CHAPTER 2: CAUCHY'S THM AND ITS APPLICATIONS

SUMMARY

↳ Cauchy's Thm: γ simple closed curve separating plane into inside and outside, and f holomorphic in open Ω containing γ, then $\int_\gamma f(z)\,dz = 0$

↳ sometimes write $\oint_\gamma f(z)\,dz$ to emphasize closed curve

APPLICATIONS

↳ Many problems in math/phys can be reduced to evaluating an integral.

EXAMPLES

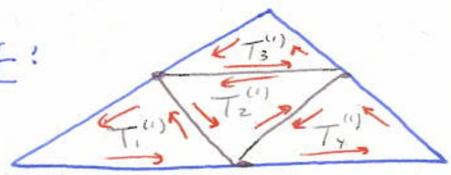
- Number Theory: Goldbach: $\int_0^1 \left(\sum_{p \leq N} e^{2\pi i p x}\right)^5 e^{-2\pi i n x}\,dx$ gives the # ways of writing n as a sum of 5 primes at most N
- Probability: Normalization Constants: $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$
- Fresnel integrals: $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$
- Cauchy → if holomorphic then infinitely differentiable
  - ↳ leads to Liouville: f entire and bounded ⇒ f constant
    - ↳ leads to a proof of the Fund Thm of Algebra

SEC 1: GOURSAT'S THM

Thm: GOURSAT
OPEN $\Omega \subset \mathbb{C}$, T TRIANGLE $\subset \Omega$, f HOLOMORPHIC IN Ω,
THEN $\int_T f(z)\,dz = 0$.

Comment! instead of triangles could use rectangles; convenient as triangularization.

Proof:



All interior integrals cancel

$$\int_{T^{(0)}} f\,dz = \sum_{j=1}^{4} \int_{T_j^{(1)}} f\,dz$$

(don't use i as dummy variable!)

Thus for at least one j, $\left| \int_{T^{(0)}} f\,dz \right| \le 4 \left| \int_{T_j^{(1)}} f\,dz \right|$

↳ JUSTIFY!

Call this triangle $T^{(1)}$, and continue

Get chain $T^{(0)} \supset T^{(1)} \supset T^{(2)} \ldots$

Triangles similar, $\operatorname{diam} T^{(n)} = 2^{-n} \operatorname{diam} T^{(0)}$

$\operatorname{perim} T^{(n)} = 2^{-n} \operatorname{perim} T^{(0)}$

$\operatorname{area} T^{(n)} = 4^{-n} \operatorname{area} T^{(0)}$

Thus there exists a unique z_0 in all $T^{(n)}$ ($\exists!\, z_0 \in T^{(n)}$ for all n)

$$\left| \int_{T^{(0)}} f\,dz \right| \le 4^n \left| \int_{T^{(n)}} f\,dz \right|$$

↳ Why useful (doing this? On $T^{(n)}$ f is approx constant

PROOF OF GOURSAT (CONT)

Now use f is holomorphic!

↳ f is differentiable at z_0, so for n large in $T^{(n)}$ have

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0)$$

$$\text{with } \lim_{z \to z_0} \psi(z) = 0$$

Now $$\int_{T^{(n)}} f\,dz = \int_{T^{(n)}} f(z_0)\,dz + \int_{T^{(n)}} f'(z_0)(z-z_0)\,dz + \int_{T^{(n)}} \psi(z)(z-z_0)\,dz$$

Note first two are integrals of polynomials on closed curve and thus vanish as have primitives z and $\frac{(z-z_0)^2}{2}$

Last integral is $\leq \max_{z \in T^{(n)}} |\psi(z)| \cdot \text{perim}\, T^{(n)}$

$$= \max_{z \in T^{(n)}} |\psi(z)| \cdot 4^{-n} \,\text{perim}\, T^{(0)}$$

Thus $$\left| \int_{T^{(0)}} f\,dz \right| \leq 4^{n} \cdot \max_{z \in T^{(n)}} |\psi(z)| \cdot 4^{-n} \,\text{perim}\, T^{(0)}$$

or $$\left| \int_{T^{(0)}} f\,dz \right| \leq \text{perim}\, T^{(0)} \cdot \max_{z \in T^{(n)}} |\psi(z)|$$

as $n \to \infty$, $\max_{z \in T^{(n)}} |\psi(z)| \to 0$ and thus $\int_{T^{(0)}} f\,dz = 0$ ∎

Comments: Note use of topology

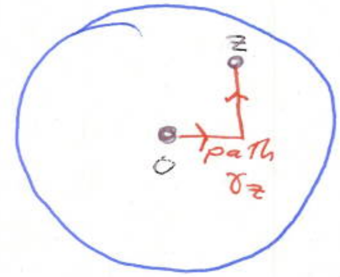
Note approx complicated fn f with linear function + error

SECTION 2: LOCAL EXISTENCE OF PRIMITIVES AND CAUCHY IN A DISK

THM: A holomorphic fn in an open disc has a primitive in the disc

Proof: Recall F is a primitive for f if F is holo on Ω and $F' = f$

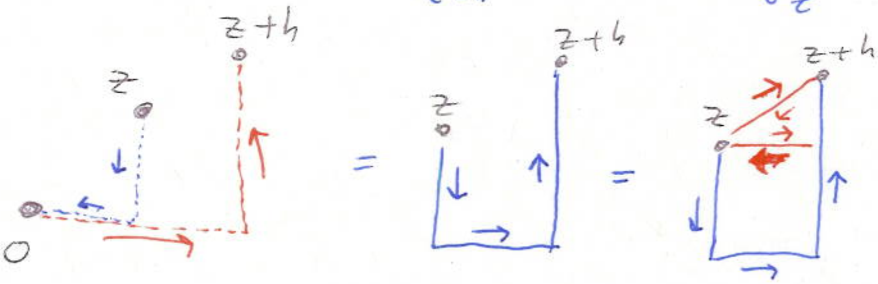
↳ Wlog assume disc centered at origin.

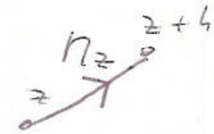


Define $F(z) := \int_{\gamma_z} f(w)\, dw$

Claim: F holo in Ω (our disc) and $F' = f$

↳ $F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw$





as integral over △ and ▭ are zero as f holo

Thus $F(z+h) - F(z) = \int_{\eta_z} f(w)\,dw$

↳ By continuity: $f(w) = f(z) + \psi(w)$, $\quad \psi(w) \to 0$ as $w \to z$

$$F(z+h) - F(z) = \int_{\eta_z} f(z)\,dw + \int_{\eta_z} \psi(w)\,dw$$

$$= f(z)\, w \Big|_z^{z+h} + \int_{\eta_z} \psi(w)\,dw$$

$$= f(z)\, h + \int_{\eta_z} \psi(w)\,dw$$

$$\Rightarrow \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \left| \int_{\eta_z} \psi(w)\,dw \right| / |h|$$

$$\leq \frac{\max_{w \in \eta_z} |\psi(w)| \cdot |h|}{|h|} \to 0$$

■

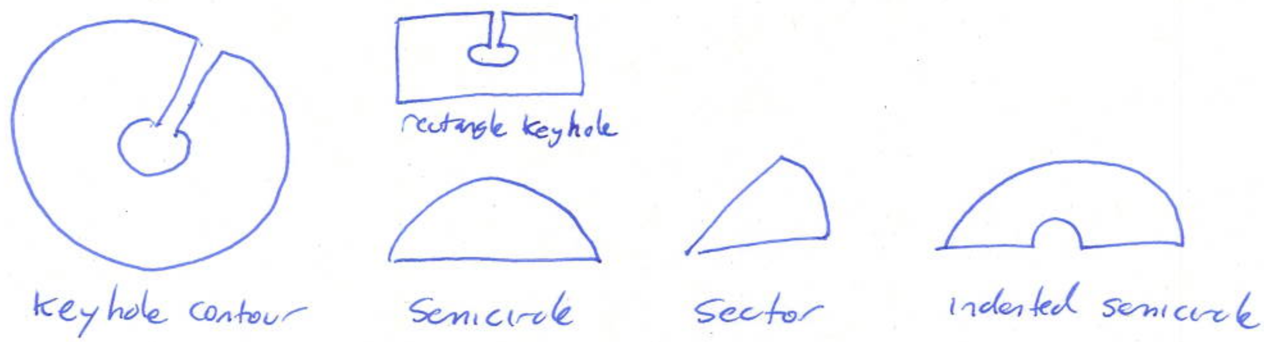
Sec 2: Continued

Cauchy's Thm: f holomorphic in a disc and γ closed curve in disc, then $\oint_\gamma f(z)\,dz = 0$

Proof: Previous theorem shows f has a primitive. Done by Cor 3.3 of Chapter 1.

Defn: Toy Contour: Any closed curve where notion of interior/exterior is clear.

Exs:



Skip Example 1

The examples make more sense after develop Residue Thm

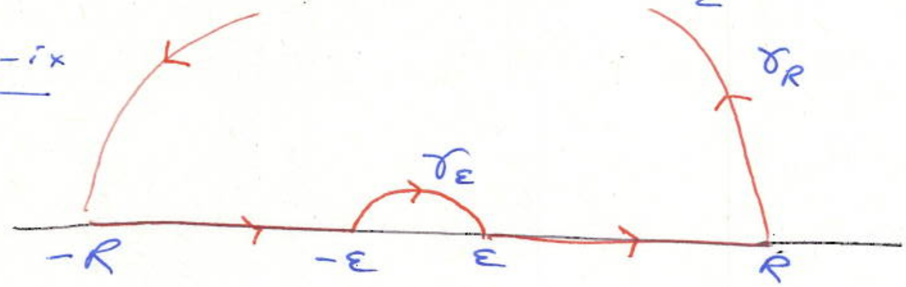
SEC 3: CONTINUED

Example 2: $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi$

First note reasonable: decays like $1/x^2$, near origin looks like $\frac{1}{2}$

Write $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$

Use following contour



$\boxed{\text{EQ } \#}$ $\displaystyle\int_{-R}^{-\varepsilon} + \int_{\gamma_\varepsilon} + \int_{\varepsilon}^{R} + \int_{\gamma_R} f(z)\,dz = 0$ if f holo on contour

Idea: $\displaystyle\int_{\gamma_\varepsilon} \to \overset{\text{known}}{a}$ as $\varepsilon \to 0$, $\displaystyle\int_{\gamma_R} \to 0$ as $R \to 0$

would leave us with $\displaystyle\int_{\varepsilon \le |x| \le R}$ of what we desire

PROBLEM: $e^{iz} = e^{ix-y}$ is small as $y \to \infty$ but e^{-iz} blows up.

If use $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ need to use upper half plane contour for e^{ix} and lower half plane for e^{-ix}

$\boxed{\text{Alternative: Take } f(z) = \dfrac{1-e^{iz}}{z^2} \text{ and take real part of } \#}$

On γ_ε have $\displaystyle\int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz = \int_{\gamma_\varepsilon} \frac{1-(1+iz+E(z))}{z^2}\,dz$ with $|E(z)| \le C|z|^2$

$$= \int_{\gamma_\varepsilon} -i\frac{dz}{z} - \underbrace{\int_{\gamma_\varepsilon} \frac{E(z)}{z^2}\,dz}$$

$$-i\int_{\pi}^{0} \frac{\varepsilon i e^{i\theta}\,d\theta}{\varepsilon e^{i\theta}}$$

integrand $\le C$

bounded by $C \cdot \pi\varepsilon \to 0$

$$-i \cdot i \cdot (-\pi)$$

SEC 3: CONTINUED

On γ_R, have $\left|\frac{1-e^{iz}}{z^2}\right| \le \frac{1+|e^{ix}|e^{-y}}{|z|^2} \le \frac{1+e^{-y}}{|z|^2} \le \frac{2}{R^2}$

as $|z| = R$ on γ_R. As length$(\gamma_R) = \pi R$, we have

$$\left|\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,dz\right| \le \frac{2}{R^2}\cdot \pi R \to 0$$

Thus $\displaystyle\int_{\varepsilon \le |x| \le R} \frac{1-e^{ix}}{x^2}\,dx - \pi +$ errors tending to zero as $\varepsilon \to 0$ and $R \to \infty$ $= 0$

so $\displaystyle\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\,dx = \pi$; result follows by taking real parts.

KEY TAKEAWAY

Choose function and contour carefully

Need to be able to exploit decay

SEC 4: CAUCHY'S INTEGRAL FORMULAS

THM: f holo in open Ω containing disc D with boundary C (with positive orientation). Then for any $z \in D$:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

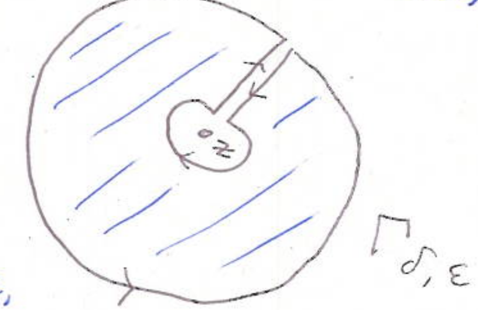
Will have <u>many</u> applications

↳ Note integral need not vanish as $\frac{f(\zeta)}{\zeta - z}$ is not holomorphic in ζ

↳ Lang: If Greeks were intelligent would have invented $\frac{1}{2\pi i}$ as constant

<u>Proof</u>: Here δ = width of corridor

ε = radius of small disc



As $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is holo in interior of $\Gamma_{\delta,\varepsilon}$,

$$\int_{\Gamma_{\delta,\varepsilon}} F(\zeta)\, d\zeta = 0$$

As $\delta \to 0$ integral over two parts of corridor cancel by continuity

$$\Rightarrow 0 = \int_C F(\zeta)\, d\zeta - \int_{C_\varepsilon} F(\zeta)\, d\zeta = 0, \quad C_\varepsilon \text{ positive orientation}$$

So
$$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Exploit the fact that f is holomorphic and ζ and z are close on C_ε:

$$\frac{f(\zeta)}{\zeta - z} = \underbrace{\frac{f(\zeta) - f(z)}{\zeta - z}}_{\text{bounded as } f \text{ holo}} + \frac{f(z)}{\zeta - z}$$

SEC 4: CAUCHY'S INTEGRAL FORMULA (CONT)

As $\dfrac{f(\zeta)-f(z)}{\zeta - z}$ is bounded on C_ε, the integral of this piece is bounded by $B \cdot 2\pi\varepsilon$ for some constant B.

Note $\displaystyle\int_{C_\varepsilon} \frac{f(z)}{\zeta - z}\, d\zeta$: Let $\zeta = z + \varepsilon e^{i\theta}$
$d\zeta = i\varepsilon e^{i\theta}\, d\theta$

$$= f(z)\int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\, d\theta}{\varepsilon e^{i\theta}} = 2\pi i f(z)$$ indep of ε!

Thus as $\varepsilon \to 0$ obtain $\displaystyle\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \cancel{\frac{1}{2\pi i}}\, f(z)$ ∎

CORR: f holo on open Ω then f is infinitely complex differentiable, and for any $z \in \Omega$, if C is a circle contained in Ω containing z,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

PROOF: Proceed by induction, already did base case $n=1$.

Can assume $\displaystyle f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^n}\, d\zeta$

Calculation (See Book)

↳ study $\dfrac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h}$

COR: CAUCHY INEQUALITIES: disc $D \subset$ open Ω, f holo on Ω, D has center z_0 and radius R, then

$$|f^{(n)}(z_0)| \le \frac{n! \sup_{z \in \partial D} |f|}{R^n}$$

SEC 4: CAUCHY'S FORMULA: CONT

Proof of Cauchy Ineq

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right|$$

Note denom is of size R^{n+1}, length$(C) = 2\pi R$

Claim follows by substitution

↳ Very important result: will prove Fund Thm of Alg!

Chapter 1: power series (analytic) $\Longrightarrow$ holomorphic: Converse true!

$\boxed{\text{Thm: } f \text{ holo on open } \Omega \supset \text{Disc } D \text{ with center } z_0. \text{ Then } \forall z \in D \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ with } a_n = \frac{f^{(n)}(z_0)}{n!}}$

Proof: $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) + (z_0 - z)} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$

$$= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

$$= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

As $f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} dz$

$$= \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \frac{f(\zeta)\,(z - z_0)^n}{(\zeta - z_0)^{n+1}} dz$$

↳ Done if $\int_C \sum_n = \sum_n \int_C$

Proof (cont)

Can interchange as absolute value uniformly converges (Fubini-Tonelli)

↳ Have $\left| \frac{z - z_0}{\zeta - z_0} \right| \leq r < 1$ for some r for $\zeta \in C$

As f continuous, f is bounded

Thus $\int_C \sum_n |\quad| < \infty$, and can interchange ∎

$$\boxed{\text{ANALYTIC} \iff \text{HOLOMORPHIC}}$$

THM: LIOUVILLE'S THM: If f is entire (holo on all of $\mathbb{C}$) and bounded then f is constant!

Proof: Choose B st $|f(z)| \leq B$

Cauchy Inequalities imply, for any circle C_R of radius R centered at the origin, that $(n \geq 1)$

$$|f^{(n)}(0)| \leq \frac{n! \sup_{z \in C_R} |f(z)|}{R^n} \leq \frac{n!\, B}{R^n}$$

as R is arbitrary must have $f^{(n)}(0) = 0 \quad \forall n \geq 1$

Thus Series expansion is $f(z) = f(0) + 0$

↳ f is constant! ∎

(different proof than book)

SEC 4: CAUCHY'S FORMULA (CONT)

FUND THM OF ALG: EVERY NON-CONSTANT POLY WITH COMPLEX COEFFS HAS A ROOT IN $\mathbb{C}$, AND THUS FACTORS COMPLETELY IN $\mathbb{C}$!

Absolutely amazing: add root of $x^2+1=0$ and can do all
↳ nothing else needed!

Proof: Assume $P(z) = a_n z^n + \dots + a_0$ has no roots, $a_n \neq 0$

↳ implies $1/P(z)$ is holo and bounded

↳ holo easy: quotient rule (denom never zero)

bounded: $|z| \geq R$ for R enormous then

$$|P(z)| \geq |a_n| R^n - n \max_{0 \leq k \leq n-1} |a_k| \cdot R^{n-1}$$

$$\geq c R^n \text{ for some } c > 0$$

By Liouville's Thm $1/P(z)$ is constant, contradiction. ☒

Last result deals with how the zeros of a holo fn can accumulate. Very different than real valued fns:

↳ $g(x) = \begin{cases} x^3 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

↳ differentiable, zeros accumulate, not identically zero.

SEC 4: CAUCHY'S FORMULA

THM: f holo on open Ω, vanishing on sequence $\{w_k\}_{k=1}^{\infty} \subset \Omega$ with limit point IN Ω. Then f is identically zero in Ω.

Proof: Let $z_0 = \lim_{k \to \infty} w_k$

Step 1: f identically 0 in small disc about z_0

Step 2: from this conclude f is zero in Ω

Step 1: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in some small disc as f holo

If f is not identically 0 then there is a smallest $m \geq 0$ st $f(z) = a_m (z - z_0)^m (1 + g(z - z_0))$, with $g(z - z_0) = 0$

just let $g(z - z_0) = \frac{1}{a_m} \sum_{n=m+1}^{\infty} a_n (z - z_0)^{n-m}$.

Contradiction as for k large $g(w_k - z_0) \approx 0$ and thus $f(w_k)$ cannot vanish.

Step 2: See book for topology argument

Homework: #1, #2, #8

Is accumulation result still true if accumulate on boundary Ω?

Suggested: Do $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$, #3, #5, #6

Add more Problems