CHAPTER 5: ENTIRE FUNCTIONS

GOALS: Studying entire fns (holo in all of $\mathbb{C}$)

↳ where can these fns vanish?

↳ how can they grow at infinity?

↳ how much is determined by zeros?

Lots of beautiful results here (Jensen's formula, Hadamard factorization), but will concentrate on Weierstrass product

↳ motivation: $f(x) = a_n x^n + \cdots + a_0 = a_n(x-r_1)\cdots(x-r_n)$

↳ something "similar" should hold for nice f.

SEC 3: INFINITE PRODUCTS

Define $\{a_n\}_{n=1}^{\infty}$, seq of complex numbers, say $\prod_{n=1}^{\infty}(1+a_n)$ converges if $\lim_{N\to\infty} \prod_{n=1}^{N}(1+a_n)$ exists.

PROP: If $\sum_{n=1}^{\infty}|a_n| < \infty$ then $\prod_{n=1}^{\infty}(1+a_n)$ converges; it converges to 0 if and only if a term is 0.

Proof: As sum converges, for n large each $|a_n| < 1/2$; wlog assume for all n.

Thus can define $\log(1+a_n)$ by usual power series

↳ have $1+z = e^{\log(1+z)}$ for $|z|<1$

Thus $\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{\sum_{n=1}^{N}\log(1+a_n)} = e^{\sum_{n=1}^{N} b_n} = e^{B_N}$

↳ As $|\log(1+z)| \le 2z$ for $|z|<\frac{1}{2}$, $|b_n| \le 2|a_n|$

Thus $B_N \to B$; as exp is continuous, $e^{B_N} \to e^{B}$ ∎

↳ $1+a_n \neq 0 \to$ product is non-zero as of form e^{B}

SEC 3: INFINITE PRODUCTS (CONT)

PROP: $\{F_n\}$ seq holo fns on open Ω with constants $c_n > 0$ st $\sum_{n=1}^{\infty} c_n < \infty$ and $|F_n(z) - 1| \leq c_n$ $\forall z \in \Omega$. Then

(1) $\prod_{n=1}^{\infty} F_n(z)$ converges unif in Ω to a holo fn $F(z)$

(2) $F'(z)/F(z) = \sum_{n=1}^{\infty} F_n'(z)/F_n(z)$

Proof: Will just do first part.

Write $F_n(z) = 1 + a_n(z)$ with $|a_n(z)| \leq c_n$.

Estimates are uniform in z as c_n's are constants

↳ just use last proposition for each $z \in \Omega$

↳ convergence is uniform.

Skipping rest of section

Can show $\dfrac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2}\right)$

SEC 4: WEIERSTRASS INFINITE PRODUCT

THM: Given any seq $\{a_n\}$ of complex numbers with $|a_n| \to \infty$, $\exists$ entire f st f vanishes at the a_n's and nowhere else. Any other such fn ~~f~~ g is of the form $f(z)e^{g(z)}$ for some entire function g.

Note the $\{a_n\}$ may include multiplicities...

Proof: Last part clear: if f_1, f_2 vanish at $\{a_n\}$ and nowhere else, then f_1/f_2 has only removable singularities and thus is entire and never zero, and thus $f_1(z)/f_2(z) = e^{g(z)}$ (see Section 6 of Chapter 3).

Could try $\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)$ but frequently product diverges for many z (as would happen for $a_n = n$).

Define Canonical factors for integer $k \geq 0$ by

$E_0(z) = 1 - z$ and $E_k(z) = (1-z)e^{z + z^2/2 + \cdots + z^k/k}$ $k \geq 1$

↳ note for E_k that exponent is start of $-\log(1-z)$.

Claim: $|z| \leq \frac{1}{2} \Rightarrow |1 - E_k(z)| \leq c|z|^{k+1}$ for some $c > 0$.

↳ Proof: $1 - z = e^{\log(1-z)}$

↳ $E_k(z) = e^{\log(1-z) + z + z^2/2 + \cdots + z^k/k} = e^{w(z)}$

with $w(z) = -\sum_{n=k+1}^{\infty} z^n/n$.

as $|z| \leq \frac{1}{2}$, $|w| \leq |z|^{k+1} \sum_{n=k+1}^{\infty} \frac{|z|^{n-k+1}}{n}$

$\leq |z|^{k+1} \cdot \sum_{j=0}^{\infty} 2^{-j} \leq 2|z|^{k+1}$

Thus $|w| < 1$ so $|1 - E_k(z)| = |1 - e^w| \leq c'|w| \leq c|z|^{k+1}$

Sec 4: Weierstrass Product (cont)

Proof (cont): important c is indep of k

↳ can take $c' = e$ and $c = 2e$

Time for overkill:

Set $f(z) = z^{\text{ord}_0(f)} \prod_{n=1}^{\infty} E_n(z/a_n)$

↳ Claim f has desired properties (right # zeros at origin, vanishes only at a_n's with right multiplicities).

↳ Suffices to show on disc @ arbitrarily large radius R.

Eventually, for $n \geq N$ say $|z/a_n| \leq \frac{1}{2}$. then

$$f(z) = \underbrace{z^{\text{ord}_0(f)} \prod_{n=1}^{N} E_n(z/a_n)}_{\text{finite prod, ok}} \underbrace{\prod_{n=N+1}^{\infty} E_n(z/a_n)}_{\text{here } |z/a_n| < 1}$$

done by previous lemma. ∎

↳ Can do better: Hadamard's replace $\prod_n E_n()$ with $\prod_n E_k$

H/W: #6, #7

Suggested: #11