CHAPTER 3: MEROMORPHIC FNS AND THE LOGARITHM

GOALS:

- Extend Cauchy's Thm to include fns with poles
- Understand singularities of fns + their consequences
  - ↳ Picard's Thm (what does the ST:TNG work on?)
    - ↳ how many values can a complex diff fn omit without being constant?
- Riemann Mapping Thm: later in semester, will see "lite" version here
  - ↳ any simply connected open set not all of $\mathbb{C}$ can be mapped to unit disc with a complex diff fn
    - ↳ what of boundary?
- Rouche's Thm:
  - ↳ Relating # zeros of f and g
- Finding maxima of fns
- Analytic continuation
  - ↳ $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

Sec 1: Zeros and Poles

Point singularity of f is a $z_0 \in \mathbb{C}$ st f is defined for all z near z_0 but not at z_0. Also call this an isolated singularity.

↳ removable singularities: $f(z) = \frac{z^2-1}{z-1}$ at $z=1$

zero/pole: f has a zero of order n if $f(z) = (z-z_0)^n g(z)$ with $g(z_0) \neq 0$; has a pole of order n at z_0 if $1/f(z)$ has a zero of order n at z_0.

↳ actual defn is just $f(z_0) = 0$. If f is holo in connected open Ω get $f(z) = (z-z_0)^n g(z)$ — see book (pg 73)

Often convenient to study deleted neighborhoods $\{z : 0 < |z-z_0| < r\}$. Similarly if pole at z_0 get $f(z) = (z-z_0)^{-n} h(z)$.

Call the integer n the multiplicity of the zero or pole; if $n=1$ say simple

Thm 1.3: If f has a pole of order n at z_0, then $\exists$ holo G st

$$\boxed{f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z)}$$

Proof: $f(z) = (z-z_0)^{-n} h(z)$

$h(z) = h_0 + h_1 z + \cdots$

Expand ∎

Call $\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}$ the principal part of f at the pole z_0, and a_{-1} the residue, often written by $\boxed{\mathrm{res}_{z_0}(f) = a_{-1}}$

Sec 1: Zeros + Poles: Continued

Lemma: If $P(z)$ is the principal part of f at z_0 and C is a ~~sufficiently small~~ circle centered at z_0, then $\frac{1}{2\pi i}\oint_C P(z)\,dz = a_{-1}$

Proof: Follows from $\oint_C z^n\,dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise} \end{cases}$

↳ Prove this: could do $r\cos n\theta + ir\sin n\theta \cdots$

$z = re^{i\theta} \quad dz = ire^{i\theta}\,d\theta$

Very important to be able to find residues (the a_{-1} term).
This part not in book.

Example: Find poles and residues of $\frac{1}{z^2+1}$

↳ Soln: clear poles are $z = i$ and $-i$

Get $\frac{1}{z^2+1} = \frac{1}{z+i}\,\frac{1}{z-i}$

note $\frac{1}{z+i}$ is holo at $z=i$

↳ $z+i = z-i+2i = 2i\left(1+\frac{z-i}{2i}\right) = 2i\left(1-\frac{i(z-i)}{2}\right)$

Thus $\frac{1}{z+i} = \frac{1}{2i\left(1-\frac{i(z-i)}{2}\right)} = \frac{1}{2i}\left[1+\frac{i(z-i)}{2}+\cdots\right]$

note residue is just $\frac{1}{2i}$, which is $\frac{1}{i+i} = \frac{1}{2i}$

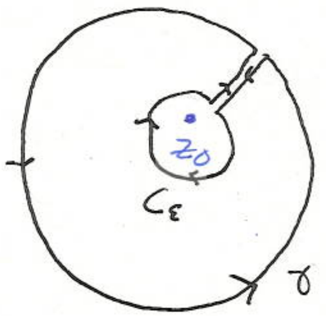
↳ often don't need infinite series.

SEC 2: THE RESIDUE FORMULA

Thm: f holo on open set containing circle C and its interior except for pole z_0 inside C. Then

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}_{z_0} f \quad \text{or} \quad \frac{1}{2\pi i}\oint_C f(z)\,dz = \operatorname{Res}_{z_0} f$$

Proof:



f holo inside contour, so $\oint_\gamma f(z)\,dz = 0$

Letting separation tend to zero gives

$$\oint_C f(z)\,dz - \oint_{C_\varepsilon} f(z)\,dz = 0$$

Write $f(z) = P_{z_0}(z) + G(z)$ with G holo

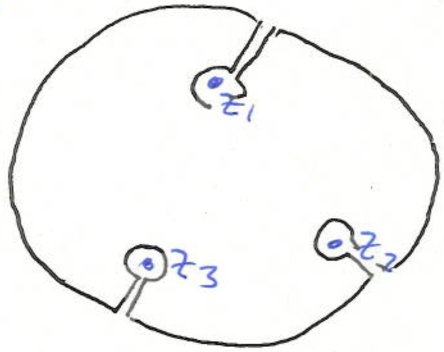
Then $\oint_{C_\varepsilon} f(z)\,dz = \oint_{C_\varepsilon} P(z) + G(z) = 2\pi i \operatorname{Res}_{z_0} f$

Cor: f holo in open set containing C and interior save at $z_0, \ldots, z_N$ inside C. Then

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_{k=1}^{N} \operatorname{Res}_{z_k} f$$

CALL THIS THE RESIDUE FORMULA

Proof:



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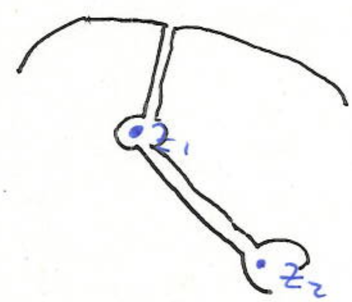


SEC 2: THE RESIDUE FORMULA (CONT)

Can use this to evaluate lots of integrals:

Example 1: $f(z) = \dfrac{1}{z^2+1}$ $\quad \displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

General Advice

- Choose appropriate f
- Choose appropriate contours so can exploit decay

Do more examples

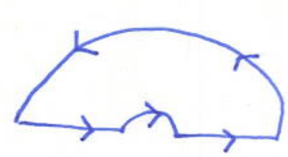
Question: What if pole on the contour?

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$:

↳ first prove $\displaystyle\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin x}{x}\,dx < \infty$ so this makes sense! Natural choice $f(z) = \dfrac{e^{iz} - e^{-iz}}{2iz}$

Problem: $e^{\pm iz}/z$ has a pole at $z=0$, want

but need to do

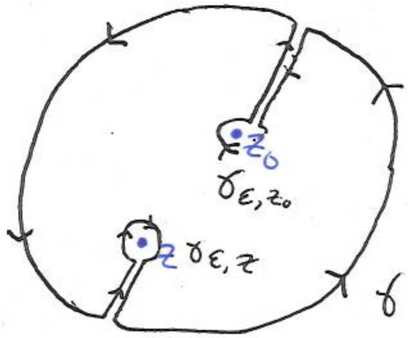


or maybe

Sec 3: Singularities and Meromorphic Fns

Thm: Riemann's Thm on Removable Singularities

f holo on open Ω save possibly at $z_0 \in \Omega$, If f is bounded on $\Omega - \{z_0\}$ Then z_0 is a removable singularity, and $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} dz$



f is holo in This Contour: $\oint_\gamma \frac{f(\zeta) d\zeta}{\zeta - z} = 0$

Letting separation tend to zero yields

$$\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta - \underbrace{\oint_{C_{\varepsilon, z_0}} \frac{f(\zeta)}{\zeta - z} d\zeta}_{\substack{\text{tends to } 0 \\ f \text{ is bounded} \\ \zeta \text{ near } z_0 \text{ and} \\ \text{Thus far from } z}} - \oint_{C_{\varepsilon, z}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0$$

Yields $\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \oint_{C_{\varepsilon, z}} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z)$

Thus regain $f(z)$ for $z \neq z_0$, use This for $z = z_0$ too.

On The curve C, ζ is far from z_0

↳ integral is Thus Continuous in z.

↳ By Thm 5.4, Chap 2 know That This integral rep means f is holo/analytic.

<u>Cor</u>: f has isolated singularity at z_0 Then z_0 is a pole IFF $|f(z)| \to \infty$ as $z \to z_0$.

See book for proof.

SEC 3: SINGULARITIES + REMOVABLE FNS

Three types of singularities

- Removable: f bounded near z_0
- pole: $|f(z)| \to \infty$ as $z \to z_0$
- Essential: anything else
  - ↳ ex: $f(z) = e^{1/z}$
    - ↳ if $z = x \to 0^+$ tends to $+\infty$
    - if $z = iy$ then absolute value is 1

THM: CASORATI-WEIERSTRASS

f holo in punctured disk $D_r(z_0) - \{z_0\}$ and has an essential singularity at $\{z_0\}$. Then f of the punctured disk is dense in $\mathbb{C}$.

Proof: By contradiction. Assume not, so $\exists \omega\ \exists \delta$ st

$|f(z) - \omega| > \delta$ for all $z \in D_r(z_0) - \{z_0\}$.

Set $g(z) = \dfrac{1}{f(z) - \omega}$

↳ holo on punctured disk, bounded by $1/\delta$

↳ Thus has removable singularity at z_0.

↳ if $g(z_0) \neq 0$ then $f(z) - \omega$ is holo at z_0, contradicting z_0 an essential sing of f.

↳ if $g(z_0) = 0$ then f has pole at z_0, again contradicting essential singularity ▨

Give refs to Picard's Thms (Big and Little)

Sec 3: Sings and Meromorphic Fns (cont)

Skipping most of rest of section

Say we have a fn f with only isolated singularities that are poles. We say f is <u>meromorphic</u> on an open Ω if there exists a seq $\{z_0, z_1, z_2, \dots\}$ that has no limit point in Ω st

(1) f is holo in $\Omega - \{z_0, z_1, z_2, \dots\}$

(2) f has poles at $z_0, z_1, z_2, \dots$

Sec 4: The Argument Principle and Applications

Logarithm is hard to define:

$$z = 5e^{2\pi i/4} \text{ or } 5e^{2\pi i/4 + 2\pi i k} \qquad k \in \mathbb{Z}$$

Sadly lose $\log f_1 f_2 = \log f_1 + \log f_2$

Do have $\dfrac{(f_1 f_2)'}{f_1 f_2} = \dfrac{f_1'}{f_1} + \dfrac{f_2'}{f_2}$ (quotient rule)

Say holo f has zero order n at z_0: $f(z) = (z-z_0)^n g(z)$

Then $\dfrac{f'(z)}{f(z)} = \dfrac{n}{z-z_0} + \dfrac{g'(z)}{g(z)}$, with $\dfrac{g'(z)}{g(z)}$ holo

↳ Note f'/f has simple pole of residue n at z_0.

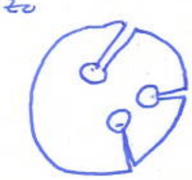
↳ Similar result if f has a pole

THM: ARGUMENT PRINCIPLE

f meromorphic in open set containing circle C and its interior, f has no zeros or poles on C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \begin{pmatrix}\#\text{ zeros of } f\\ \text{inside } C\end{pmatrix} - \begin{pmatrix}\#\text{ poles of } f\\ \text{inside } C\end{pmatrix}$$

Proof: If f has a zero or pole of order n at z_0 then by previous $\frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + \text{holo}$ in small nbhood

use appropriate contour: 

Cor: Of course C need not be a circle...

Question: Can you use this to prove any poly of deg 2 has exactly two roots?

What about a deg n poly?

Applications

- Rouche's Thm
- Open Mapping Thm: holo f maps open sets to open sets
- Maximum modulus principle

Sec 4: Arg Principle and App (cont)

Thm: Rouche's Thm: f, g holo on open set containing circle C and its interior. If $|f(z)| > |g(z)| \ \forall z \in C$ Then f and $f+g$ have same # of zeros inside C.

Proof: Will continuously pass from f to $f+g$

Main idea: if a cont fn is always an integer then it's constant

Let $f_t(z) = f(z) + t\, g(z)$

↳ Note $f_0(z) = f(z)$ and $f_1(z) = f(z) + g(z)$

↳ As $|f| > |g|$ on C, f_t is non-zero on C

Argument Principle: # zeros of $f_t = n(t) = \frac{1}{2\pi i} \oint_C \frac{f_t'(z)}{f_t(z)}\, dz$

↳ Proof follows by showing integral is continuous as $n(t) \in \mathbb{Z}$

Note $\frac{f_t'(z)}{f_t(z)}$ is cont for $(t, z) \in [0,1] \times C$, compact set

↳ num and denom cont, denom is non-zero

$\Rightarrow n(0) = n(1)$ ∎

Mapping is **OPEN** if it sends open sets to open sets.

Consider $f(x) = x^2$ on $(-1,1)$. Then $f((-1,1)) = [0,1)$ is not open, so the nice real-valued fn $f(x) = x^2$ is not open

Thm: Open Mapping Thm

f holo and non-constant in open Ω then f is open.

Proof: Say $w_0 \in Im(f)$ so $f(z_0) = w_0$

Must show some ball about $w_0 \subset Im(f)$

Set $g_w(z) = f(z) - w$ for some fixed w near w_0

↳ $g_w(z) = (f(z) - w_0) + (w_0 - w)$

$= F(z) + G(z)$

Choose $\delta > 0$ st disc $|z - z_0| \le \delta$ contained in Ω

and $f(z) \ne w_0$ on circle $|z - z_0| = \delta$.

↳ Must prove can do: proof not in book.

↳ If can't, for all radii $r \le \delta$ have a z_r on C_r st $f(z_r) = w_0$, thus $f(z) - w_0$ has an accumulation point and is locally constant.

↳ Thus $\exists \varepsilon > 0$ st $|f(z) - w_0| > \varepsilon$ on circle $|z - z_0| = \delta$

As $|w - w_0| < \varepsilon$, have $|F(z)| > |G(z)|$ on circle $|z - z_0| = \delta$.

Thus by Rouche F and G have same # of zeros.

↳ As F has a zero (z_0), $F+G$ has a zero and thus $\exists z$ st $f(z) = w$

THM: MAXIMUM MODULUS PRINCIPLE

If f non-const holo fn in open Ω then f cannot attain a maximum in Ω.

Proof: Follows from open mapping Thm.

Note by "maximum" we mean a maximum for $|f|$

↳ If $|f|$ is max at $z_0 \in \Omega$, then since open map we have an open ball $B_{\tilde{r}}(|f(z_0)|) \subset f(B_r(z_0))$

Thus have larger absolute value.

COR: Ω st $\bar{\Omega}$ is compact, f holo on Ω and cont on $\bar{\Omega}$, then $\sup_{z \in \Omega} |f| \le \sup_{z \in \bar{\Omega} - \Omega} |f|$ (ie, max on boundary)

Proof: by real analysis have max of $|f|$ on $\bar{\Omega}$, by above cannot attain in Ω, only $\bar{\Omega} - \Omega$ left. ∎

Note: Compactness important:

Consider e^{-iz^2} on first quadrant

SEC 5: HOMOTOPIES AND SIMPLY CONN DOMAINS

Skipping proofs: see book

γ_0, γ_1 curves in open Ω with common endpoints. Say <u>homotopic in Ω</u> if $\exists$ jointly cont $\gamma_s(t)$ for $(s,t) \in [0,1] \times [a,b]$ $(\gamma_1, \gamma_2 : [a,b] \to \mathbb{C})$ with $\gamma_s(t)|_{s=0} = \gamma_0(t)$ and $\gamma_s(t)|_{s=1} = \gamma_1(t)$.

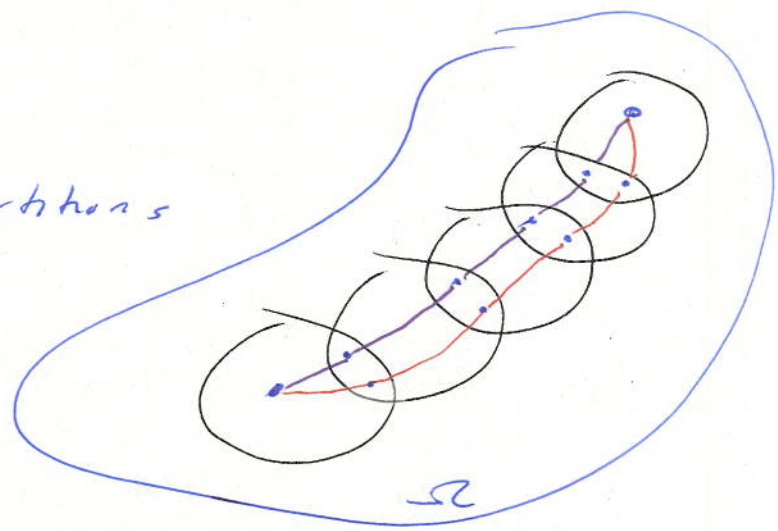
<u>THM</u>: f holo on Ω, γ_0 and γ_1 homotopic, then
$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

<u>Idea of proof</u>

↳ Partition

↳ Use primitives in partitions

  ↳ use points in overlap

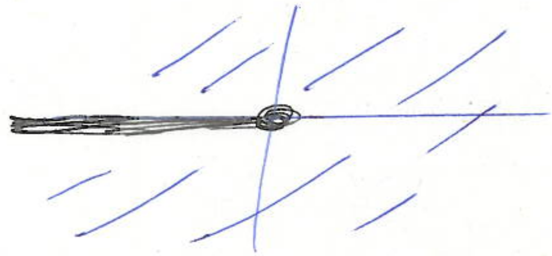


Sec 6: The Complex Logarithm

$z = re^{i\theta} = re^{i\theta + 2\pi i k} \quad k \in \mathbb{Z}$

natural to set $\log z = \log r + i\theta$, but ...

Need to make a branch cut.

Most common:  $\mathbb{C} - (-\infty, 0]$

Principle branch

THM 6.1: Ω simply conn with $1 \in \Omega$, $0 \notin \Omega$. Then in Ω

There is a branch of the logarithm st $F(z) = \log_\Omega(z)$ with

(1) F is holo in Ω

(2) $e^{F(z)} = z \quad \forall z \in \Omega$

(3) $F(r) = \log r$ when r is a real number near 1.

Proof: Idea: construct a primitive (ie anti-deriv) for $f(z) = \frac{1}{z}$

Define $\log_\Omega z := F(z) = \int_\gamma f(w)\,dw$ for any path γ from 1 to z

↳ as Ω is simply conn, integral indep of γ (Cauchy's Thm)

↳ Yields $F'(z) = 1/z \quad \forall z \in \Omega$

↳ (2) equiv to $ze^{-F(z)} = 1$

↳ note $(ze^{-F(z)})' = (1 - zF'(z))e^{-F(z)} = 0$

so deriv constantly and zero $\Rightarrow ze^{-F(z)}$ constant

and know $1 \cdot e^{-F(1)} = 1 \cdot e^0 = 1$

↳ (3): choose a path from 1 to r in $\mathbb{R}$, so $F(r) = \int_1^r \frac{dx}{x} = \log r$ ▨

Sec 6: Complex Logarithm (Cont)

- Can Prove Taylor Expansion:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

↳ alternate proof from book!

↳ know true for $z \in \mathbb{R}$

look at difference, accumulation points...

- Ω simply conn, $1 \in \Omega$, $0 \notin \Omega$, choose branch with $\log 1 = 0$ and for $\alpha \in \mathbb{C}$ set $z^\alpha = e^{\alpha \log z}$

↳ need this in calculus for deriv of z^r general r

↳ combinatorics for z^n, $z^{p/q}$ not available

$$\boxed{\text{THM: } f \text{ nowhere vanishing holo fn on simply conn } \Omega \text{ then } \exists \text{ holo } g \text{ on } \Omega \text{ st } f(z) = e^{g(z)}}$$

See book for proof.

HW: #1, #2 (can you do $\frac{1}{1+x^{2k}}$), #10, #12, #13, #15abd
#16, #17

Suggested: #3, #6 (see Wallis' Formula), #9, #15c, #19,
#21, #22