NOTES

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Lagrange Multipliers and the Fundamental Theorem of Algebra

Theo de Jong

Consider a polynomial

$$F(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

with complex coefficients $a_i \in \mathbb{C}$ of degree $n > 0$. The Fundamental Theorem of Algebra states that there exists a $z \in \mathbb{C}$ with $F(z) = 0$. This paper presents a proof of this fundamental theorem, a proof which the author has not been able to find in the extensive literature on this subject.

1. THE PROOF. Write $z = x + iy$ and

$$F(x + iy) = P(x, y) + iQ(x, y)$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials. We will study for $c \in \mathbb{R}$ the level curves

$$L_c := \{(x, y) : P(x, y) = c\} \quad \text{and} \quad M_c := \{(x, y) : Q(x, y) = c\}.$$

Notice that $P(x, 0) = x^n + \cdots$, so $P(x, 0)$ is certainly not a constant function. By elementary calculus, one has that $P(x, 0)$ takes infinitely many values; in particular the level curves L_c are nonempty for infinitely many c.

We wish to apply the theory of Lagrange multipliers to the curve L_c. We thus need that L_c is nonsingular, that is, for all points (a, b) on L_c we have

$$\nabla P(a, b) = \left(P_x(a, b), P_y(a, b)\right) \neq (0, 0).$$

Here, as usual, P_x and P_y denote the partial derivatives. Notice that there are only finitely many level curves L_c that are singular. Indeed, the derivative $F'(z)$ has only finitely many roots. If one considers the map F' as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, one has that the deriviative of $F = (P, Q)$ at (a, b) is given by the matrix

$$\begin{pmatrix} P_x(a, b) & P_y(a, b) \\ Q_x(a, b) & Q_y(a, b) \end{pmatrix} = \begin{pmatrix} P_x(a, b) & P_y(a, b) \\ -P_y(a, b) & P_x(a, b) \end{pmatrix}.$$

The equality of the matrices follows from the Cauchy-Riemann equations[1]

$$P_x = Q_y \qquad P_y = -Q_x.$$

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[1]The Cauchy-Riemann equations hold for general holomorphic functions defined on an open set of $\mathbb{C}$; for polynomials these equations can be proved in a simple algebraic way.

It follows that there are only finitely many level curves L_c that contain a singular point. Notice, moreover, that if L_c is nonsingular at (a, b), then with $d = Q(a, b)$, the level curve M_d is also nonsingular at (a, b).

The main idea of the proof is in the following lemma.

Lemma. *Suppose the level curve L_c is nonempty and contains no singular points. Then there exists an $(a, b) \in L_c$ with $Q(a, b) = 0$.*

Proof. We look at the function

$$Q^2 \colon L_c \to \mathbb{R}_{\geq 0}.$$

It has a global minimum. Indeed, it is easy to see, and used in many proofs of the Fundamental Theorem of Algebra, that

$$|F(z)|^2 = P^2(x, y) + Q^2(x, y)$$

goes to infinity when z goes to infinity. If we take a fixed point $(p, q) \in L_c$, we thus can find an $R \gg 0$ such that

$$P^2(x, y) + Q^2(x, y) = c^2 + Q^2(x, y) > c^2 + Q^2(p, q)$$

for all $(x, y) \in L_c$ with $x^2 + y^2 > R^2$. By elementary calculus, on the nonempty compact set

$$\{(x, y) \colon x^2 + y^2 \leq R\} \cap L_c$$

the function Q^2 has a minimum attained, say, at (a, b). This minimum is certainly smaller than or equal to $Q^2(p, q)$. Thus Q^2 has a global minimum at $(a, b) \in L_c$.

We can now apply the theorem on Lagrange multipliers to the function Q^2 and the nonsingular nonempty curve L_c: it says that there exists a $\lambda \in \mathbb{R}$ with $\nabla Q^2(a, b) = \lambda \nabla P(a, b)$. Written out, this means

$$2Q(a, b)\big(Q_x(a, b), Q_y(a, b)\big) = \lambda\big(P_x(a, b), P_y(a, b)\big).$$

Using the Cauchy-Riemann equations we get

$$2Q(a, b)\big(-P_y(a, b), P_x(a, b)\big) = \lambda\big(P_x(a, b), P_y(a, b)\big).$$

Thus either $Q(a, b) = 0$ or the vector $\big(-P_y(a, b), P_x(a, b)\big)$ is perpendicular to itself. But as is easy to see, a vector can only be perpendicular to itself if it is the zero vector. Since $\big(P_x(a, b), P_y(a, b)\big) \neq (0, 0)$, as we assumed the level curve L_c to be nonsingular, we conclude that $Q(a, b) = 0$. ■

Everything that has been said about L_c and Q also holds with the roles of L and M, and Q and P, reversed. We apply this freely in the sequel.

Proof of the Fundamental Theorem of Algebra. As noted before there are infinitely many values of c such that L_c is nonempty and nonsingular. Applying the lemma shows that for some $(a, b) \in L_c$, $Q(a, b) = 0$, so the level curve M_0 is not empty. If M_0 is also nonsingular, we can apply the lemma with the roles of M and L and Q and P exchanged to show that there exists a point $(p, q) \in M_0$ with $P(p, q) = 0$. Thus $(p, q) \in L_0 \cap M_0$ and therefore $F(p + iq) = 0$.

If M_0 is singular, we notice that Q takes infinitely many values, as M_0 is nonsingular at (a, b). Thus we can take a sequence (c_n) converging to 0 such that M_{c_n} is nonsingular and nonempty. Applying the lemma, we find $(a_n, b_n) \in M_{c_n}$ with $P(a_n, b_n) = 0$. As $P^2 + Q^2$ goes to infinity when (x, y) goes to infinity, we see that (a_n, b_n) is bounded. By passing to a subsequence, we may assume that (a_n, b_n) converges to a limit which we denote by $(\bar{a}, \bar{b})$. Then

$$P(\bar{a}, \bar{b}) = \lim_{n\to\infty} P(a_n, b_n) = 0;$$

$$Q(\bar{a}, \bar{b}) = \lim_{n\to\infty} Q(a_n, b_n) = \lim_{n\to\infty} c_n = 0.$$

Thus we have achieved our goal of finding an intersection point of L_0 and M_0, and the Fundamental Theorem of Algebra is proved. ■

2. SOME REMARKS. The idea of finding an intersection point of L_0 and M_0 first appeared in the first proof of Gauß. In this proof he used some "obvious" properties of real algebraic curves, whose proofs he promised to give on demand, but which he never did prove. Those properties of real algebraic curves can be proved by an application of the implicit function theorem for the case of two variables. This theorem can be proved by elementary calculus, essentially using only the intermediate and mean value theorems. For complete details of the proof we refer to the paper of Ostrowski [**3**]. Other presentations of the first proof of Gauß can be found in [**4**] and [**1**]. In the third proof of Gauß the existence of the intersection point is proved as an application of Fubini's theorem for a rectangular region. The theorem of Lagrange multipliers, which is used in our proof, can be proved as an application of the implicit function theorem and some further elementary calculus.

Our proof can also be seen as a variant of the elementary proof which goes back to Legendre and Argand: here one proves that the function $|F(z)|: \mathbb{C} \to \mathbb{R}_{\geq 0}$ has an absolute minumum. If this minimum $|F(z_0)|$ is not zero, one arrives at a contradiction by showing that there exists a $z_1 \in \mathbb{C}$ with $|F(z_1)| < |F(z_0)|$.

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REFERENCES

1. B. Fine and G. Rosenberger, *The Fundamental Theorem of Algebra*, Undergraduate Texts in Mathematics, Springer Verlag, Berlin, 1997
2. E. Netto, *Die vier Gauss'schen Beweise für die Zerlegung ganzer algebraischer Functionen in reelle Factoren ersten oder zweiten Grades*, Engelmann, Leipzig, 1913.
3. A. Ostrowski, *Über den ersten und vierten Gauss'schen Beweis des Fundamentalsatzes der Algebra*, appendix to A. Fraenkel, *Zahlbegriff und Algebra bei Gauß*, Nachrichten der Gesellschaft der Wissenschaften Göttingen, Mathematisch-Physikalische Klasse, 1919, 1–58.
4. J. V. Uspensky, *The Theory of Equations*, McGraw-Hill, New York, 1963.

Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, Mainz, Germany
dejong@mathematik.uni-mainz.de