A Simple Counterexample to Havil's "Reformulation" of the Riemann Hypothesis

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The Riemann Hypothesis (RH) is "the greatest mystery in mathematics" [3]. It is a conjecture about the Riemann zeta function. The zeta function allows us to pass from knowledge of the integers to knowledge of the primes.

In his book *Gamma: Exploring Euler's Constant* [4, p. 207], Julian Havil claims that the following is "a tantalizingly simple reformulation of the Riemann Hypothesis."

Havil's Conjecture. If

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \cos(b \ln n) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \sin(b \ln n) = 0$$

for some pair of real numbers a and b, then $a = 1/2$.

In this note, we first state the RH and explain its connection with Havil's Conjecture. Then we show that *the pair of real numbers* $a = 1$ *and* $b = 2\pi/\ln 2$ *is a counterexample to Havil's Conjecture, but not to the RH.* Finally, we prove that Havil's Conjecture becomes a *true* reformulation of the RH if his conclusion "then $a = 1/2$" is weakened to "then $a = 1/2$ or $a = 1$."

The Riemann Hypothesis In 1859 Riemann published a short paper *On the number of primes less than a given quantity* [6], his only one on number theory. Writing s for a complex variable, he assumes initially that its real part $\Re(s)$ is greater than 1, and he begins with *Euler's product-sum formula*

$$\prod_p \frac{1}{1-\frac{1}{p^s}} = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\Re(s) > 1).$$

Here the product is over all primes p. To see the equality, expand each factor $1/(1-1/p^s)$ in a geometric series, multiply them together, and use unique prime factorization.

Euler proved his formula only for real s (> 1). He used it to give a new proof of Euclid's theorem on the infinitude of the primes: if there were only finitely many, then as $s \to 1^+$ the left-hand side of the formula would approach a finite number while the right-hand side approaches the harmonic series $\sum 1/n = \infty$. Going further than Euclid, Euler also used his formula to show that, unlike the squares, the primes are so close together that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{23}+\frac{1}{29}+\frac{1}{31}+\frac{1}{37}+\cdots=\infty.$$

Aiming to go even further, Riemann develops the properties of the *Riemann zeta function* $\zeta(s)$. He defines it first as the sum of the series in Euler's formula,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad (\Re(s) > 1), \tag{1}$$

which converges by comparison with the p-series (here $p = \Re(s)$)

$$\sum \left| \frac{1}{n^s} \right| = \sum \frac{1}{n^{\Re(s)}}.$$

Using other formulas for $\zeta(s)$, Riemann extends its definition to the whole complex plane, where it has one singularity, a simple pole at $s = 1$, reflecting Euler's observation that $\zeta(s) \to \sum 1/n = \infty$ as $s \to 1^+$.

Riemann analyzes the *zeros of* $\zeta(s)$, which he calls the roots of the equation $\zeta(s) = 0$. He shows that there are none in the right half-plane $\{s : \Re(s) > 1\}$, and that the only ones in the left half-plane $\{s : \Re(s) < 0\}$

are the negative even integers $s = -2, -4, -6, \ldots$. (These real zeros had been found by Euler more than a century earlier—see [1].)

Turning his attention to the zeros in the closed strip $\{s : 0 \le \Re(s) \le 1\}$, Riemann proves that they are symmetrically located about the vertical line $\Re(s) = 1/2$. Using an integral, he estimates the number of them with imaginary part between 0 and some bound T. Then he says:

> One finds in fact about this many [on the line] within these bounds and it is very likely that all of the [zeros in the strip are on the line]. One would of course like to have a rigorous proof of this; however, I have tentatively put aside the search for such a proof after some fleeting vain attempts ....

Thus was born the now famous and still unproven RH.

Riemann Hypothesis. If $\zeta(s) = 0$ and $s \ne -2, -4, -6, \ldots$, then $\Re(s) = 1/2$.

Around 1896 Hadamard and de la Vallée Poussin, independently, took a step in the direction of the RH by proving that $\zeta(s) \ne 0$ on the line $\Re(s) = 1$. This was a crucial ingredient in their proofs of the *Prime Number Theorem* (PNT), which estimates $\pi(x)$, *the number of primes p with* $2 \le p \le x$. Conjectured by Gauss in 1792 at the age of 15, the PNT says that $\pi(x)$ is asymptotic to $x/\ln x$ as $x \to \infty$. More precisely, Gauss's guess was that $\pi(x)$ is asymptotic to the *logarithmic integral*

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{t},$$

which is equal to $x/\ln x$ plus a smaller quantity. If the RH is true, then the PNT's asymptotic formula $\pi(x) \sim \mathrm{Li}(x)$ comes with a good bound on the error $\pi(x) - \mathrm{Li}(x)$. Namely, the RH implies: for any $\epsilon > 0$, the inequality

$$|\pi(x) - \mathrm{Li}(x)| < Cx^{\frac{1}{2}+\epsilon} \qquad (x \ge 2)$$

holds, with some positive number C that depends on ϵ but not on x. In fact, this statement is equivalent to the RH, and the appearance of $1/2$ in both is not a coincidence.

Since $\zeta(s) \neq 0$ when $\Re(s) = 1$, the symmetry of the zeros in the strip $\{s : 0 \leq \Re(s) \leq 1\}$ implies that they lie in the *open* strip $\{s : 0 < \Re(s) < 1\}$. In 1914 Hardy proved that infinitely many of them lie on the line $\Re(s) = 1/2$. That lends credence to the RH, but of course does not prove it. Hardy's result was later improved by Selberg and others, who showed that a positive percentage of the zeros in the strip lie on its center line. Using computers and further theory, the first 10^{13} zeros have been shown to lie on the line. For more on the RH, the PNT, and their historical background, consult [2], [3], and [4, Chapter 16].

In order to relate the RH to Havil's Conjecture, we need to introduce a function closely related to $\zeta(s)$.

The alternating zeta function For s with positive real part, the *alternating zeta function* $\zeta_*(s)$ (also known as the *Dirichlet eta function* $\eta(s)$) is defined as the sum of the alternating series

$$\zeta_*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots \quad (\Re(s) > 0),$$

which we now show converges. Since $\Re(s) > 0$, the nth term approaches 0 as $n \to \infty$, so that we only need to show convergence for the series formed by grouping the terms in odd-even pairs. Writing each pair as an integral

$$\frac{1}{n^s} - \frac{1}{(n+1)^s} = s \int_n^{n+1} \frac{dx}{x^{s+1}}$$

with n odd, we set $\sigma = \Re(s)$ and bound the integral:

$$\left| \int_n^{n+1} \frac{dx}{x^{s+1}} \right| \leq \int_n^{n+1} \frac{dx}{|x^{s+1}|} = \int_n^{n+1} \frac{dx}{x^{\sigma+1}} < \frac{1}{n^{\sigma+1}}.$$

As the series $\sum n^{-\sigma-1}$ converges, so does the alternating series for $\zeta_*(s)$.

When $\Re(s) > 1$ the alternating series converges absolutely, and so we may rewrite it as the difference

$$\begin{aligned}
\zeta_*(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right) \\
&= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots - \frac{2}{2^s}\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) \\
&= \zeta(s) - \frac{2}{2^s}\zeta(s).
\end{aligned}$$

Thus the alternating zeta function is related to the Riemann zeta function by the simple formula

$$\zeta_*(s) = \left(1 - 2^{1-s}\right)\zeta(s). \tag{2}$$

We derived it for s with $\Re(s) > 1$, but a theorem in complex analysis guarantees that the formula then remains valid over the whole complex plane.

At the point $s = 1$, the simple pole of $\zeta(s)$ is cancelled by a zero of the factor $1-2^{1-s}$. This agrees with the fact that $\zeta_*(s)$ is finite at $s = 1$. Indeed, $\zeta_*(1)$ is equal to Mercator's alternating harmonic series

$$\zeta_*(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2.$$

The product formula (2) shows that $\zeta_*(s)$ vanishes at each zero of the factor $1 - 2^{1-s}$ with the exception of $s = 1$. (This can also be proved without using (2)—an elementary proof is given in [5], [7], and [8].) It is a nice exercise to show that the zeros of $1 - 2^{1-s}$ lie on the line $\Re(s) = 1$, and occur at the points s_k given by

$$s_k = 1 + i\frac{2\pi k}{\ln 2} \qquad (k = 0, \pm 1, \pm 2, \pm 3, \ldots).$$

Thus s_k is also a zero of $\zeta_*(s)$ if $k \neq 0$.

Since $1 - 2^{1-s} \neq 0$ when $\Re(s) \neq 1$, relation (2) also shows that $\zeta_*(s)$ and $\zeta(s)$ have the same zeros in the strip $\{s : 0 < \Re(s) < 1\}$. The first one is

$$\rho_1 = 0.5 + 14.134725141734693790457251983562470270784257115699 2\ldots i,$$

the Greek letter ρ (rho) standing for root. Using a calculator, the reader can see it is likely that $\zeta_*(\rho_1) = 0$. But be patient: at $s = \rho_1$ the alternating series for $\zeta_*(s)$ converges very slowly, because its nth term has modulus $|(-1)^{n-1}n^{-\rho_1}| = n^{-1/2}$. For example, to get $n^{-1/2} < 0.1$, you need $n > 100$.

If we substitute the series for $\zeta_*(s)$ into equation (2) and solve for $\zeta(s)$, we obtain the formula

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \qquad (\Re(s) > 0,\ s \neq 1). \tag{3}$$

Since the series converges whenever $\Re(s) > 0$, the right-hand side makes sense for all $s \neq 1$ with positive real part, the first factor's poles at $s = s_k \neq 1$ being cancelled by zeros of the second factor. Thus the formula extends the definition (1) of $\zeta(s)$ to a larger domain.

We can now explain the relation between the RH and Havil's Conjecture.

The counterexample Let's write $s = a + ib$, where a and b are real numbers. Euler's famous formula

$$e^{ix} = \cos x + i \sin x$$

shows that

$$\frac{1}{n^s} = \frac{1}{n^{a+ib}} = \frac{1}{n^a} e^{-ib\ln n} = \frac{1}{n^a} \left(\cos(b\ln n) - i\sin(b\ln n)\right).$$

Now if $a = \Re(s) > 0$, then

$$\begin{aligned} \zeta_*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^a} \left(\cos(b\ln n) - i\sin(b\ln n)\right) \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \cos(b\ln n) + i \sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \sin(b\ln n). \end{aligned} \tag{4}$$

Up to sign, the last two series are the real and imaginary parts of $\zeta_*(s)$. Hence $\zeta_*(s) = 0$ if and only if both series vanish. Since they are the same series as in Havil's Conjecture and $\zeta_*(s) = 0$ at $s = s_1 = 1 + 2\pi i/\ln 2$, *the* *pair of real numbers $a = 1$ and $b = 2\pi/\ln 2$ is a counterexample to Havil's Conjecture.*

On the other hand, since the theorem of Hadamard and de la Vallée Poussin says that $\zeta(s)$ has no zeros with real part equal to 1, *the point $s_1 = 1 + 2\pi i/\ln 2$ is not a counterexample to the RH.* Therefore, Havil's Conjecture is *not* a reformulation of the RH.

Note that from looking at the two series in his conjecture it is not at all clear that they are equal to 0 when $a = 1$ and $b = 2\pi/\ln 2$. This shows the power of the alternate formulation $\zeta_*(s_1) = 0$. Havil gives formula (3) for $\zeta(s)$, but not the equivalent formula (2) for $\zeta_*(s)$. That may account for his having overlooked the zeros of the factor $1 - 2^{1-s}$ in (2).

To conclude, we give a *true* reformulation of the RH.

The RH without tears Here is a corrected version of Havil's Conjecture.

New Conjecture. If

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \cos(b \ln n) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \sin(b \ln n) = 0 \tag{5}$$

for some pair of real numbers a and b, then $a = 1/2$ or $a = 1$.

Let's show that this is indeed a reformulation of the RH.

Proposition. *The New Conjecture is true if and only if the RH is true.*

Proof. Suppose the New Conjecture is true. Assume $\zeta(s) = 0$ and $s \neq -2, -4, -6, \ldots$. By Riemann's results and the Hadamard-de la Vallée Poussin theorem, s lies in the open strip $\{s : 0 < \Re(s) < 1\}$. Then relation (2) gives $\zeta_*(s) = 0$. Writing $s = a + ib$, equation (4) yields the equalities in (5), and so by the New Conjecture $a = 1/2$ or $a = 1$. But $a = \Re(s) < 1$. Hence $\Re(s) = 1/2$. Thus the New Conjecture implies the RH.

Conversely, suppose the RH is true. Assume a and b satisfy condition (5). In particular, both series in (5) converge, and so their nth terms tend to 0 as $n \to \infty$. It follows that the sum of the squares of the nth terms, namely, n^{-2a}, also tends to 0. Hence $a > 0$. Then with $s = a+ib$ equation (4) applies, and (5) yields $\zeta_*(s) = 0$. Now relation (2) shows that s is a zero of $\zeta(s)$ or of $1 - 2^{1-s}$. In the first case the RH says $a = 1/2$, and in the second case we know $a = 1$. Thus the RH implies the New Conjecture.

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