}$$

Homework for Wednesday, do all N

Math 344: Mathematics of Sports: Spring 2023:

Lecture 16: Guest Speaker

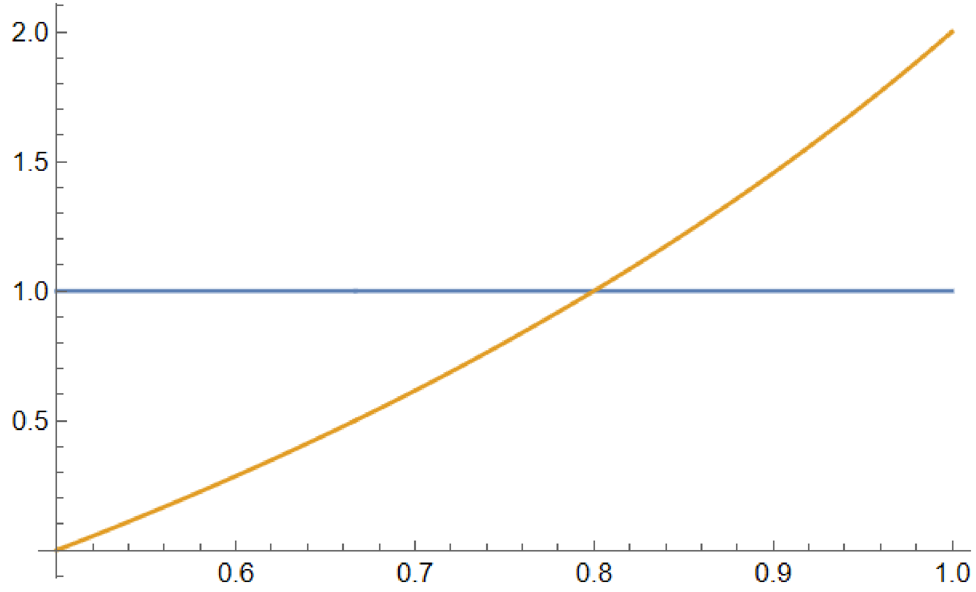
Lecture 17: Finishing Backgammon Doubling: https://youtu.be/UkTrEvBeIPo

Plan for the day.

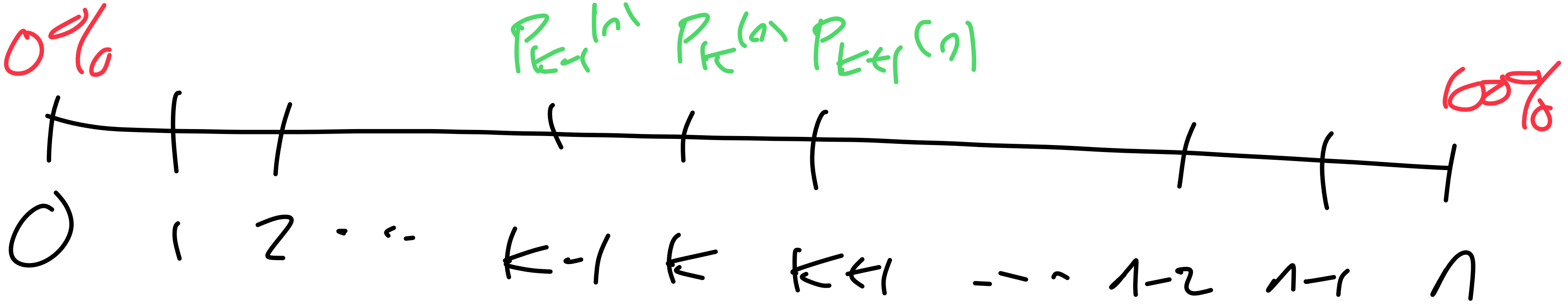
- Backgammon Doubling: Theory and Coding

https://tse3.mm.bing.net/th?id=OIP.1OEv_w_SVogPsuRAaCfZDQHaEc&pid=Api&P=0

```
expectedvaluedoubling[p_] := (12 p^2 - 14 p + 4) / (-3 p^2 + 8 p - 4);
Plot[{1, expectedvaluedoubling[p]}, {p, .5, 1}]
```



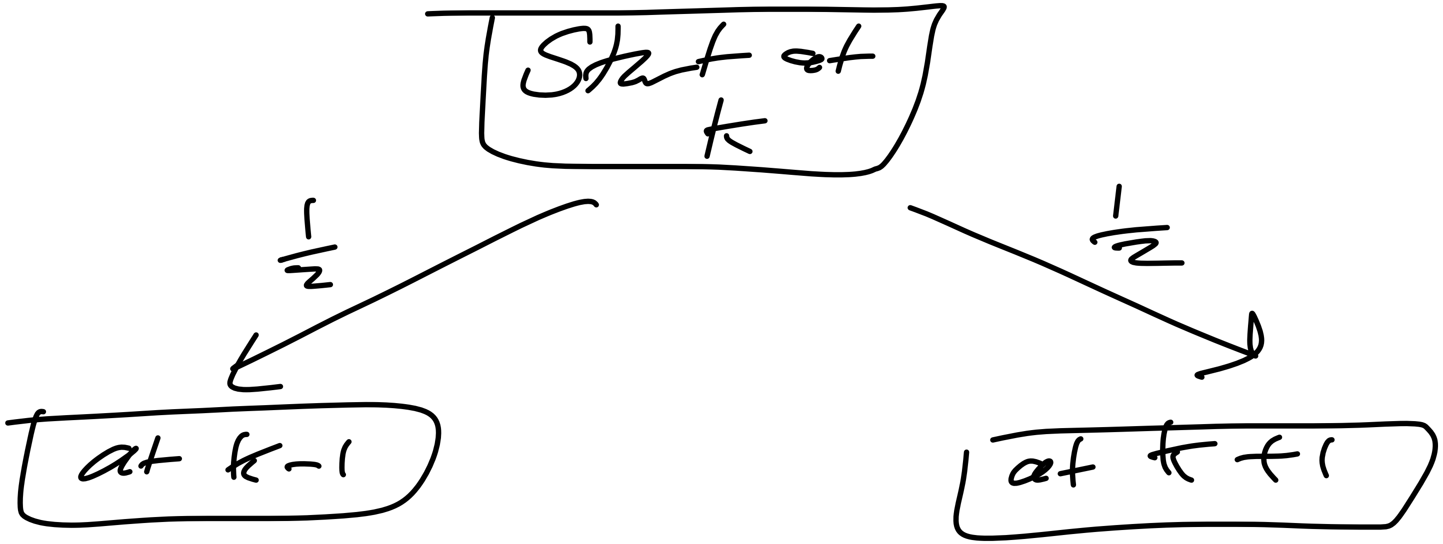
Random Walk: Start at k, always move up 1 with prob 1/2, down 1 with prob 1/2, what is the prob hit n before 0?



all prob are for hitting n before 0

Lemma: $P_k(n) = \frac{1}{2} P_{k+1}(n) + \frac{1}{2} P_{k-1}(n) = \frac{P_{k+1}(n) + P_{k-1}(n)}{2}$

Proof

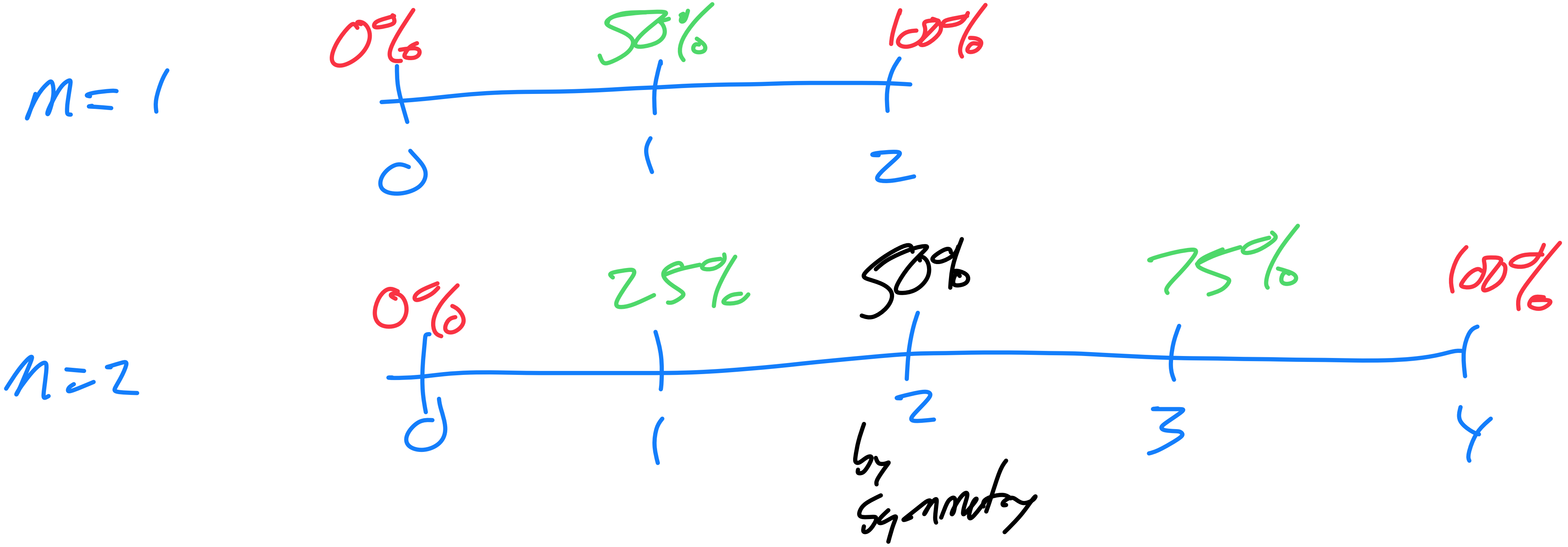


by defn, prob win here is $P_{k-1}(n)$

Similarly $P_{k+1}(n)$

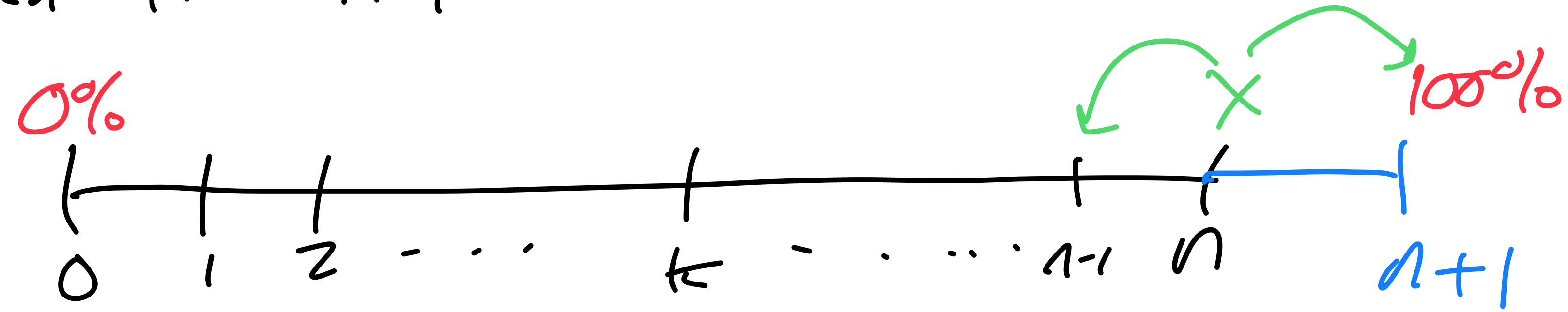
$$P_k(n) = \frac{1}{2} P_{k-1}(n) + \frac{1}{2} P_{k+1}(n)$$

Cor: Can do $n = 2^m$ easily



Could also argue by moving in next steps of 2

What if $n \neq 2^m$?



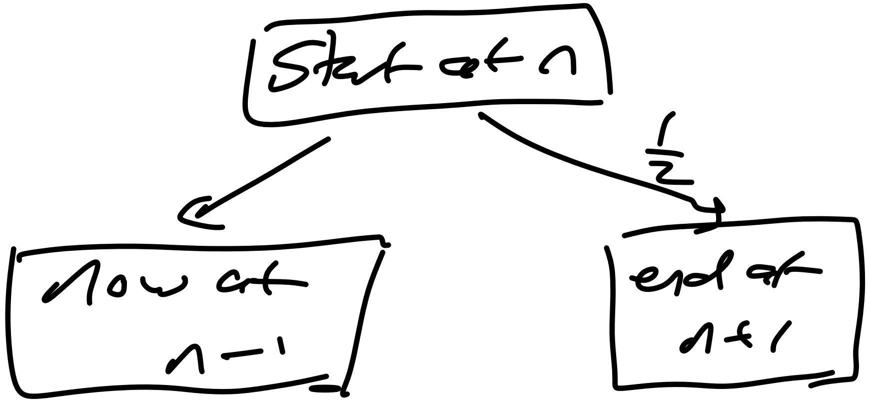
Assume know for n, try to find for n+1

Start at n, what is $P_n(n+1)$: go to n+1 from n before we hit zero

$X =$

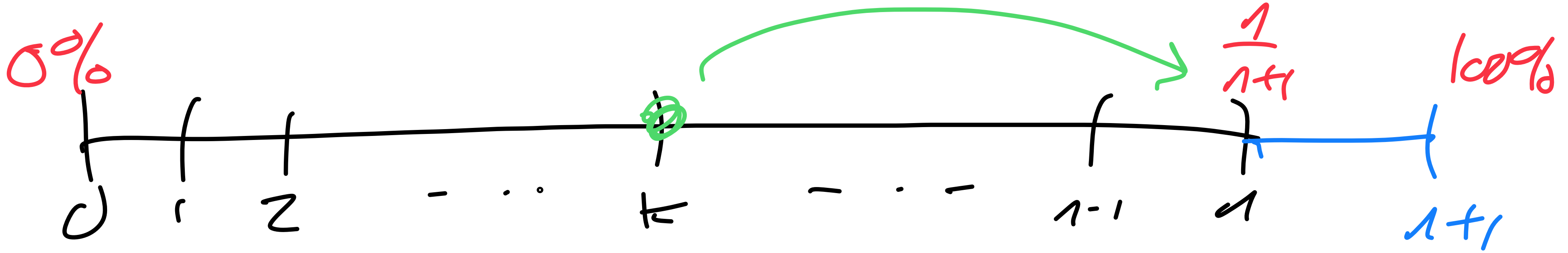
$$\frac{2n}{2n}X = \frac{1}{2} + \frac{1}{2}\frac{n-1}{n}X$$

$$\frac{n+1}{2n}X = \frac{1}{2} \quad \text{so} \quad X = \frac{n}{n+1}$$ ✓



know with prob $\frac{n-1}{n}$ hit n before 0

now have prob X

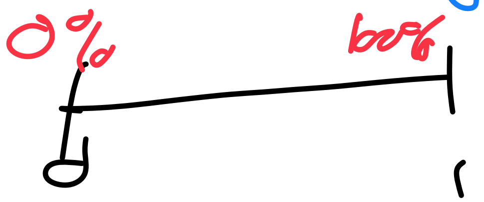


If at k, prob hit n before 0 is $\frac{k}{n}$

So $P_k(n+1) = \underset{\text{Get to } n}{\frac{k}{n}} \cdot \frac{n}{n+1} = \frac{k}{n+1}$

prob get to $n+1$ before get to 0 if start at n (just proved)

Enough to do



Basketball: Bird vs Magic

Problem: 1st basket wins

Bird always hits prob p, Magic with q

Bird shoots first

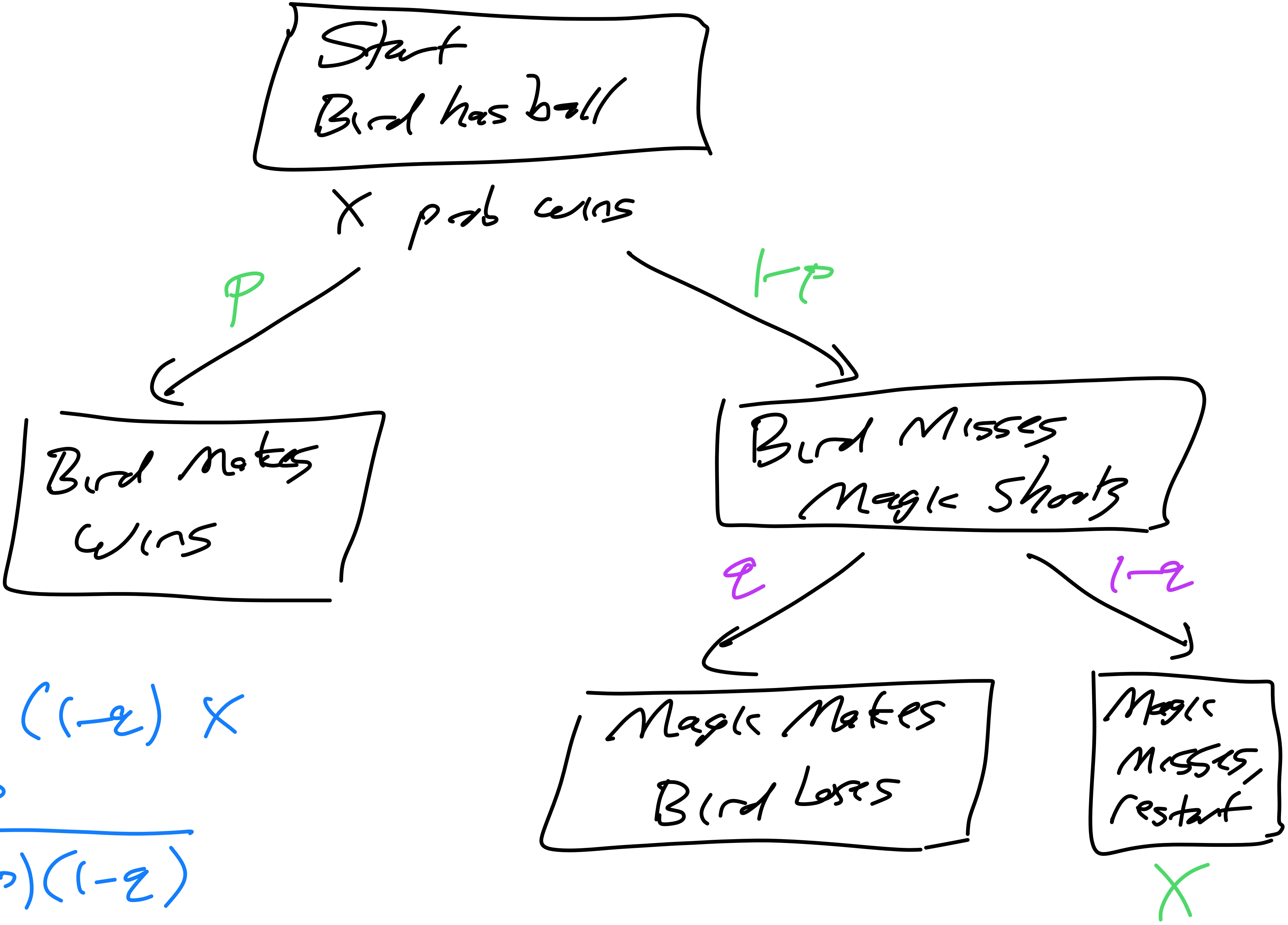
Let $x = x(p,q)$ = prob Bird wins, shooting first

$y = y(p,q)$ = prob Bird wins, shooting second

Relationship b/w x and y?

Conj: $y = (1-q)x$ YES!

Solving!



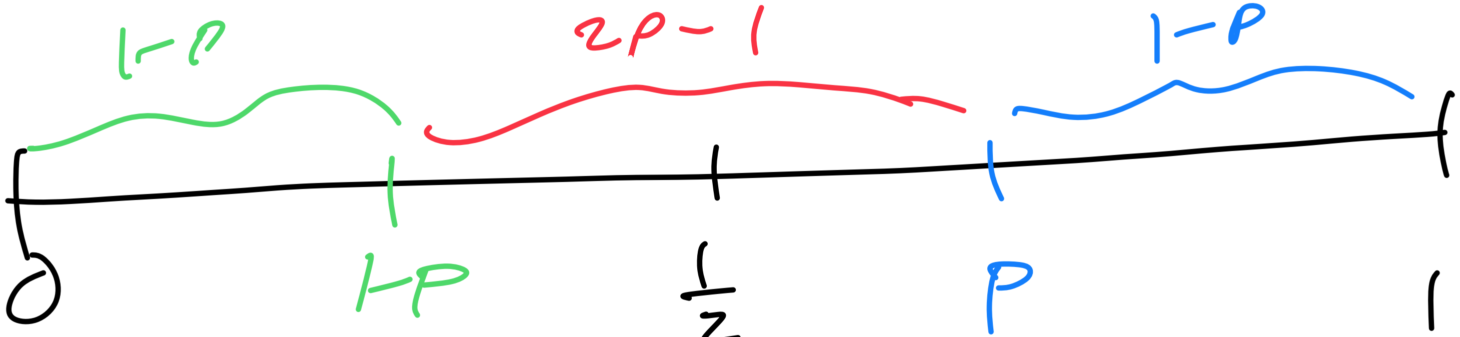
$$X = p + (1-p)(1-q)X$$

$$X = \frac{p}{1-(1-p)(1-q)}$$

Backgammon Doubling

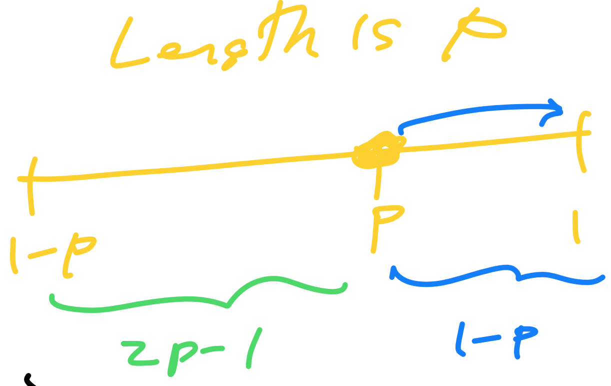
Caveat: If first to 100 wins, only change prob by 1/100: not big

First to hit n points wins



player 2 would double

doubling threshold



prob p hit 1 before 0

prob hit 1 from p before hit $1-p$: $\frac{2p-1}{p}$

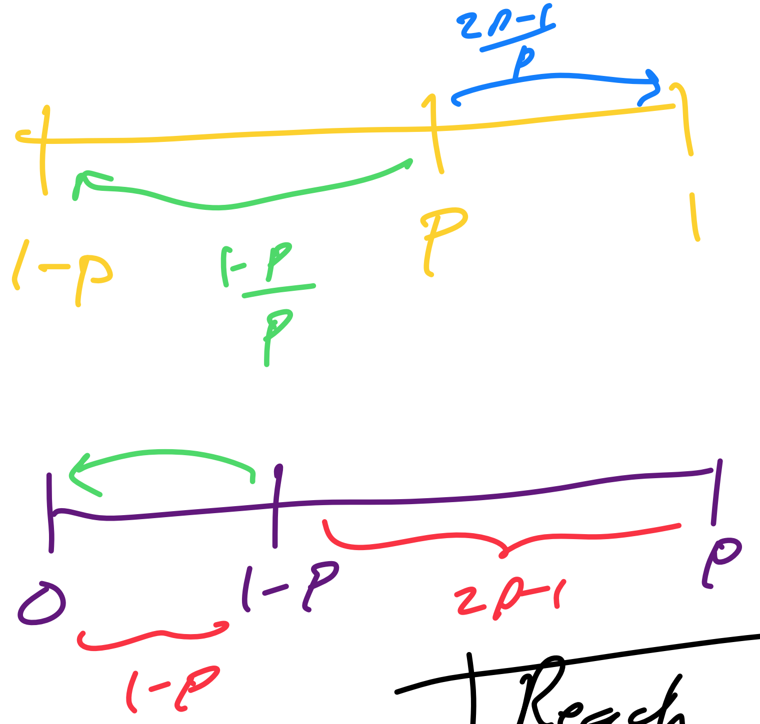
prob hit $1-p$ from p before hit 1 : $\frac{1-p}{p}$

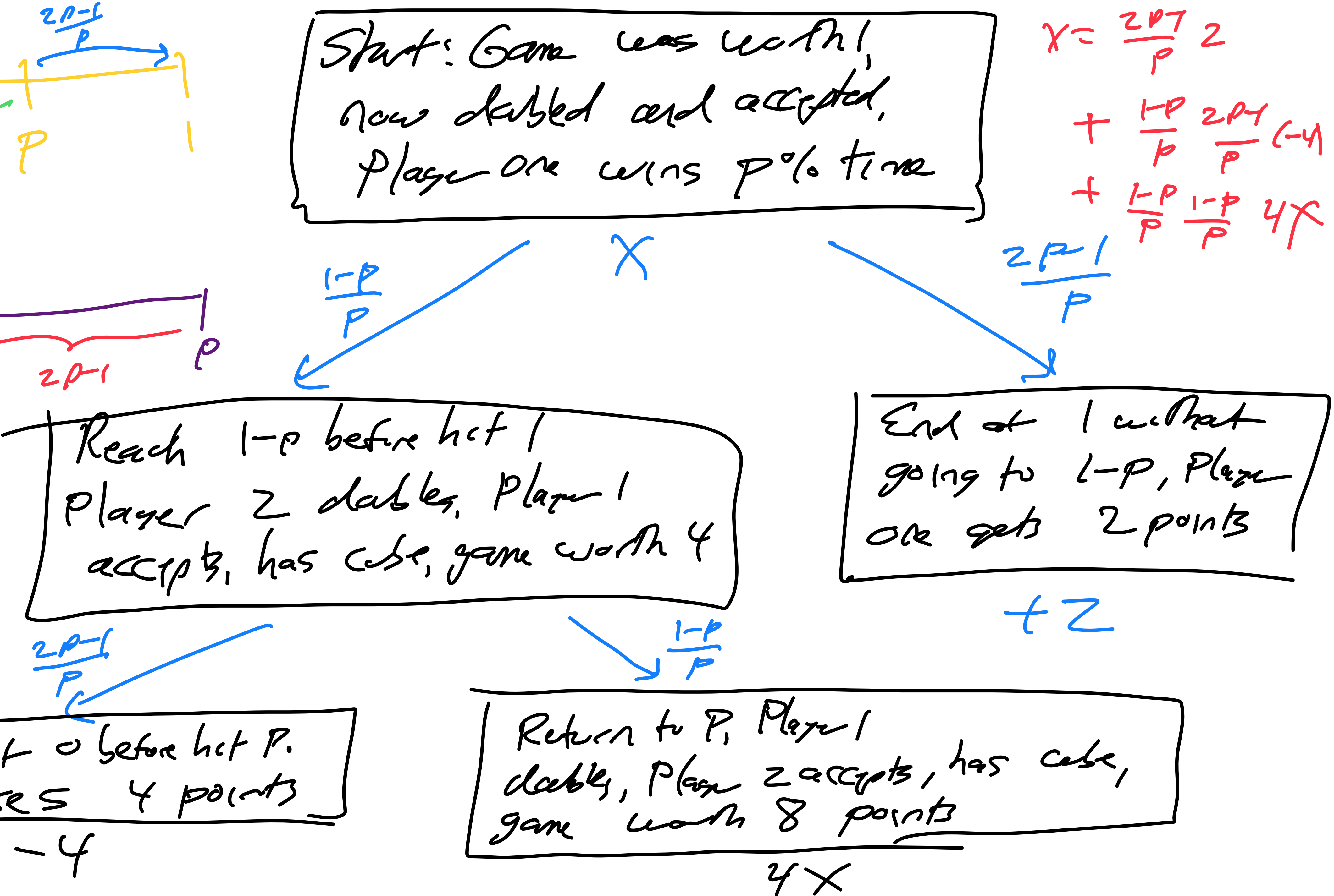
→ sums to 1

Start at 1 point, start with Player 1 has prob p of winning and doubles. If accepted worth 2 points, Player 2 has the doubling cube, and find the expected value. If Player 2 declines then Player 1 gets 1 point.

So X = Expected Value for Player 1 after doubling a game worth 1 point with Player 2 accepting and player one winning p % of the time.

Game is worth 2 now, Player 2 has the cube.





Start: Game was worth 1, now doubled and accepted, Player one wins p% time

Reach 1-p before hit 1 Player 2 doubles, Player 1 accepts, has cube, game worth 4

End at 1 without going to 1-p, Player one gets 2 points

Hit 0 before hit P. loses 4 points

Return to P, Player 1 doubles, Player 2 accepts, has cube, game worth 8 points

$$x = \frac{2p-1}{p} 2 + \frac{1-p}{p} \frac{2p-1}{p} (-4) + \frac{1-p}{p} \frac{1-p}{p} 4x$$

$$x = \frac{2p-1}{p} 2$$

$$+ \frac{1-p}{p} \frac{2p-1}{p} (-4)$$

$$+ \frac{1-p}{p} \frac{1-p}{p} 4x$$

$$X = 2 \frac{2p-1}{p} - 4 \frac{(1-p)(2p-1)}{p^2} + 4x \frac{(1-p)^2}{p^2}$$

$$X = \frac{2 \frac{2p-1}{p} - 4 \frac{(1-p)(2p-1)}{p^2}}{1 - 4 \frac{(1-p)^2}{p^2}} * \frac{p^2}{p^2}$$

↓ alg

$$X = \frac{12p^2 - 14p + 4}{-3p^2 + 8p - 4}$$

If decline, Player one gets...

1 point

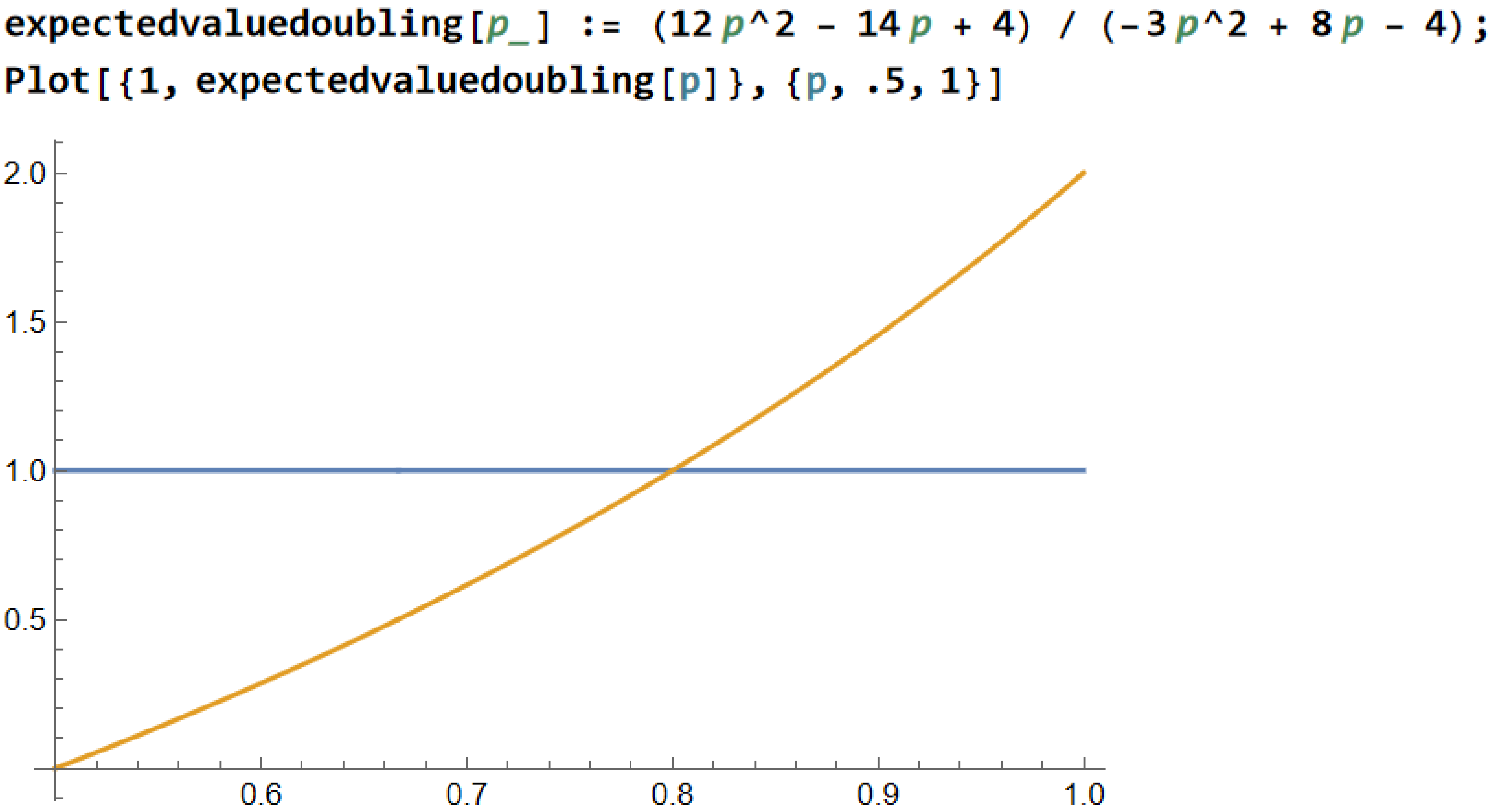
Compare:

$$1 \quad \text{vs} \quad \underbrace{\frac{12p^2 - 14p + 4}{-3p^2 + 8p - 4}}$$

If > 1, Player 2 Should decline, and

If < 1, Player 2 shall accept

Critical threshold: When equal



```
expectedvaluedoubling[p_] := (12 p^2 - 14 p + 4) / (-3 p^2 + 8 p - 4);
Plot[{1, expectedvaluedoubling[p]}, {p, .5, 1}]
```

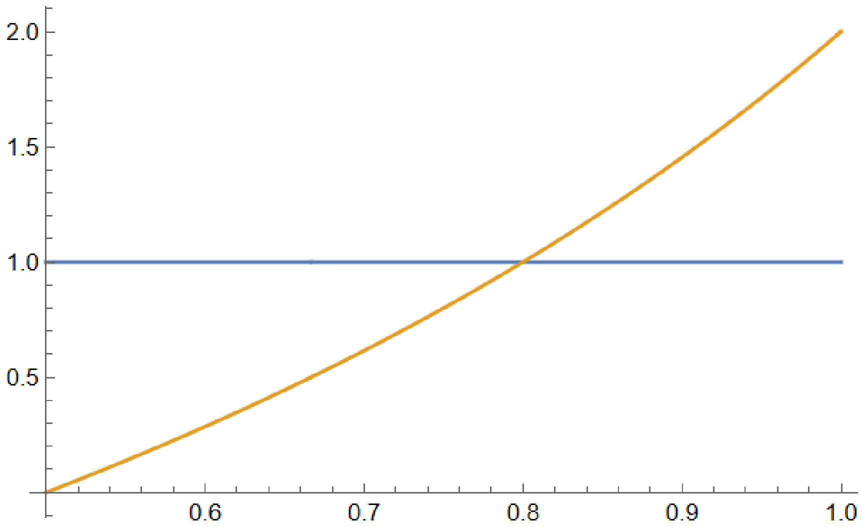
Math 344: Mathematics of Sports: Spring 2023:

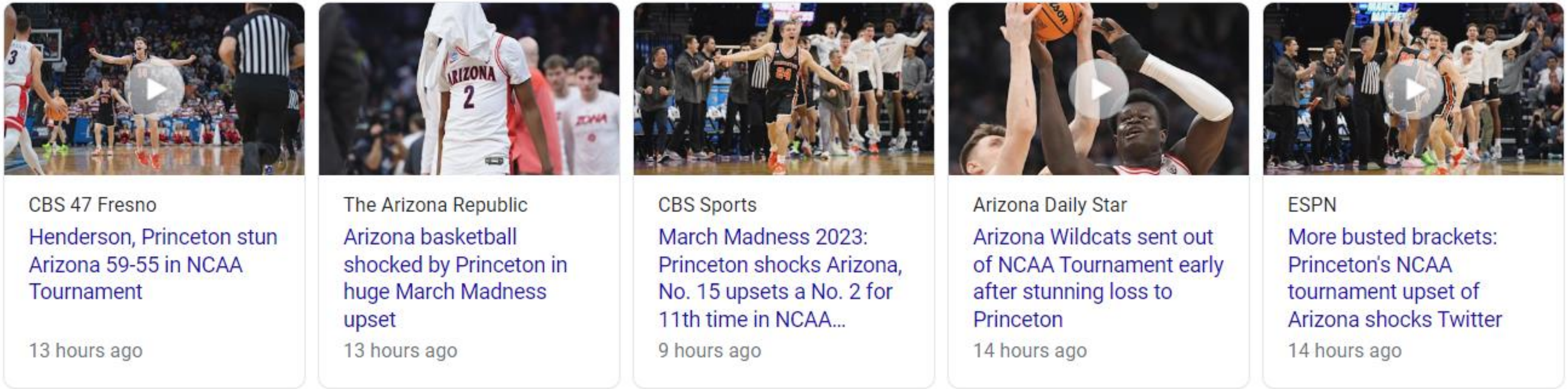
Lecture 18: Backgammon Discussion, Upsets: https://youtu.be/clcZFy3SNBE

Plan for the day.

- Backgammon Doubling: Theory and Coding
- Upsets

```
expectedvaluedoubling[p_] := (12 p^2 - 14 p + 4) / (-3 p^2 + 8 p - 4);
Plot[{1, expectedvaluedoubling[p]}, {p, .5, 1}]
```





CBS 47 Fresno

Henderson, Princeton stun Arizona 59-55 in NCAA Tournament

13 hours ago

The Arizona Republic

Arizona basketball shocked by Princeton in huge March Madness upset

13 hours ago

CBS Sports

March Madness 2023: Princeton shocks Arizona, No. 15 upsets a No. 2 for 11th time in NCAA...

9 hours ago

Arizona Daily Star

Arizona Wildcats sent out of NCAA Tournament early after stunning loss to Princeton

14 hours ago

ESPN

More busted brackets: Princeton's NCAA tournament upset of Arizona shocks Twitter

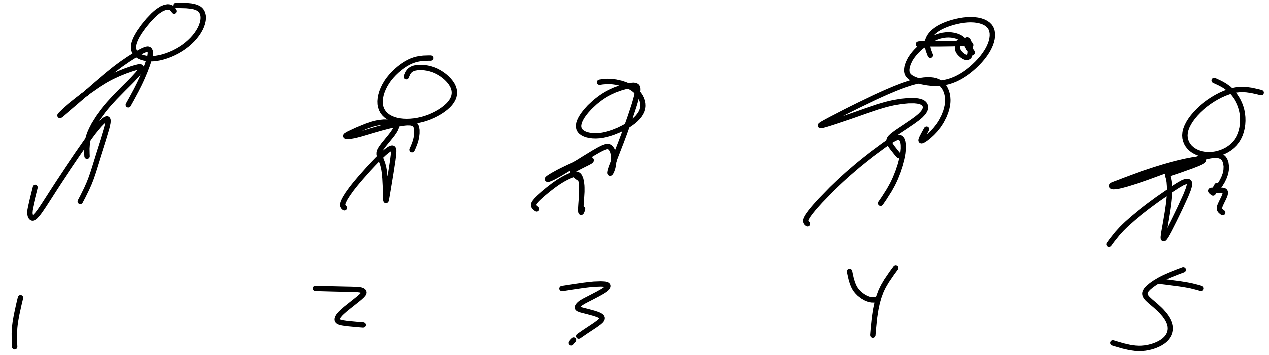
14 hours ago

$$\begin{cases} +1 & 1/2 \\ -1 & 1/2 \end{cases}$$

New Q's

1) Finite # doubles

5 Pirates: need >50% to agree on a split



100

Backward Analysis

4 and 5: 4 gets 100, 5 gets 0

3, 4, and 5: 3 gets 99, 4 gets 0, 5 gets 1

2, 3, 4 and 5: 2 gets 99, 3 gets 0, 4 gets 1, 5 gets 0

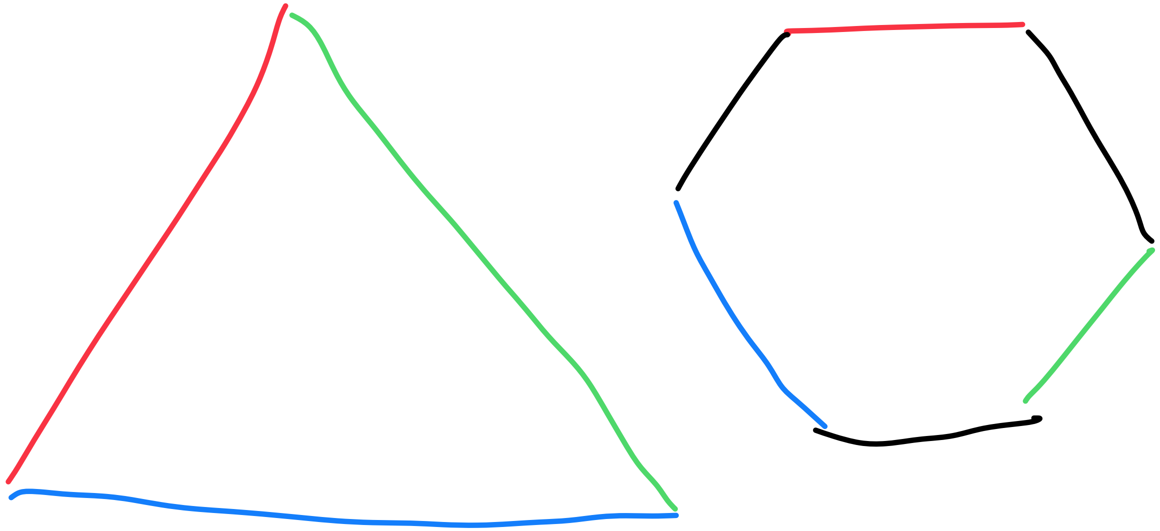
(1, 2, 3, 4, 5 : 1 gets 98, 3 and 5 get 1, 2 and 4 get 0

Generalize

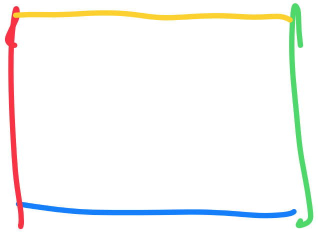
a) more than just ±1, maybe ±1, ±2, ... with differing probabilities

b) change # players

If 3 players....?



4 players



Math 344: Mathematics of Sports: Spring 2023:
Lecture 19: New Shift in Baseball: https://youtu.be/gq_rUfYiSsI

Plan for the day.

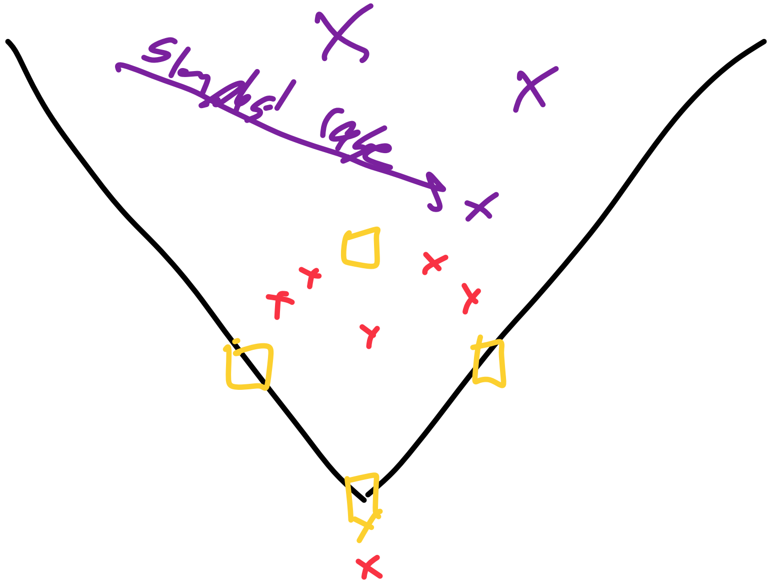
- Two outfield shift…

https://thecomeback.com/mlb/absurd-outfield-shift-joey-gallo.html

How to shift?

Players hitting profile is indep of shift

Infield fixed; can vary 3 outfielders...



1) Distribution of balls in play to outfield
   - ↳ foul balls? YES
   - ↳ home runs? 1-2 rows in matter

2) time of ball in flight

3) Pitcher can influence (2).

4) ground balls to outfield: could get out at 1st

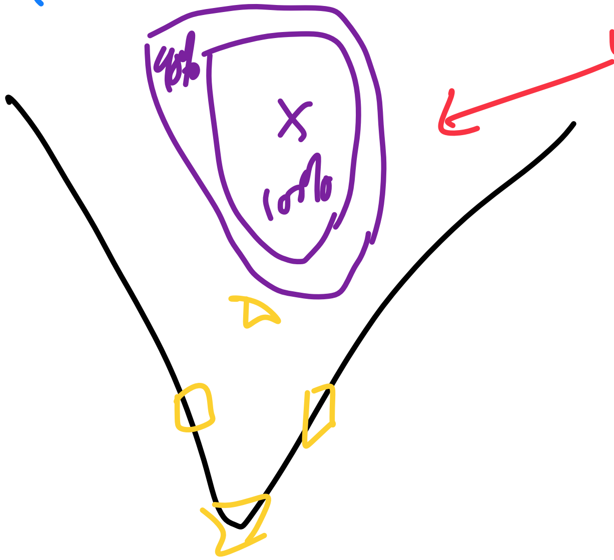
How "effective" is a configuration?

What Metric?

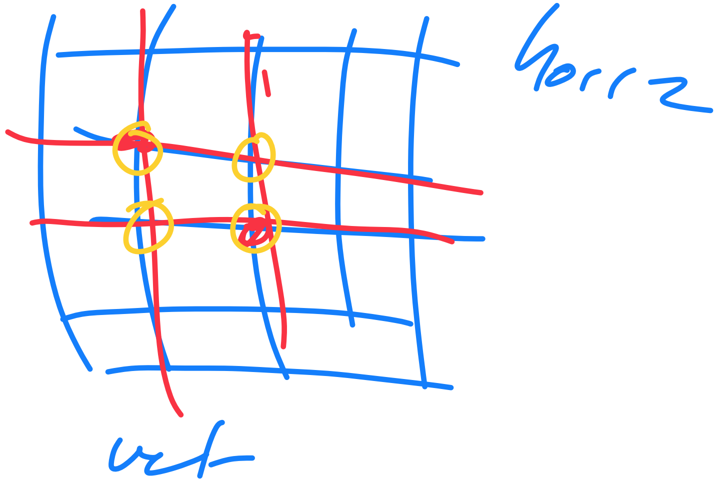
Compare run expectancy

win prob

Data on effectiveness



valued ball landing here based on dist to nearest fielder



Math 344: Mathematics of Sports: Spring 2023:
Lecture 29: Monte Carlo Integration: https://youtu.be/XmOjxxxv15A

Plan for the day.

- Integral of Error Function: The Search for Closure
- Monte Carlo Integration
- How to Extend Monte Carlo: Pure Math to the Rescue

RECENT LECTURES:

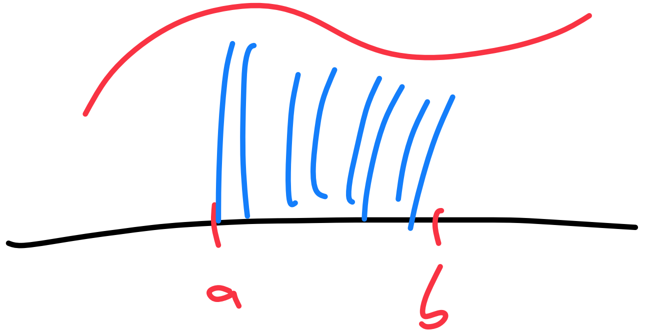
- Class 28: Presentation on Home Field Advantage: https://youtu.be/ruM7yrJS-gk
- Class 27: Visit: Analytics in Baseball
- Class 26: Visit: History of Baseball (especially in the Berkshires)
- Class 25: 4/17: German Tank Problem: Slides, Video: https://youtu.be/RjAX_nzOA10 (papers: not starting at 1, and also higher dimensions)
- Class 24: 4/14: Presentation on Chess Rankings: https://youtu.be/V5AnFgx0rz4
- Class 23: 4/12: Presentation on Basketball Steals: https://youtu.be/m2fMUc0nVpM
- Class 22: 4/10: Presentation on Hockey Statistics: https://youtu.be/2SZ5GNTrJF8
- Class 21: 4/7: Presentation on the Great Shootout: https://youtu.be/p6IOULH9w2s
- Class 20: 4/5: Presentation by Dick Quinn

$f(x)$ is a density for the Random Variable X if

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Say $\text{Prob}(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

or $\text{Prob}(X \in [a,b]) = \int_{a}^{b} f(t)\,dt$



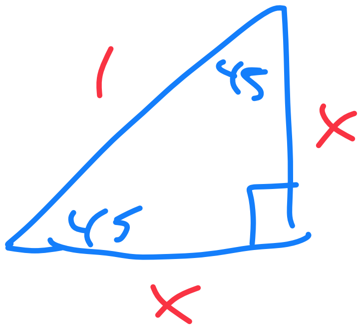
$(\cos x)' = -\sin x$ $(\sin x)' = \cos x$ in radians

$e^x = \sum_{n=0}^{\infty} x^n/n! = 1 + x + x^2/2! + \dots$

$e^{ix} = \cos x + i\sin x$ $i = \sqrt{-1}$

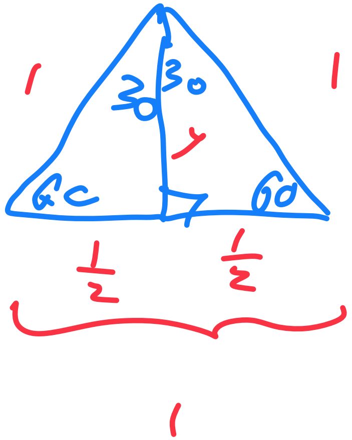
$\cos x = 1 - x^2/2! + x^4/4! - \dots$

$\sin x = x - x^3/3! + x^5/5! - \dots$



$1^2 = x^2 + x^2$

$\Rightarrow x = \sqrt{2}/2$



$1^2 = (\frac{1}{2})^2 + y^2$

$\sqrt{\frac{3}{4}} = y$

or $y = \frac{\sqrt{3}}{2}$

Normal with mean μ and st deviation σ has density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Standard Normal: $\mu = 0$, $\sigma = 1$

$$\mathrm{Prob}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \sum_{n=0}^{\infty} \frac{(-t^2/2)^n}{n!} \, dt$$

$$\overset{?}{=} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \int_{-\infty}^{x} t^{2n} \, dt$$

BAD!

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{t^{2n+1}}{2n+1} \Big|_{-\infty}^{x} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \left(\frac{x^{2n+1}}{2n+1} - \frac{(-\infty)^{2n+}}{2n+1} \right)$$

Fubini's Thm

If f is nice then

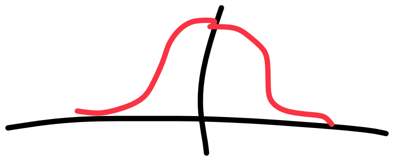
$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx = \int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dx$$

Rectangle

$[a,b] \times [c,d]$

ok if $\iint_R |f(x,y)|\,dA < \infty$

Study $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$



If $x = \infty$, integral is 1, if $x = 0$, integral is $\frac{1}{2}$

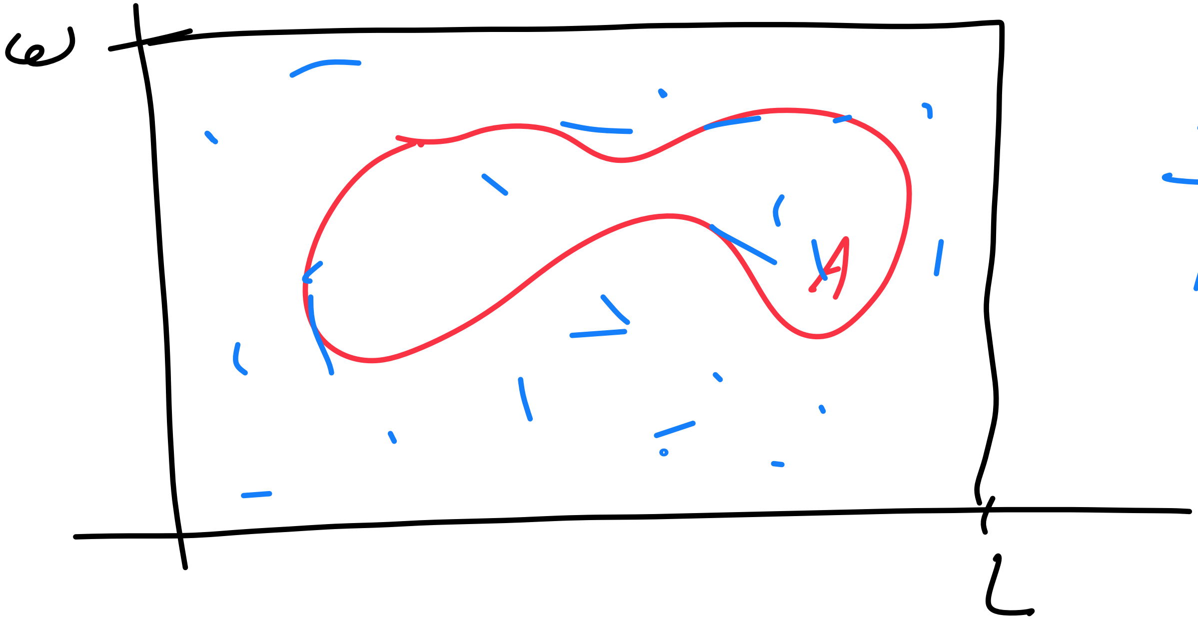
$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = \underbrace{\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt}_{1/2} + \int_{0}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \left(\frac{x^{2n+1}}{2n+1} - \frac{0^{2n+1}}{2n+1} \right)$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{x^{2n+1}}{2n+1}$$

"related to the error function erf"

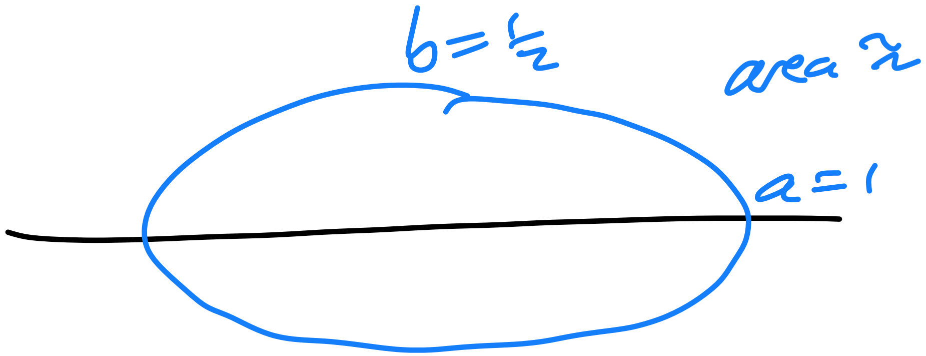
Monte Carlo Integration



$$\frac{\text{Area}(A)}{\text{Area}(\text{Rect})} \approx \frac{\#\text{ darts in } A}{\#\text{ darts Thrown}}$$

$$\text{Area}(A) \approx \frac{\#\text{ hit}}{\#\text{ toss}}(L \times w)$$

if toss N darts, error is on the order of $\frac{1}{\sqrt{N}}$



area $\approx \pi \frac{1}{2}$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

Area Guesses

$a=1, b=\frac{1}{2}$

πab $\quad \pi \frac{8}{16}$

$\pi\left(\frac{a+b}{2}\right)^2$ $\quad \pi \frac{9}{16}$

$a=b=r$

Circle

$$\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2=1$$

area πr^2

Math 344: Mathematics of Sports: Spring 2023:
Lecture 30: Numerical Techniques: Grid Search vs Greedy Algorithm:
https://youtu.be/afBzGwwKtLM

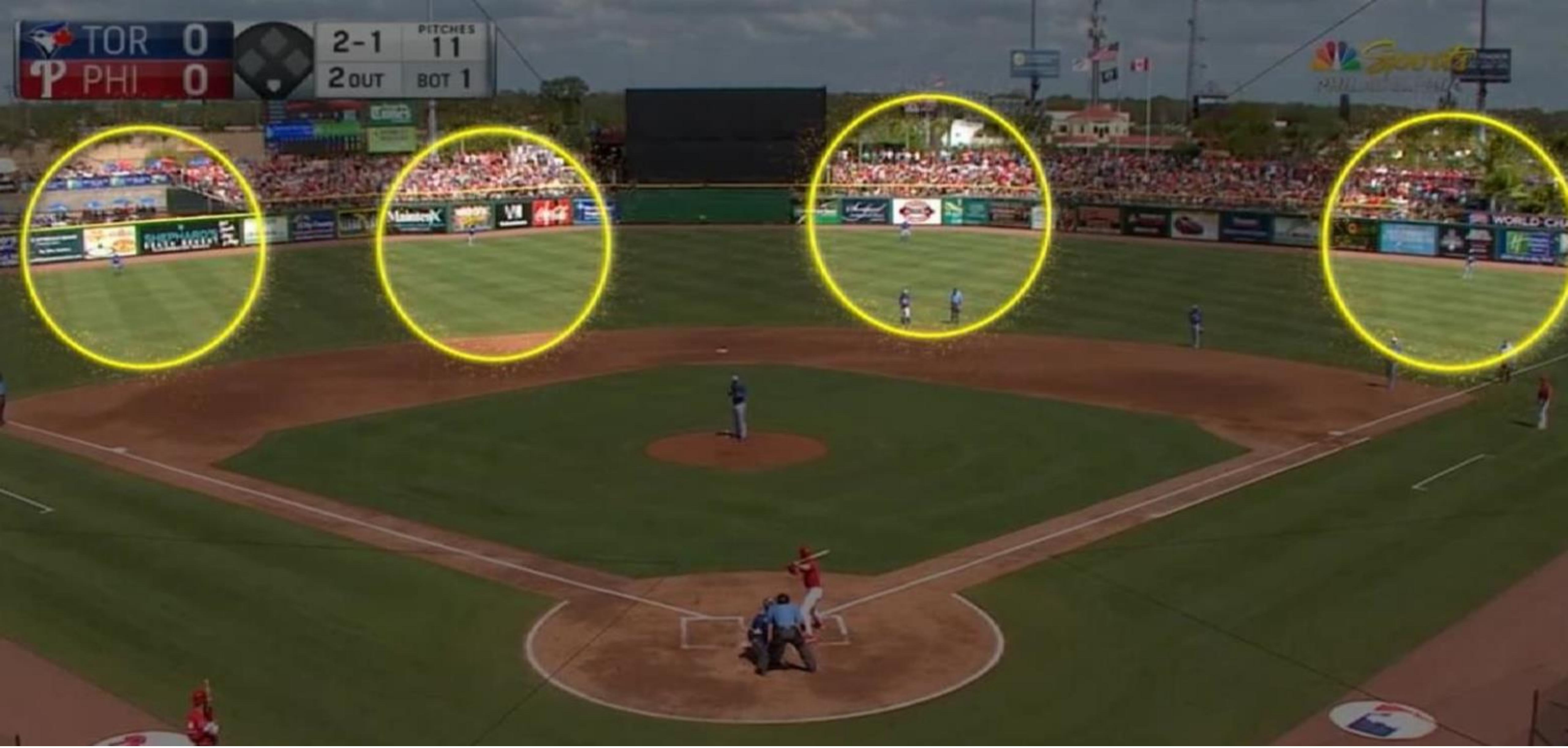
Plan for the day.

- How to model optimizing outfielder locations
- Finding candidates for extrema

Papers:

- Do dogs know calculus?
  https://www.csun.edu/~dgray/BE528/Pennigs2003Dogs_Calculus.pdf

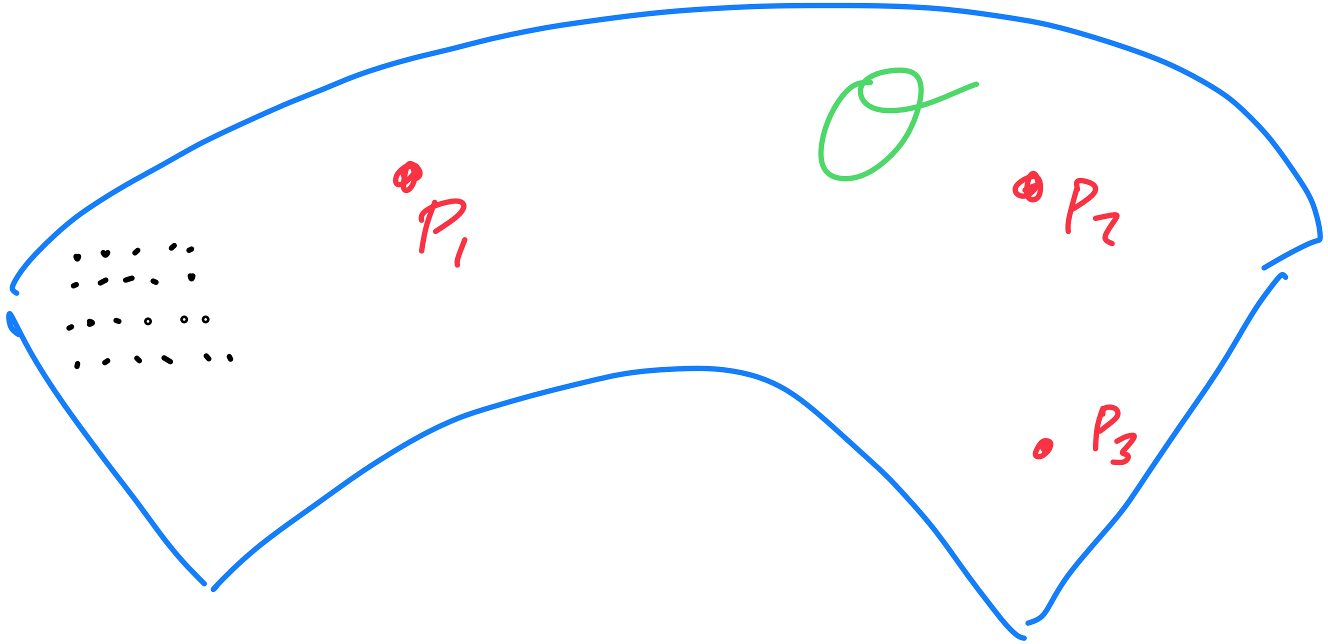
- Do dogs know bifurcations?
  https://www.maa.org/sites/default/files/pdf/upload_library/22/Polya/minton356.pdf



https://img.mlbstatic.com/mlb-images/image/private/ar_16:9,g_auto,q_auto:good,w_1536,c_fill,f_jpg/mlb/ruc791mtwe3pav0nczun

Joey Gallo still seeing the shift. Red Sox with the two-deep outfield. #MNTwins

https://theathletic.com/4336237/2023/03/23/mlb-two-man-outfield-shift/

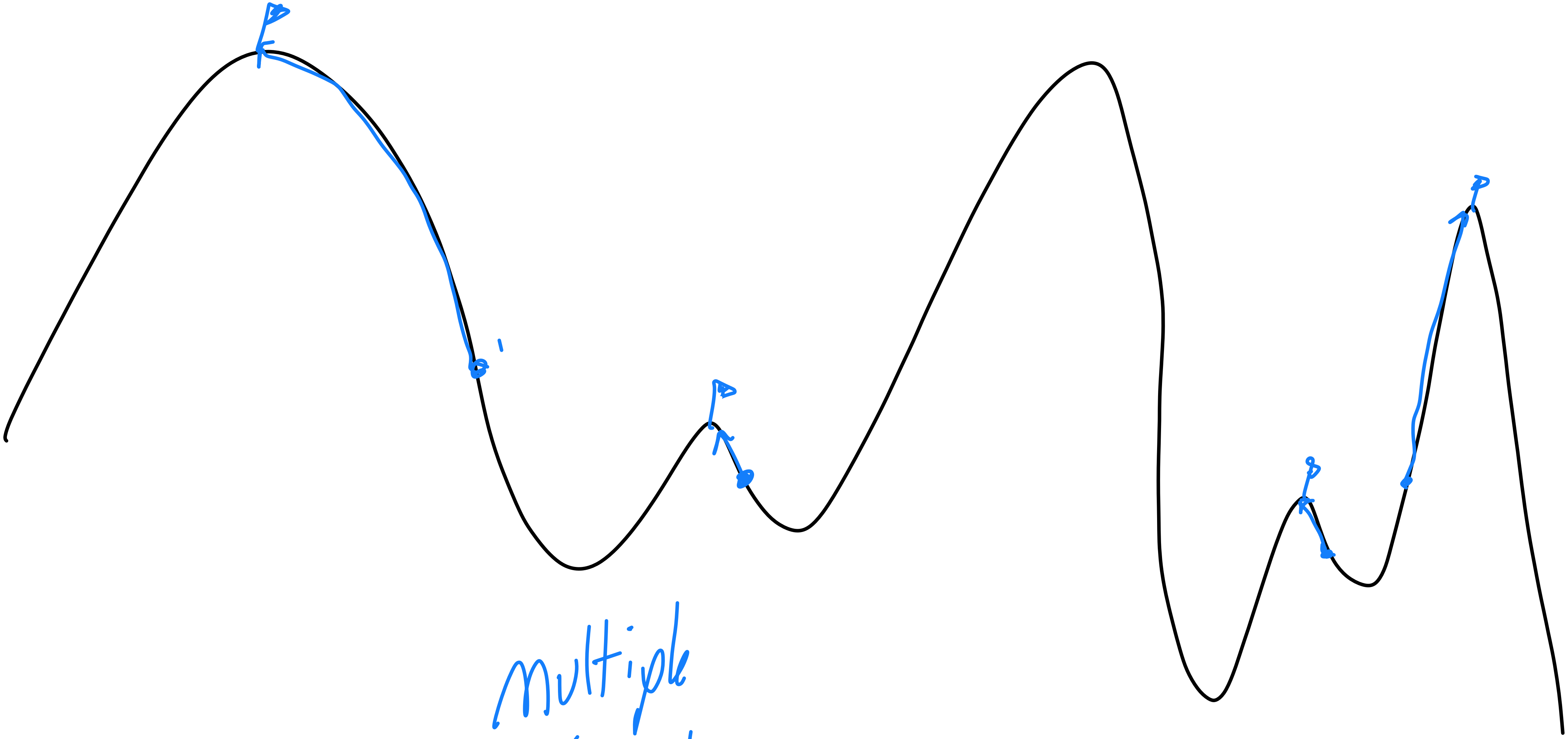


Approaches

(1) Grid search

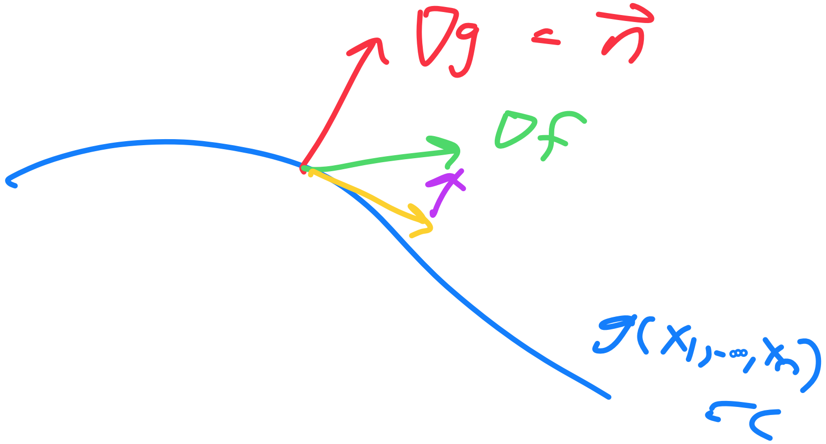
(2) Continuous/ gradient approach

$$\max_{(p_1,p_2,p_3)\in\Theta^3} \Delta W_{B,P,G}(p_1,p_2,p_3)$$



multiple
starts

Lagrange Multipliers



Constraint $g(x_1,\dots,x_n) = c$

function $f(x_1,\dots,x_n)$

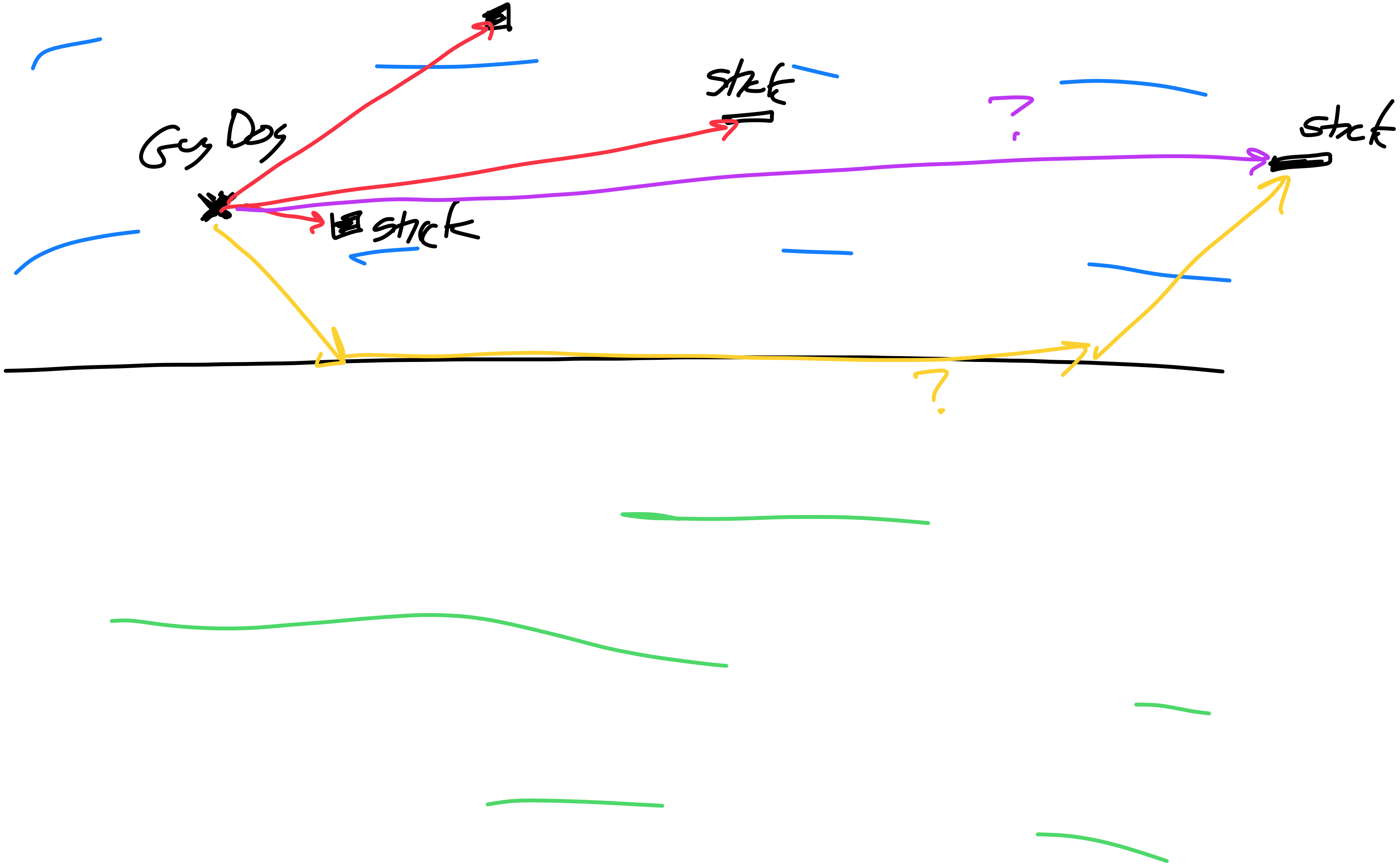
Candidates for max/min of f for $(x_1,\dots,x_n)$ on surface

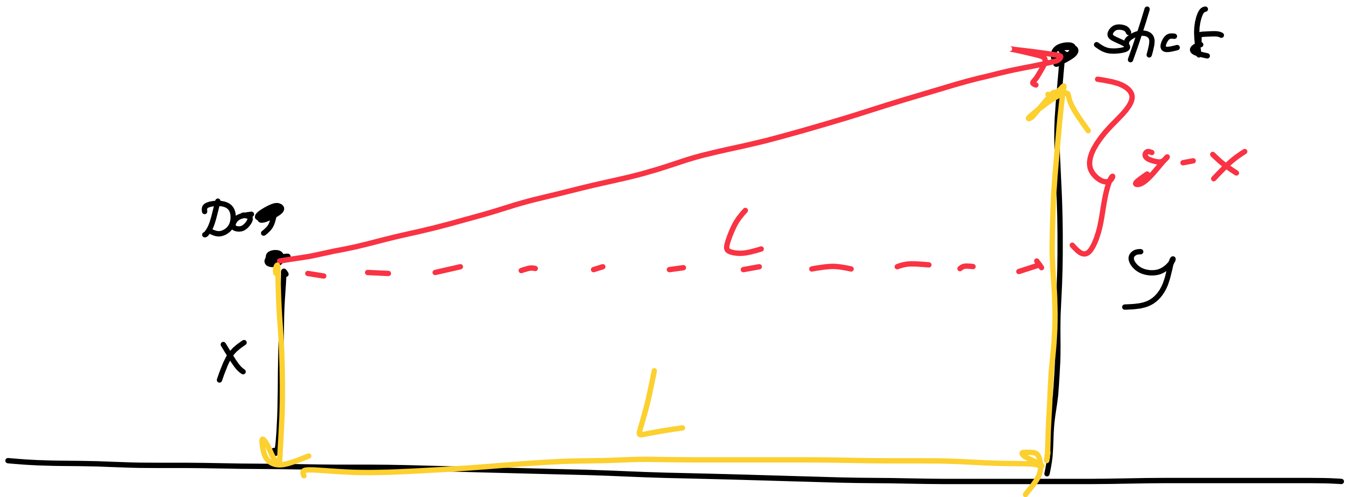
Satisfy! $\nabla f = \lambda \nabla g$ $\left(\frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1} \quad \dots \right)$

$g(x_1,\dots,x_n) = c$

So $n+1$ eqs

$n+1$ unknowns


Guy Dog
stick
stick
stick
?
?



$V_{water} = 1$ m/sec

$V_{land} = \infty$ m/sec

$time_{water\ route} = \sqrt{L^2 + (y-x)^2}$

$t_{land\ route} = x+y$

Set squares equal

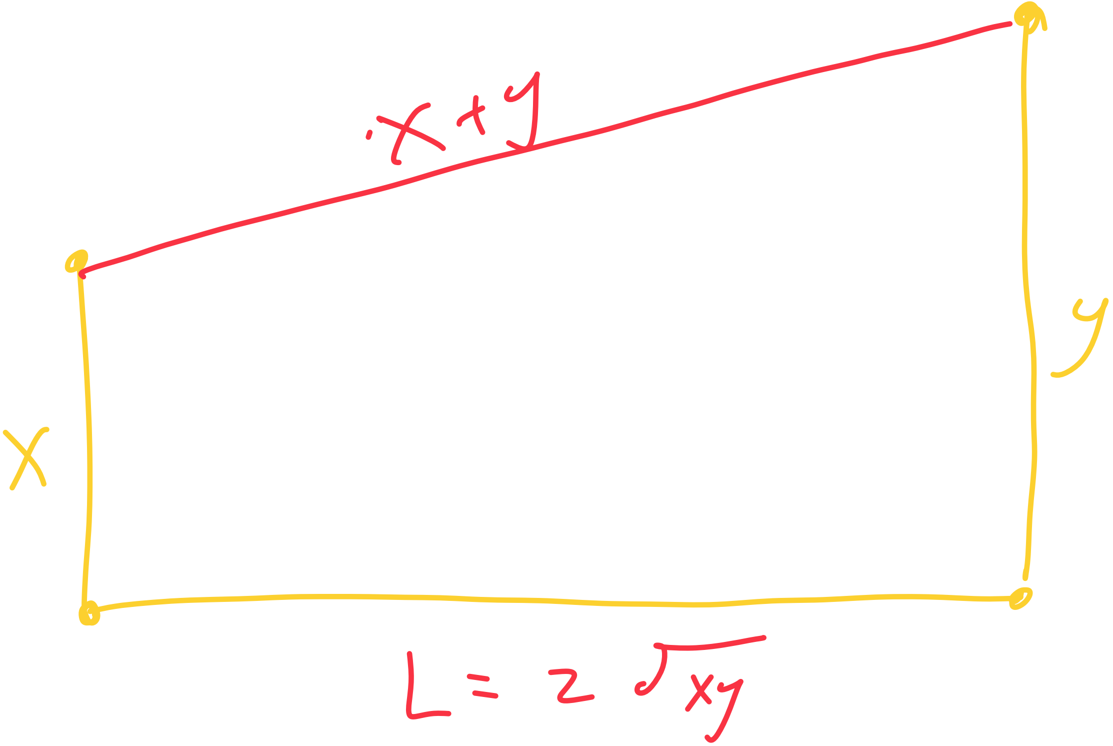
$L^2 + (y-x)^2 = (x+y)^2$

$L^2 + \cancel{x^2} - 2xy + \cancel{y^2} = \cancel{x^2} + 2xy + \cancel{y^2}$

$L^2 = 4xy$ or $L = 2\sqrt{xy}$

If $L > 2\sqrt{xy}$ then land is faster: water time > land time

Yields the Arithmetic Mean-Geometric Mean Inequality!



Based on the trapezoid, clearly diagonal side is at least as long as bottom,

so $x+y \geq 2\sqrt{xy}$

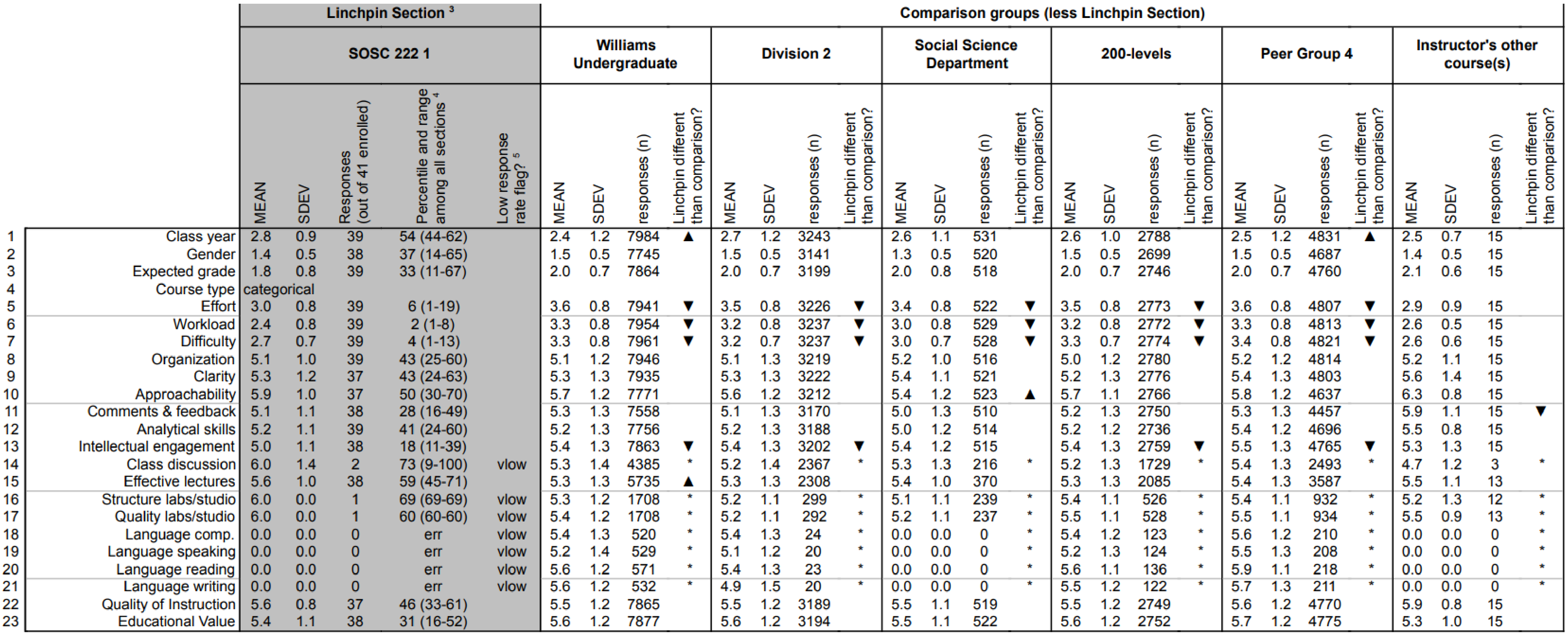
or $\frac{x+y}{2} \geq \sqrt{xy}$

Math 344: Mathematics of Sports: Spring 2023:
Lecture 31: No lecture, discussing projects
Lecture 32: Evaluating Teaching Metrics

Plan for the day.

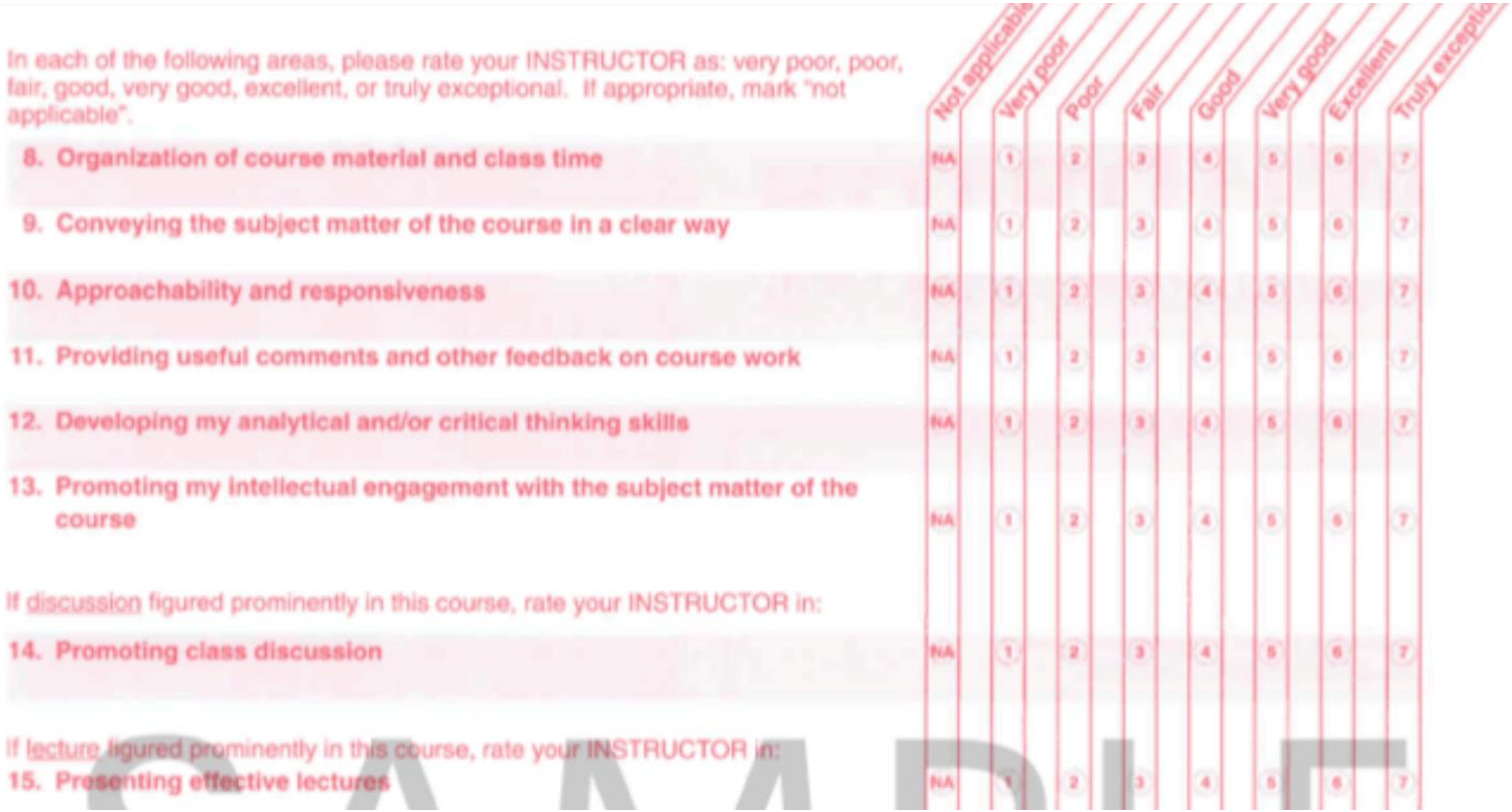
- Examine old SCS metrics
- Examine new SCS metrics
- Discuss what these metrics measure, versus what they should measure…

https://www.williams.edu/institutional-research/files/2018/07/guide_to_interpreting_scs_reports.pdf



		Linchpin Section [3]					Comparison groups (less Linchpin Section)																							
		SOSC 222 1					Williams Undergraduate				Division 2				Social Science Department				200-levels				Peer Group 4				Instructor's other course(s)			
		MEAN	SDEV	Responses (out of 41 enrolled)	Percentile and range among all sections [4]	Low response rate flag? [5]	MEAN	SDEV	responses (n)	Linchpin different than comparison?	MEAN	SDEV	responses (n)	Linchpin different than comparison?	MEAN	SDEV	responses (n)	Linchpin different than comparison?	MEAN	SDEV	responses (n)	Linchpin different than comparison?	MEAN	SDEV	responses (n)	Linchpin different than comparison?	MEAN	SDEV	responses (n)	Linchpin different than comparison?
1	Class year	2.8	0.9	39	54 (44-62)		2.4	1.2	7984	▲	2.7	1.2	3243		2.6	1.1	531		2.6	1.0	2788		2.5	1.2	4831	▲	2.5	0.7	15	
2	Gender	1.4	0.5	38	37 (14-65)		1.5	0.5	7745		1.5	0.5	3141		1.3	0.5	520		1.5	0.5	2699		1.5	0.5	4687		1.4	0.5	15	
3	Expected grade	1.8	0.8	39	33 (11-67)		2.0	0.7	7864		2.0	0.7	3199		2.0	0.8	518		2.0	0.7	2746		2.0	0.7	4760		2.1	0.6	15	
4	Course type	categorical																												
5	Effort	3.0	0.8	39	6 (1-19)		3.6	0.8	7941	▼	3.5	0.8	3226	▼	3.4	0.8	522	▼	3.5	0.8	2773	▼	3.6	0.8	4807	▼	2.9	0.9	15	
6	Workload	2.4	0.8	39	2 (1-8)		3.3	0.8	7954	▼	3.2	0.8	3237	▼	3.0	0.8	529	▼	3.2	0.8	2772	▼	3.3	0.8	4813	▼	2.6	0.5	15	
7	Difficulty	2.7	0.7	39	4 (1-13)		3.3	0.8	7961	▼	3.2	0.7	3237	▼	3.0	0.7	528	▼	3.3	0.7	2774	▼	3.4	0.8	4821	▼	2.6	0.6	15	
8	Organization	5.1	1.0	39	43 (25-60)		5.1	1.2	7946		5.1	1.3	3219		5.2	1.0	516		5.0	1.2	2780		5.2	1.2	4814		5.2	1.1	15	
9	Clarity	5.3	1.2	37	43 (24-63)		5.3	1.3	7935		5.3	1.3	3222		5.4	1.1	521		5.2	1.3	2776		5.4	1.3	4803		5.6	1.4	15	
10	Approachability	5.9	1.0	37	50 (30-70)		5.7	1.2	7771		5.6	1.2	3212		5.4	1.2	523	▲	5.7	1.1	2766		5.8	1.2	4637		6.3	0.8	15	
11	Comments & feedback	5.1	1.1	38	28 (16-49)		5.3	1.3	7558		5.1	1.3	3170		5.0	1.3	510		5.2	1.3	2750		5.3	1.3	4457		5.9	1.1	15	▼
12	Analytical skills	5.2	1.1	39	41 (24-60)		5.2	1.3	7756		5.2	1.3	3188		5.0	1.2	514		5.2	1.2	2736		5.4	1.2	4696		5.5	0.8	15	
13	Intellectual engagement	5.0	1.1	38	18 (11-39)		5.4	1.3	7863	▼	5.4	1.3	3202	▼	5.4	1.2	515		5.4	1.3	2759	▼	5.5	1.3	4765	▼	5.3	1.3	15	
14	Class discussion	6.0	1.4	2	73 (9-100)	vlow	5.3	1.4	4385	*	5.2	1.4	2367	*	5.3	1.3	216	*	5.2	1.3	1729	*	5.4	1.3	2493	*	4.7	1.2	3	*
15	Effective lectures	5.6	1.0	38	59 (45-71)		5.3	1.3	5735	▲	5.3	1.3	2308		5.4	1.0	370		5.3	1.3	2085		5.4	1.3	3587		5.5	1.1	13	
16	Structure labs/studio	6.0	0.0	1	69 (69-69)	vlow	5.3	1.2	1708	*	5.2	1.1	299	*	5.1	1.1	239	*	5.4	1.1	526	*	5.4	1.1	932	*	5.2	1.3	12	*
17	Quality labs/studio	6.0	0.0	1	60 (60-60)	vlow	5.4	1.2	1708	*	5.2	1.1	292	*	5.2	1.1	237	*	5.5	1.1	528	*	5.5	1.1	934	*	5.5	0.9	13	*
18	Language comp.	0.0	0.0	0	err	vlow	5.4	1.3	520	*	5.4	1.3	24	*	0.0	0.0	0	*	5.4	1.2	123	*	5.6	1.2	210	*	0.0	0.0	0	*
19	Language speaking	0.0	0.0	0	err	vlow	5.2	1.4	529	*	5.1	1.2	20	*	0.0	0.0	0	*	5.2	1.3	124	*	5.5	1.3	208	*	0.0	0.0	0	*
20	Language reading	0.0	0.0	0	err	vlow	5.6	1.2	571	*	5.4	1.3	23	*	0.0	0.0	0	*	5.6	1.1	136	*	5.9	1.1	218	*	0.0	0.0	0	*
21	Language writing	0.0	0.0	0	err	vlow	5.6	1.2	532	*	4.9	1.5	20	*	0.0	0.0	0	*	5.5	1.2	122	*	5.7	1.3	211	*	0.0	0.0	0	*
22	Quality of Instruction	5.6	0.8	37	46 (33-61)		5.5	1.2	7865		5.5	1.2	3189		5.5	1.1	519		5.5	1.2	2749		5.6	1.2	4770		5.9	0.8	15	
23	Educational Value	5.4	1.1	38	31 (16-52)		5.6	1.2	7877		5.6	1.2	3194		5.5	1.1	522		5.6	1.2	2752		5.7	1.2	4775		5.3	1.0	15	

1. My class year is:	fr ①	so ②	jr ③	sr ④	other ⑤
2. I am:	female ①	male ②			
3. Based on my performance in the course so far, the grade I expect to receive is:	A/A+ ①	A-/B+ ②	B/B- ③	C ④	D or lower ⑤
4. I took this course primarily as a:	pure elective ①	college/ divisional requirement ②	graduate/ professional school requirement ③	major requirement ④	elective within the major ⑤
5. I would describe the effort I put into this course as:	very little ①	little ②	moderate ③	great ④	very great ⑤
6. Compared to other courses I've taken at Williams, the workload in this course was:	much lighter than average ①	lighter than average ②	average ③	heavier than average ④	much heavier than average ⑤
7. Compared to other courses I've taken at Williams, the difficulty of this course was:	much less than average ①	less than average ②	average ③	greater than average ④	much greater than average ⑤



In each of the following areas, please rate your INSTRUCTOR as: very poor, poor, fair, good, very good, excellent, or truly exceptional. If appropriate, mark "not applicable".

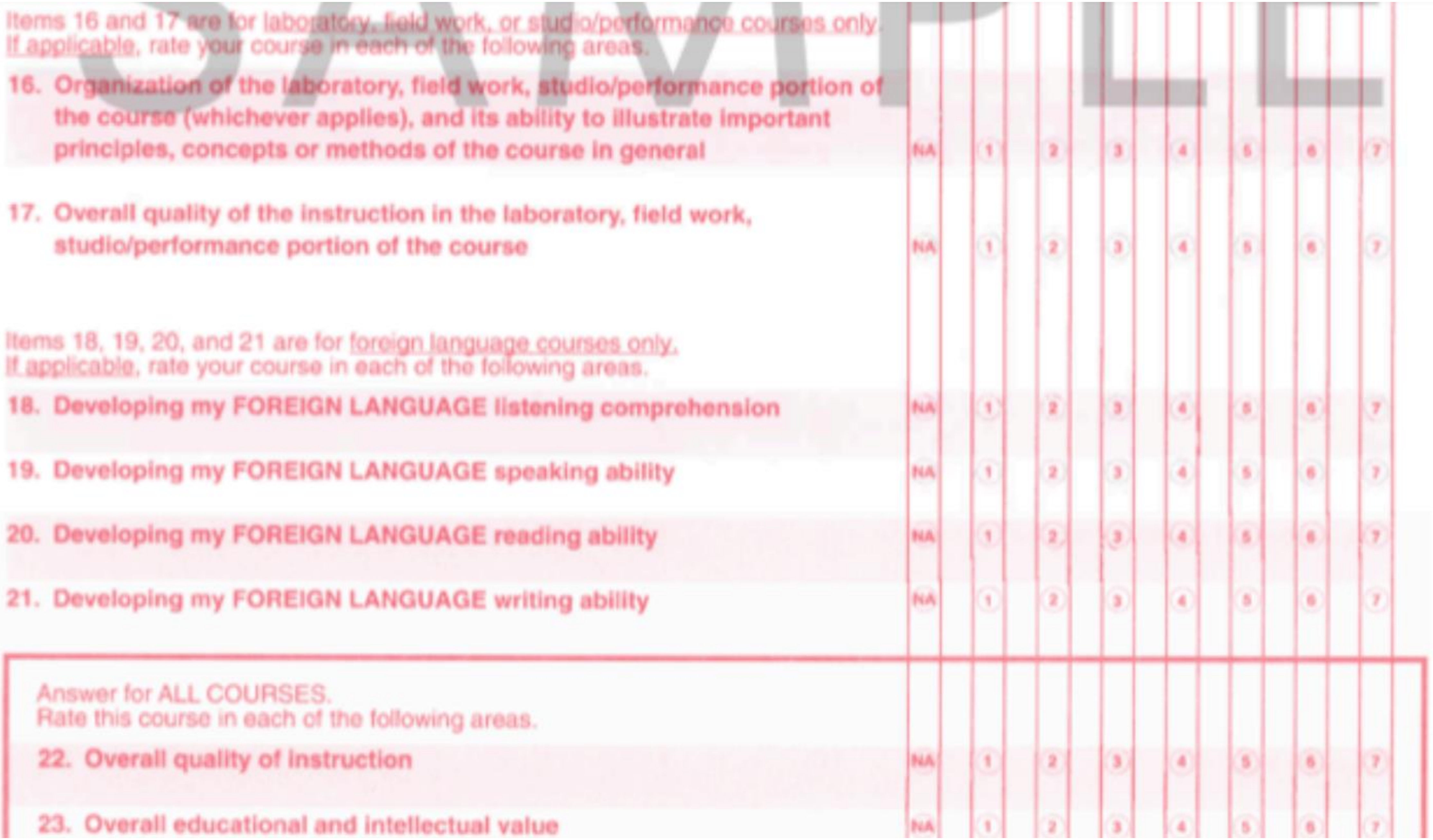
	Not applicable	Very poor	Poor	Fair	Good	Very good	Excellent	Truly exceptio
8. Organization of course material and class time	NA	1	2	3	4	5	6	7
9. Conveying the subject matter of the course in a clear way	NA	1	2	3	4	5	6	7
10. Approachability and responsiveness	NA	1	2	3	4	5	6	7
11. Providing useful comments and other feedback on course work	NA	1	2	3	4	5	6	7
12. Developing my analytical and/or critical thinking skills	NA	1	2	3	4	5	6	7
13. Promoting my intellectual engagement with the subject matter of the course	NA	1	2	3	4	5	6	7

If <u>discussion</u> figured prominently in this course, rate your INSTRUCTOR in:

14. Promoting class discussion	NA	1	2	3	4	5	6	7

If <u>lecture</u> figured prominently in this course, rate your INSTRUCTOR in:

15. Presenting effective lectures	NA	1	2	3	4	5	6	7



Items 16 and 17 are for laboratory, field work, or studio/performance courses only.
If applicable, rate your course in each of the following areas.

16. Organization of the laboratory, field work, studio/performance portion of the course (whichever applies), and its ability to illustrate important principles, concepts or methods of the course in general	NA	1	2	3	4	5	6	7
17. Overall quality of the instruction in the laboratory, field work, studio/performance portion of the course	NA	1	2	3	4	5	6	7

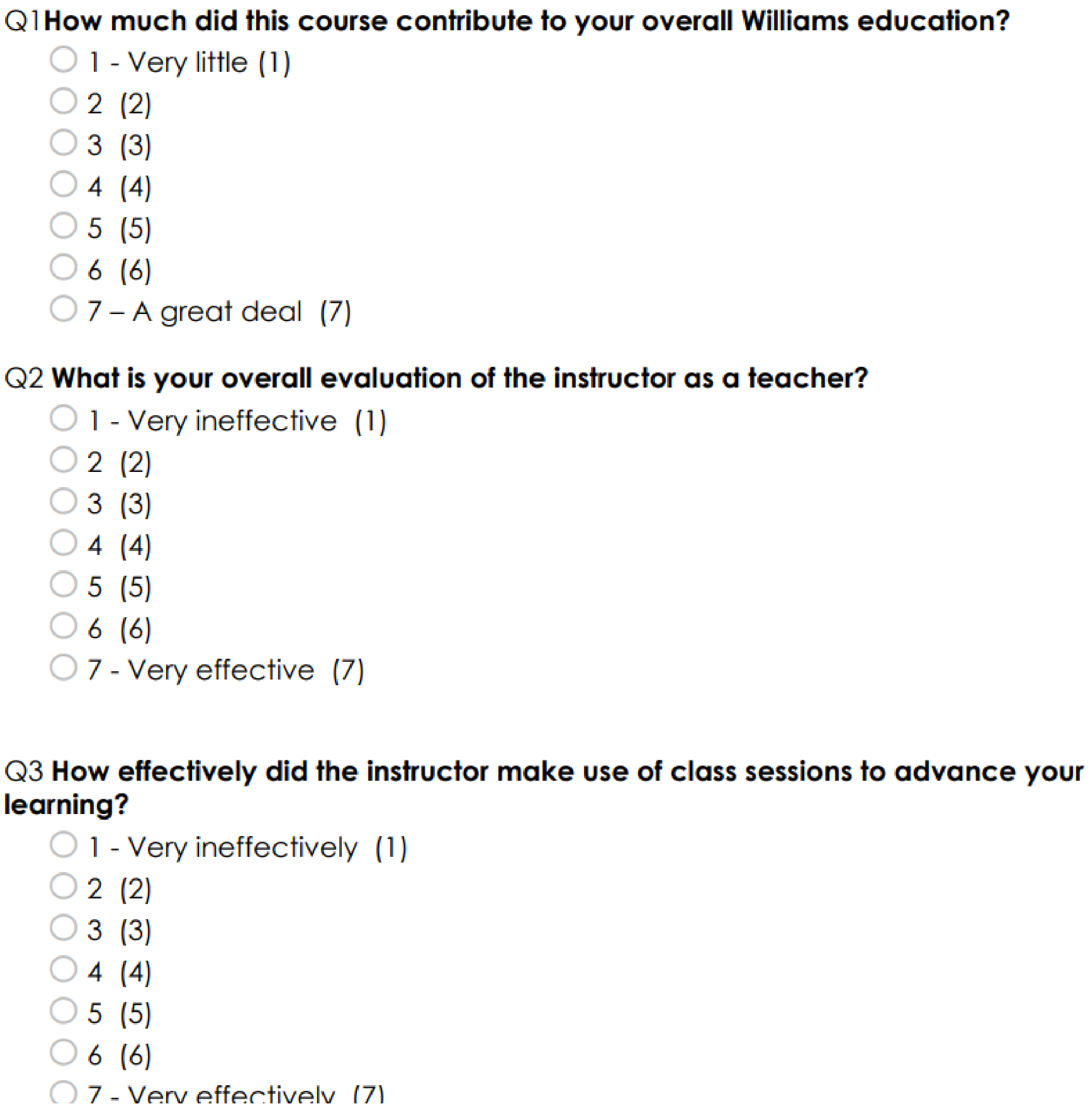
Items 18, 19, 20, and 21 are for foreign language courses only.
If applicable, rate your course in each of the following areas.

18. Developing my FOREIGN LANGUAGE listening comprehension	NA	1	2	3	4	5	6	7
19. Developing my FOREIGN LANGUAGE speaking ability	NA	1	2	3	4	5	6	7
20. Developing my FOREIGN LANGUAGE reading ability	NA	1	2	3	4	5	6	7
21. Developing my FOREIGN LANGUAGE writing ability	NA	1	2	3	4	5	6	7

Answer for ALL COURSES.
Rate this course in each of the following areas.

22. Overall quality of instruction	NA	1	2	3	4	5	6	7
23. Overall educational and intellectual value	NA	1	2	3	4	5	6	7

https://www.williams.edu/institutional-research/files/2020/10/Williams_College_Student_Course_Survey_Form_Effective_Fall_2020.pdf



Q1 **How much did this course contribute to your overall Williams education?**

- ○ 1 - Very little (1)
- ○ 2 (2)
- ○ 3 (3)
- ○ 4 (4)
- ○ 5 (5)
- ○ 6 (6)
- ○ 7 – A great deal (7)

Q2 **What is your overall evaluation of the instructor as a teacher?**

- ○ 1 - Very ineffective (1)
- ○ 2 (2)
- ○ 3 (3)
- ○ 4 (4)
- ○ 5 (5)
- ○ 6 (6)
- ○ 7 - Very effective (7)

Q3 **How effectively did the instructor make use of class sessions to advance your learning?**

- ○ 1 - Very ineffectively (1)
- ○ 2 (2)
- ○ 3 (3)
- ○ 4 (4)
- ○ 5 (5)
- ○ 6 (6)
- ○ 7 - Very effectively (7)

Q4 **Did you seek help with course material from the instructor outside of class?**

○ Yes (1)
○ No (0)

Display This Question:

If Did you seek help with course material from the instructor outside of class? = Yes

Q5 **How helpful was the instructor in discussing course material outside of class?**

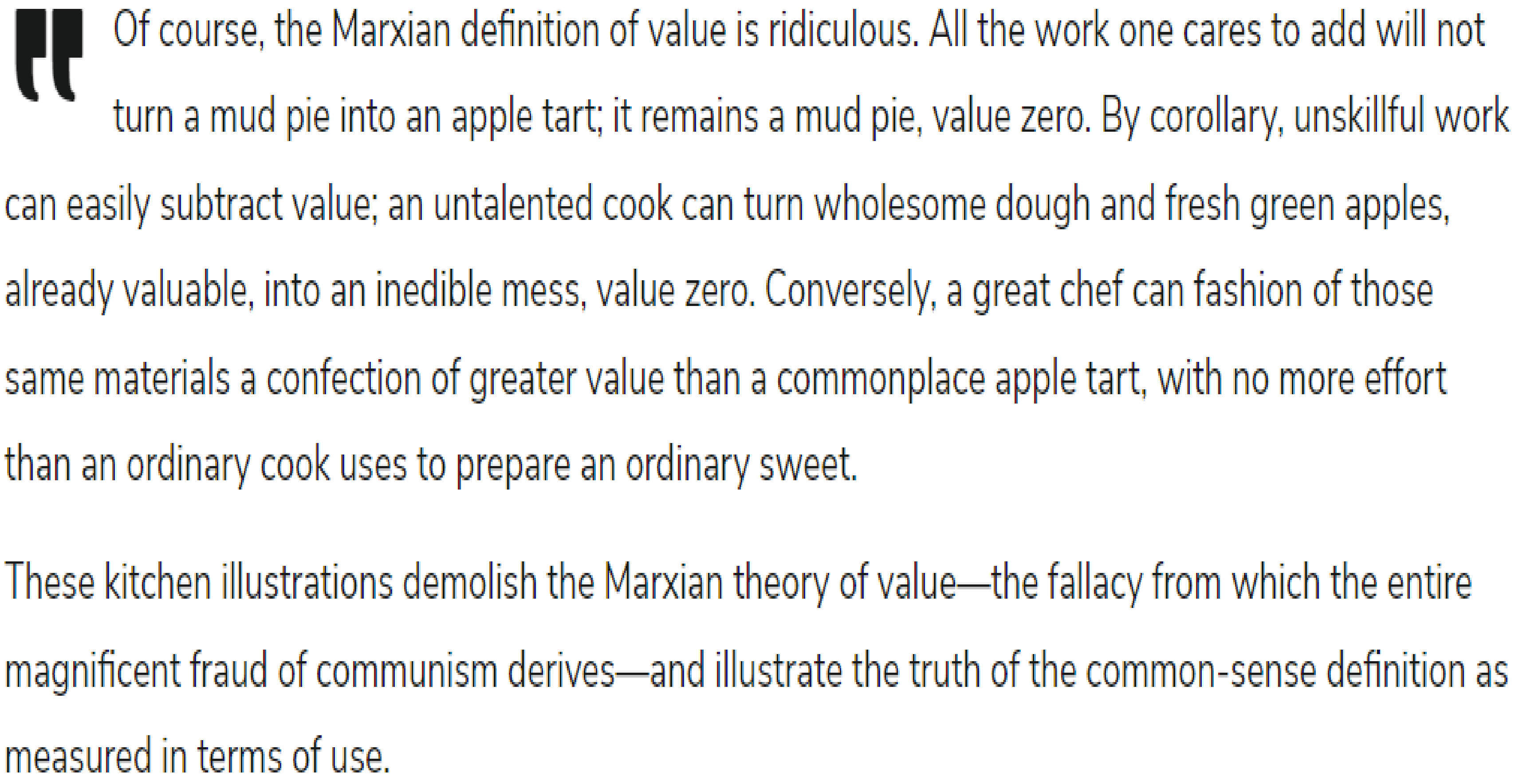
○ 1 - Very unhelpful (1)
○ 2 (2)
○ 3 (3)
○ 4 (4)
○ 5 (5)
○ 6 (6)
○ 7 - Very helpful (7)

Q6 **How helpful to your learning was the feedback you received from the instructor in the course?**

○ 1 - Very unhelpful (1)
○ 2 (2)
○ 3 (3)
○ 4 (4)
○ 5 (5)
○ 6 (6)
○ 7 - Very helpful (7)

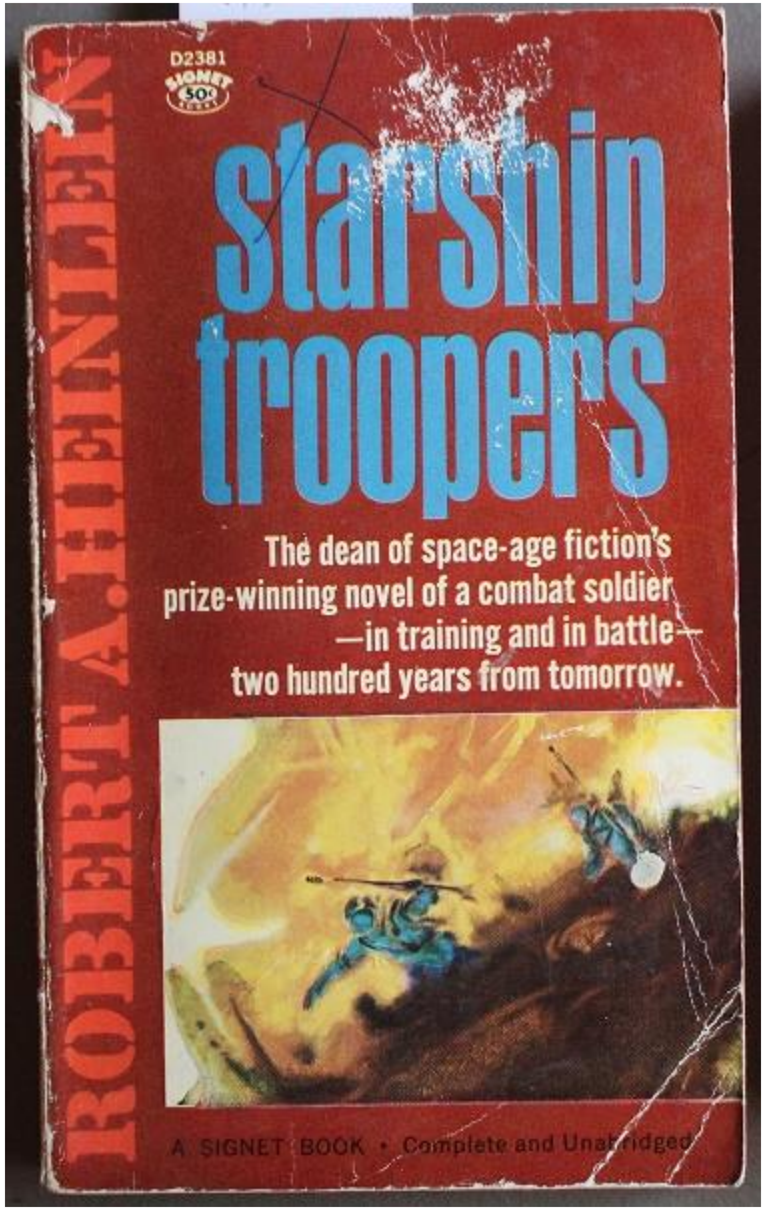
Q7 **How would you evaluate the workload in this course?**

○ 1 - Very low workload (1)
○ 2 (2)
○ 3 (3)
○ 4 (4)
○ 5 (5)
○ 6 (6)
○ 7 - Very high workload (7)



> Of course, the Marxian definition of value is ridiculous. All the work one cares to add will not turn a mud pie into an apple tart; it remains a mud pie, value zero. By corollary, unskillful work can easily subtract value; an untalented cook can turn wholesome dough and fresh green apples, already valuable, into an inedible mess, value zero. Conversely, a great chef can fashion of those same materials a confection of greater value than a commonplace apple tart, with no more effort than an ordinary cook uses to prepare an ordinary sweet.
>
> These kitchen illustrations demolish the Marxian theory of value—the fallacy from which the entire magnificent fraud of communism derives—and illustrate the truth of the common-sense definition as measured in terms of use.

https://www.litcharts.com/lit/starship-troopers/characters/mr-dubois



Math 344: Mathematics of Sports: Spring 2023:

Lecture 32: Egg Drop Mathematics: It *IS* all it's cracked up to be:

Plan for the day.

- Analyze the Egg Drop Problem
- Discuss Optimization Lessons

Egg Drop Mathematics: It *IS* all it's cracked up to be.

You have a building with N floors and you have 2 golden eggs. These are very special eggs. There is some floor n such that if you drop either egg from below n there is no damage; you can drop as many times as you wish. HOWEVER, if you drop even once from floor n or higher they immediately break. Find in as few drops as you can what n is; in other words, in as few drops as you can, what is the lowest floor where if you drop from there it breaks? Note it doesn't matter if you have any of the golden eggs at the end - we just want to know n.

http://techieme.in/wp-content/uploads/buildingeggs.png

https://cdn-media-1.freecodecamp.org/images/0*Uq3d3wn7B8yOPGIK.jpg

Two Codes/algorithms

1: runs 1 sec for all but one out of 10^6 inputs, which takes $10^6 - 1$ seconds

average run-time is ≈ 2 seconds

2: runs in 1000 seconds for all inputs

average run-time is 1000 seconds

Switch: Do 1, if doesn't terminate in 2 seconds switch to 2, average run-time is ≈ 1

One egg! Drop floor 1, then 2, then 3 . . .

Two eggs! ?

- First at $\frac{N}{2}$, if breaks then 1, 2, 3 . . .
  - ↳ if not then at $\frac{3N}{4}$, if breaks $\frac{N}{2}+1$

- Start at 2, if breaks do 1
  - ↳ if not then 4, if breaks then 3 . . .

More generally

Drop at floor f, if break do 1, 2, ...

↳ else drop at 2f, if break do f+1, ...

Extreme Cases!

f = N/2

Can eliminate many but danger of having to do a lot of floor by floor instead

and

f = 2

once break know within 1 floor, but could take awhile to break

Worst Case Scenario

$f = N/2$

$\#drops = 1 + \left(\frac{N}{2} - 1\right)$

$= \frac{N}{2}$

f

$\#drops =$

$\frac{N}{f} + f - 1$

use x

not f:

$\frac{N}{x} + x - 1$

$f = 2$

$\#drops: \frac{N}{2} + 1$

Minimize $\left(\frac{N}{x}+x\right)-1$ with $x \in \{1,2,\dots,N\}$

Calculus: $x = 1$ or 2, N (endpoints)

$g(x) = \frac{N}{x}+x$ $\quad g'(x) = -\frac{N}{x^2}+1$

so $g'(x)=0 \Rightarrow x=\sqrt{N}$

Cost $\approx \frac{N}{\sqrt{N}}+\sqrt{N}-1 \approx 2\sqrt{N}$

Non Calc Heuristic: $\underbrace{\frac{N}{x}}_{x \text{ large}}$ and $\underbrace{x}_{x \text{ small}}$: Set $\frac{N}{x}=x$

yields $x=\sqrt{N}$

$N = 105$ 1st drop at 14

drop at 14: if crack need 13 more, total is 14

↳ if crack next at 14+13, if crack need 12, total is 14

↳ if crack next at 14+13+12, if crack need 11, total is 14

$$14+13+12+\cdots+1 = \frac{14 \cdot 15}{2} = 7 \cdot 15 = 105$$

"dynamic dropping"

3 eggs!

Once egg cracks have the 2-egg problem

Drop at X, worse case: $\frac{N}{X} + \underbrace{\overset{\text{2 eggs to do}}{X}}$

$$\text{Cost}(X) \simeq \frac{N}{X} + 2\sqrt{X}$$

Heuristic

$$\frac{N}{X} \simeq 2X^{1/2}$$

so $N \sim X^{3/2}$

$X \sim N^{2/3}$

$$C'(X) = -\frac{N}{X^2} + X^{-1/2}$$

so $N \simeq X^{3/2}$

or $X \simeq N^{2/3}$

run time $\sim N^{1/3}$

# eggs	Drop	Run-time
1	1	N
2	$N^{1/2}$	$N^{1/2}$
3	$N^{2/3}$	$N^{1/3}$
k	$N^{\frac{k-1}{k}}$	$N^{1/k}$

$k+1$ eggs! Cost: $\frac{N}{x} + x^{1/k}$

$c'(x) = -\frac{N}{x^2} + \frac{1}{k} x^{\frac{1}{k}-1}$

So $N = x^{1+1/k}$

So $x = N^{k/k+1}$

Math 344: Mathematics of Sports: Spring 2023:
Lecture 34: Golf Statistics: https://youtu.be/MRlcDVvVwOI
Lecture 35: NBA Analytics: http://youtu.be/osTQa4Utd_g

Plan for the day.

- Watch video