CHAPTER 8: CONFORMAL MAPPINGS

GOAL: Study when $\exists$ holomorphic bijection b/w two open sets.

↳ When can we generalize to boundary?

↳ when can we explicitly describe mapping?

Why care: Say holo bijection $f: \mathcal{U} \to V$ and $f^{-1}: V \to \mathcal{U}$
If understand fns/problems on one, can "pass" or "transfer" to the other

↳ Big Example: Solving Dirichlet Problem

SEC 1: CONFORMAL EQUIVALENCE + EXAMPLES

Bijective (1-1, onto) holomorphic fn f: open $\mathcal{U} \to$ open V is called a conformal map (or biholomorphism), and say $\mathcal{U}$ and V are conformally equivalent (or biholomorphic)

KEY PROP: $f: \mathcal{U} \to V$ holomorphic and injective, then $f'(z) \neq 0$ for all $z \in \mathcal{U}$, and the inverse of f is defined on its range and is holomorphic, and thus the inverse of a conformal map is holomorphic.

Remember Open Mapping Thm (pg 92): non-constant f maps open sets to open sets. In above situation, f is a conformal map from $\mathcal{U}$ to $f(\mathcal{U})$.

Sec 1: Conformal Maps (cont)

Proof of Prop (By contradiction)

↳ Suppose $\exists z_0 \in U$ st $f'(z_0) = 0$

Then $f(z) - f(z_0) = a(z-z_0)^k + G(z)$

for all z near z_0 with $a \neq 0$, $k \geq 2$

write $f(z) - f(z_0) - w = F(z) + G(z)$ for small w

Note $F(z) = a(z-z_0)^k - w$, z close to z_0 Then $|F(z)| \approx |w|$

$G(z)$ is $b(z-z_0)^{k+1} + \cdots$ and thus $|G(z)| < |F(z)|$

if $|w|$ is small and z close to z_0

↳ careful if $w=0$, but still true

Note $F(z)=0$ is $a(z-z_0)^k - w = 0 \rightarrow z = z_0 + \left(\frac{w}{a}\right)^{1/k}$

↳ Thus F has at least two solns

↳ By Rouche's Thm, $F+G$ has at least two zeros

↳ Thus $f(z) - f(z_0) - w$ has at least two solns in small ball

Thus two different z mapped to $f(z_0) + w$

↳ roots distinct as f' non-zero: if same then expanding at double root gives f' vanishes there

↳ Contradicts f injective ∎

(See book for proof inverse is holomorphic)

Sec 1: Conformal Maps (cont)

Let's Do Examples!

$\mathbb{D}$ = unit disc $= \{z : |z| < 1\}$

$\mathbb{H}$ = upper half plane $= \{z = x+iy : y > 0\}$

Ex 1: $F(z) = \frac{i-z}{i+z}$ and $G(w) = i\,\frac{1-w}{1+w} = G(w)$

Then $F : \mathbb{H} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{H}$ conformal maps

↳ Why? Only problem point of F is $-i$, of G is -1; neither in domain

Claim $F(\mathbb{H}) \subset \mathbb{D}$

↳ $z \in \mathbb{H} \to z = x+iy,\ y>0$

$$F(z) = \frac{-x + i(1-y)}{x + i(1+y)} \quad \text{so } |F(z)| < 1$$

↳ $$F'(z) = \frac{-(i+z) - (i-z)}{(i+z)^2} = \frac{-2i}{(i+z)^2} \neq 0 \text{ for } z \in \mathbb{H}$$

Thus $f : \mathbb{H} \to f(\mathbb{H})$ is conformal map

Say want $F(z) = w \in \mathbb{D}$

↳ Then $\frac{i-z}{i+z} = w \to i - z = wi + wz$

↳ $i(1-w) = (1+w)z$ or $z = i\,\frac{1-w}{1+w}$ $(= G(w))$

Done if $z \in \mathbb{H}$: algebra: see book $\boxtimes$

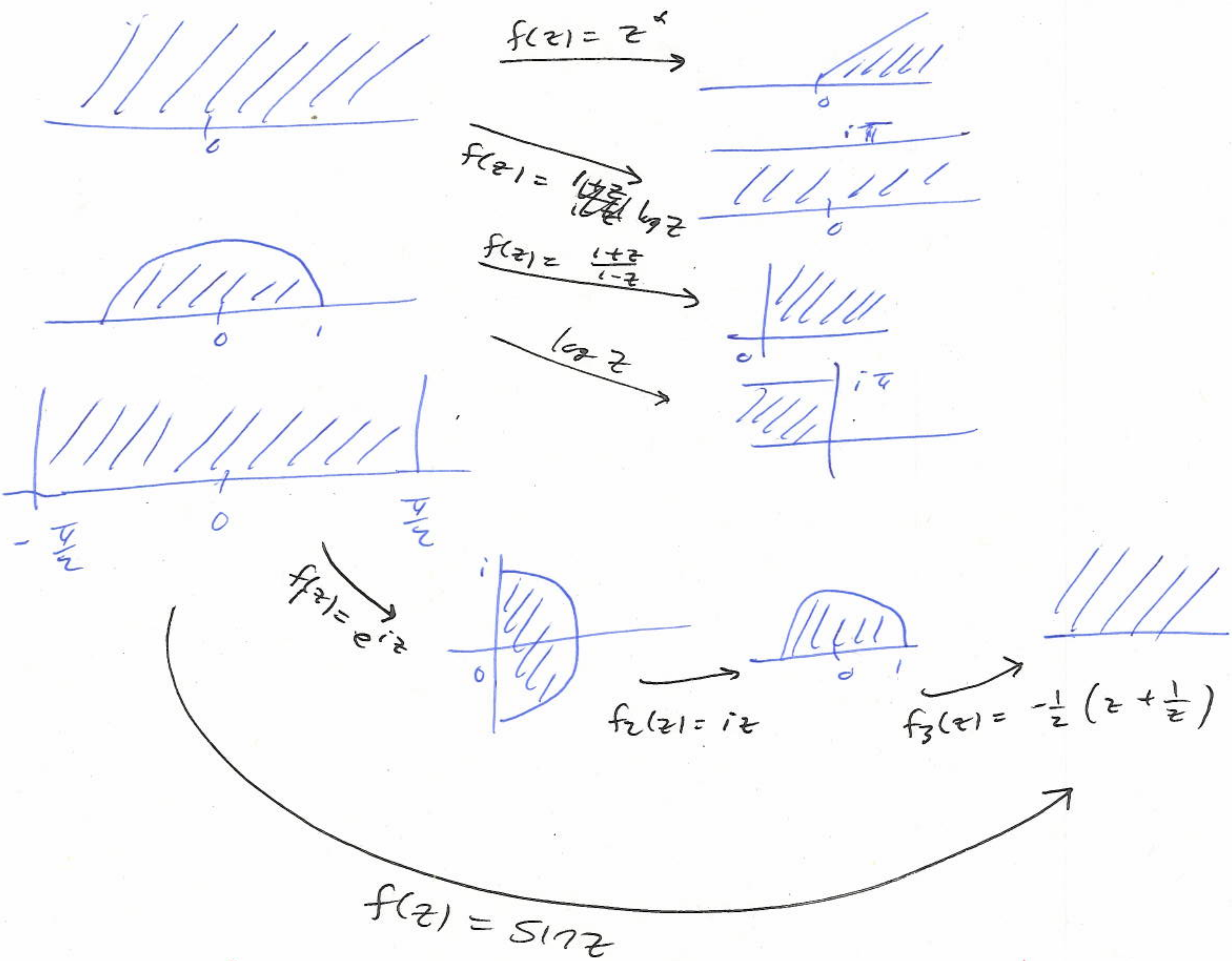
~~↳ $\mathrm{Re}(z) = \left(i\,\frac{1-w}{1+w}\right)(-i)$~~

Behavior on boundary interesting: See Book

Sec 1: Conformal Maps (cont)

Examples

- Translations: $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z + h$
- Rotation: $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^{i\varphi} z \quad \varphi \in \mathbb{R}$
- Dilation: $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = cz \quad c > 0$
- $z \mapsto z^n$: sector $\{z : 0 < \arg(z) < \frac{\pi}{n}\} \to \mathbb{H}$
- $z \mapsto e^{iz}$ maps $\{z : -\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}\}$ to upper halfdisc



See rest of §1.3 for comments on Dirichlet Problem

SEC 2: SCHWARZ LEMMA, AUTOMORPHISMS OF $\mathbb{D}$ AND $\mathbb{H}$

SCHWARZ LEMMA: $f: \mathbb{D} \to \mathbb{D}$ holomorphic with $f(0)=0$. Then
(1) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
(2) If $\exists z_0$ st $|f(z_0)| = |z_0|$ then f is a rotation.
(3) $|f'(0)| \leq 1$; if equality then f is a rotation.

Proof: Write $f(z) = 0 + a_1 z + a_2 z^2 + \cdots$ valid in $\mathbb{D}$
Thus $f(z)/z$ holo in $\mathbb{D}$ (removable singularity as $f(0)=0$)

(1) If $|z|=r$ as $|f(z)| \leq 1$ have $\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$
By Maximum Modulus Principle (pg 92) true for all $|z| \leq r$ as max on boundary.
Letting $r \to 1^-$ implies $|f(z)| \leq |z|$

(2) If $\exists z_0$ st $|f(z_0)| = |z_0|$ then attains max in interior
(Consider closed set $\{z : |z| < |z_0| + \frac{1-|z_0|}{2}\} \subset \mathbb{D}$)
Thus by Max Modulus Principle $f(z)/z = c$ a constant
Hence $|f(z_0)| = |c||z_0| \to |c| = 1 \to$ rotation

(3) Let $g(z) = \frac{f(z)}{z}$; have $|g(z)| \leq 1$
Note $g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0) \to |f'(0)| \leq 1$
$\hookrightarrow$ if $|f'(0)| = 1$ then $|g(0)| = 1$ and g attains max at interior point. By Max Modulus Principle, g constant so $f(z) = cz$ with $|c| = 1$. $\blacksquare$

Sec 2.1: Automorphisms of the Disk

Conformal map from open Ω to Ω is called an automorphism (of Ω); and denote set of all by $Aut(\Omega)$.

$\boxed{\text{Thm: } Aut(\Omega) \text{ is a group under composition.}}$

Proof: • identity element is $Id(z) = z$; $f \circ Id = Id \circ f = f$

- Assoc: true by properties of fns
- closure: $f \circ g$ also automorphism
- inverse: clear: $f \in Aut(\Omega) \Rightarrow f$ conformal $\rightarrow$ bijection, and showed earlier inverse also conformal ▨

Ex: $\varphi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ $\alpha \in \mathbb{C}$, $|\alpha| < 1$

Claim: φ_α is an $Aut(\mathbb{D})$

↳ Proof: As $|\alpha| < 1$, φ_α holo in $\mathbb{D}$

If $|z| = 1 \rightarrow z = e^{i\theta}$ and

$\varphi_\alpha(e^{i\theta}) = \text{algebra} = e^{-i\theta} \dfrac{\alpha - e^{i\theta}}{\overline{\alpha - e^{i\theta}}}$

↳ Thus $|\varphi_\alpha(e^{i\theta})| = 1$

By Maximum Modulus Principle, $|\varphi_\alpha(z)| < 1$ $z \in \mathbb{D}$

Thus $\varphi_\alpha(\mathbb{D}) \subset \mathbb{D}$

Need only show surjective

Algebra: $(\varphi_\alpha \circ \varphi_\alpha)(z) = z$ so own inverse! ▨

↳ ie, to get z take $\varphi_\alpha(z)$ as input.

Note φ_α exchanges 0 and α.

Sec 2.1: AUTO OF $\mathbb{D}$

$\boxed{\text{Thm: If } f \in Aut(\mathbb{D}) \text{ then } \exists \theta \in \mathbb{R}, \alpha \in \mathbb{C} \text{ with } |\alpha| < 1 \text{ such that } f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}.}$

Proof: As automorphism of $\mathbb{D}$, $\exists! \alpha \in \mathbb{D}$ st $f(\alpha) = 0$.

Consider automorphism $g = f \circ \varphi_\alpha$

$\hookrightarrow g(0) = f(\varphi_\alpha(0)) = f(\alpha) = 0$

By Schwarz Lemma, $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$.

Key idea: g^{-1} also in $Aut(\mathbb{D})$, $g^{-1}(0) = 0$ as $g(0) = 0$.

Applying Schwarz Lemma to g^{-1} gives $|g^{-1}(w)| \leq |w|$ $\forall w \in \mathbb{D}$

$\hookrightarrow$ take $w = g(z)$, get $|g^{-1}(g(z))| \leq |g(z)|$ $\forall z \in \mathbb{D}$,

or $|z| \leq |g(z)|$, but showed $|g(z)| \leq |z|$.

$\hookrightarrow$ Thus $|g(z)| = |z|$ and by Schwarz Lemma

Conclude $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

Thus $g(z) = (f \circ \varphi_\alpha)(z) = e^{i\theta} z$

Let $z = \varphi_\alpha(w)$ get $(f \circ \varphi_\alpha)(\varphi_\alpha(w)) = e^{i\theta} \varphi_\alpha(w)$

or $f(w) = e^{i\theta} \frac{\alpha - w}{1 - \bar{\alpha} w}$ as claimed ∎

Have a Complete Characterization of $Aut(\mathbb{D})$!

Sec 2.2: Aut of $\mathbb{H}$

As know $Aut(\mathbb{D})$, know $Aut(\mathbb{H})$

Key: $F(z) = \frac{i-z}{i+z}$ conformally maps $\mathbb{H}$ to $\mathbb{D}$: have $F: \mathbb{H} \to \mathbb{D}$
$F^{-1}: \mathbb{D} \to \mathbb{H}$

Consider $\Gamma: Aut(\mathbb{D}) \to Aut(\mathbb{H})$ with $\Gamma(\varphi) = F^{-1} \circ \varphi \circ F$

↳ clear $\Gamma(\varphi) \in Aut(\mathbb{H})$

$\Gamma^{-1}(\psi) = F \circ \psi \circ F^{-1}$ is inverse map

↳ shows surjective

↳ given $\psi \in Aut(\mathbb{H})$, $F \circ \psi \circ F^{-1} \in Aut(\mathbb{D})$

and $\Gamma(F \circ \psi \circ F^{-1}) = F^{-1} \circ F \circ \psi \circ F^{-1} \circ F = \psi$

↳ to show injective:

Say $\Gamma(\varphi_1) = \Gamma(\varphi_2)$

↳ Then $F^{-1} \circ \varphi_1 \circ F = F^{-1} \circ \varphi_2 \circ F$

Apply F on left and F^{-1} on right $\to \varphi_1 = \varphi_2$

Thm: Every automorphism of $\mathbb{H}$ of the form $f(z) = \frac{az+b}{cz+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (2x2 integer matrices @ $ad-bc=1$), and any matrix of this form an automorphism

Proof: Algebra (see book).

Call such maps <u>Fractional Linear Transformations (FLT)</u>

or <u>Möbius Transformations</u>.

Sec 3: Riemann Mapping Thm

Riemann Mapping Thm: Suppose Ω is a non-empty subset of $\mathbb{C}$ that isn't all of $\mathbb{C}$, and is simply connected. If $z_0 \in \Omega$ then $\exists!$ conformal map $F : \Omega \to \mathbb{D}$ st $F(z_0) = 0$ and $F'(z_0) > 0$.

↳ Corr: Any two proper simply connected subsets of $\mathbb{C}$ are conformally equivalent.

Recall (page 98) Ω is simply connected if any two curves with same endpoints are homotopic (can be continuously deformed to each other in Ω); ie, no holes.

YES

NO

Ex 3: U, V conformally equiv, U simply conn $\to$ V simply conn

If $\mathbb{C}$ and $\mathbb{D}$ conformally equiv, say $F : \mathbb{C} \to \mathbb{D}$, then F is entire and by Liouville's Thm must be constant; impossible.

Turns out these only obstructions.

Rest is "standardization" to make unique.

Sec 3: Riemann Mapping Thm: Sec 3.2: Montel's Thm

Proof Sketch

- Uniqueness clear by study of $Aut(\mathbb{D})$
  - ↳ say $F_i : \Omega \to \mathbb{D}$ conformal map with $F_i(z_0) = 0$, $F_i'(z_0) > 0$

  Then $F_1 \circ F_2^{-1} : \mathbb{D} \to \mathbb{D}$ and

  ↳ $(F_1 \circ F_2^{-1})(0) = F_1(z_0) = 0$

  ↳ Chain rule: $(F_1 \circ F_2^{-1})'(z) = F_1'(F_2^{-1}(z)) \cdot (F_2^{-1})'(z)$

  As $F_2(F_2^{-1}(z)) = z$,

  $F_2'(F_2^{-1}(z)) \cdot (F_2^{-1})'(z) = 1$ so $(F_2^{-1})'(z) = \dfrac{1}{F_2'(F_2^{-1}(z))}$

  Thus $(F_1 \circ F_2^{-1})'(0) = F_1'(z_0) \cdot \dfrac{1}{F_2'(z_0)} > 0$

As automorphism of disk, $(F_1 \circ F_2^{-1})(z) = e^{i\theta} \dfrac{\alpha - z}{1 - \bar{\alpha} z}$

↳ as map 0 to 0, must have $\alpha = 0$

↳ as $-e^{i\theta} \in \mathbb{R}^+$, must have $\theta = \pi$

$\Longrightarrow (F_1 \circ F_2^{-1})(z) = -1 \cdot \dfrac{0 - z}{1 - 0z} = z$

Thus taking $z = F_2(z)$ gives $F_1(z) = F_2(z)$

Proof Sketch: Continued

↳ Uniqueness clear

↳ Idea: study all injective holo maps $f: \Omega \to \mathbb{D}$ with $f(z_0) = 0$.

Choose f st image fills all of $\mathbb{D}$

Can be achieved by making $f'(z_0)$ as large as possible

Need to extract f from limit of seq of fns

Let $\mathcal{F}$ be a family of holomorphic fns on $\mathbb{C}$.

- <u>Normal</u> if every seq in $\mathcal{F}$ has a subseq in $\mathcal{F}$ that converges uniformly on compact sets (called uniformly on compacta).

- <u>Uniformly on compacta</u>: $\forall$ compact $K \subset \Omega$, $\exists B_K > 0$ st $|f(z)| \leq B_K$ $\forall z \in K$ and $f \in \mathcal{F}$.

- <u>Equicontinuous</u> on a compact K if $\forall \varepsilon > 0$ $\exists \delta > 0$ st $z, w \in K$ with $|z-w| < \delta$ then $|f(z) - f(w)| < \varepsilon$ $\forall f \in \mathcal{F}$.

3.2: MONTEL'S THM

Strong conds: equicontinuous means uniform continuity holds uniformly in family; note δ is independent of point and function!

↳ Ex: MVT ⇒ $\mathcal{F}$ subset of diff fns on $[0,1]$ with bounded deriv then $\mathcal{F}$ is equicont.

↳ note: $\mathcal{F} = \{f_n : f_n(x) = x^n\}_{n=1}^{\infty}$ not equicont on $[0,1]$

↳ problem: $\lim_{n\to\infty} |f_n(1) - f_n(x_0)| \to 1 \quad \forall x_0 \neq 1$

MONTEL'S THM: $\mathcal{F}$ family of holo fns on Ω that are uniformly bounded on compacta. Then:

(1) $\mathcal{F}$ is equicontinuous on every compact subset of Ω.

(2) $\mathcal{F}$ is normal: every ~~sub~~ seq in $\mathcal{F}$ has a subseq that converges uniformly on compacta.

Proof of (1) uses Cauchy integral formula, and thus complex analysis / differentiability is essential ingredient.

↳ fails for fns from $[0,1] \to \mathbb{R}$: ex: $\mathcal{F} = \{f_n : f_n(x) = \sin nx\}_{n=1}^{\infty}$

Proof of (2) doesn't need complex analysis; often called the Arzela-Ascoli Theorem.

Sec 3.2: Montel's Thm (cont)

Final Preliminary: Sequences $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of Ω is an exhaustion if

(1) $K_\ell \subset \text{Interior}(K_{\ell+1})$ for all ℓ

(2) For any compact $K \subset \Omega$, $\exists\, \ell$ st $K \subset K_\ell$. In particular, $\Omega = \bigcup_{\ell=1}^{\infty} K_\ell$.

Lemma: Any open $\Omega \subset \mathbb{C}$ has an exhaustion.

Proof: Case 1: Ω bounded

↳ $K_\ell = \{z \in \Omega \text{ st } dist(z, \partial\Omega) \geq 1/\ell\}$ where $\partial\Omega$ is boundary of Ω.

Case 2: Ω unbounded

↳ just add in condition that also $|z| \leq \ell$.

Sec 3.2: Montel's Thm

Proof of first part

Compact $K \subset \Omega$, small r st $D_{3r}(z) \subset \Omega \ \forall z \in K$
(take $r < \frac{\operatorname{dist}(z, \partial\Omega)}{4}$ for example).

Must show $\forall \varepsilon \ \exists \delta$ st $z, w \in K$ & $|z-w| < \delta$ then $|f(z)-f(w)| < \varepsilon \ \forall f \in \mathcal{F}$

↳ Assume $z, w \in K$ with $|z-w| < r$ (can always take r smaller)

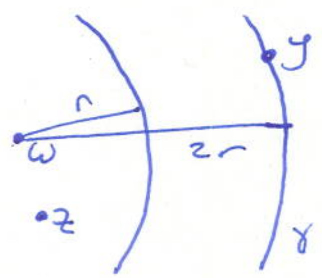
Let $\gamma = \partial D_{2r}(w)$ circle 

↳ Cauchy Integral Formula

$$f(z) - f(w) = \frac{1}{2\pi i} \oint_\gamma f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right] d\zeta$$

$$|f(z) - f(w)| = \frac{1}{2\pi} \oint_\gamma B_K \frac{|z-w|}{|\zeta - z|\,|\zeta - w|}\, |d\zeta|$$

algebra and uniformly bounded on compacta on $B_{2r}(w)$



note $|\zeta - z| \geq r$
$|\zeta - w| \geq r$

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \cdot (2\pi \cdot 2r) \cdot B_K \cdot \frac{|z-w|}{r^2}$$

Fix r small, given small ε let $\delta < \min(r, \frac{2B_K}{r}) \frac{\varepsilon}{2}$

↳ yields $|f(z) - f(w)| < \varepsilon$ when $|z-w| < \delta$.

3.2: Montel's Thm (cont)

Proof of Second Part

Multiple uses of Diagonal Argument: Talk About Cantor

Need denseness & countability

↳ Let $\{w_j\}_{j=1}^{\infty}$ dense subset of Ω

↳ Must prove exists: take $\Omega \cap \mathbb{Q}^2$

↳ Let $\{f_n\}_{n=1}^{\infty}$ be a seq in $\mathcal{F}$ and compact $K \subset \Omega$.

↳ As seq is uniformly bounded, $\exists$ subseq say

$\{f_{n,1}\} = \{f_{1,1}, f_{2,1}, f_{3,1} \ldots\}$ st $f_{n,1}(w_1)$ converges as $n \to \infty$

Then subseq of this seq $\{f_{n,2}\} = \{f_{1,2}, f_{2,2}, f_{3,2}, \ldots\}$

st $f_{n,2}(w_2)$ converges as $n \to \infty$.

Continue: ~~g_n~~ $\{f_{n,j}\}$ subseq of $\{f_{n,j-1}\}$ st

$f_{n,j}(w_\ell)$ converges as $n \to \infty$ for $\ell = 1, 2, \ldots, j$.

Let $g_n = f_{n,n}$, consider $\{g_n\} \subset \mathcal{F}$.

↳ By construction, $g_n(w_j)$ converges $\forall j$

By Part (1) know $\{g_n\}$ is equicont on Ω, must show

converges uniformly on K.

↳ Given ε choose δ by equicont

↳ Result from Analysis: $\exists J$ st $K \subset \bigcup_{j=1}^{J} D_\delta(w_j)$

Choose N big st $|g_m(w_j) - g_n(w_j)| < \varepsilon \ \forall j \leq J$

(Cauchy Convergence).

Sec 9.2: Montel's Thm (cont)

Proof (cont)

$z \in K \to z \in D_\delta(w_j)$ for some $j \leq J$ (note disk not necc $\subset \Omega$)

Thus $|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(w_j)|$

$$+ |g_n(w_j) - g_m(w_j)|$$

$$+ |g_m(w_j) - g_m(z)| < 3\varepsilon$$

(first and third from equicont, second from g_n converges at $\{w_j\}$)

Shows $\{g_n\}$ converges uniformly on the arbitrary compact K.

One more diagonalization to get subseq converging uniformly on every compact subset of Ω.

↳ Take exhaustion $K_1 \subset K_2 \subset \cdots$ of Ω.

$K = K_1$ in above, get seq labeled $\{g_{n,1}\}$ converges unif on K_1

↳ applying method again, get subseq $\{g_{n,2}\}$ " " " K_2

↳ Continue, ~~let~~ consider subseq $\{g_{n,n}\}$ ▤

Final preliminary to proving Riemann

Prop: Connected, open $\Omega \subset \mathbb{C}$, $\{f_n\}$ seq of injective holo fns on Ω converging uniformly on compacta to a holo fn f. Then f is either injective or constant.

Proof: Assume f not injective: $f(z_1) = f(z_2)$ with $z_1 \neq z_2$.

Set $g_n(z) = f_n(z) - f_n(z_1)$: As f_n injective, z_1 only zero.

Thus $\{g_n\}$ converges uniformly on compacta to $g(z) = f(z) - f(z_1)$.

↳ If g not identically zero, z_2 isolated zero as Ω connected (otherwise accumulation point....)

Thus $\exists$ small circle γ about z_2 st $g(\partial\gamma) \neq 0$ and

$$1 = \frac{1}{2\pi i} \oint_\gamma \frac{g'(\zeta)}{g(\zeta)}\, d\zeta$$

As $g(\partial\gamma) \neq 0$, $1/g_n$ conv unif to $1/g$ on γ, and as $g_n' \xrightarrow{\text{unif}} g'$ on γ have

$$\underbrace{\frac{1}{2\pi i} \oint_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)}\, d\zeta}_{\text{equals } 0 \text{ as } g_n \text{ has no zero inside } \gamma \text{ (only zero is at } z_1)} \longrightarrow \underbrace{\frac{1}{2\pi i} \oint_\gamma \frac{g'(\zeta)}{g(\zeta)}\, d\zeta}_{\text{equals } 1 \text{ by assump}}$$

Contradiction. ∎

3.3: PROOF OF RIEMANN MAPPING THM

Step 1: simply connected, open proper subset Ω of $\mathbb{C}$
Claim Ω conformally equivalent to subset of $\mathbb{D}$ containing 0

Step 2: $\mathcal{F} = \{f : f \text{ holo, injective, } f(0)=0,\ f:\Omega \to \mathbb{D}\}$
$\exists f \in \mathcal{F}$ that maximizes $|f'(0)|$

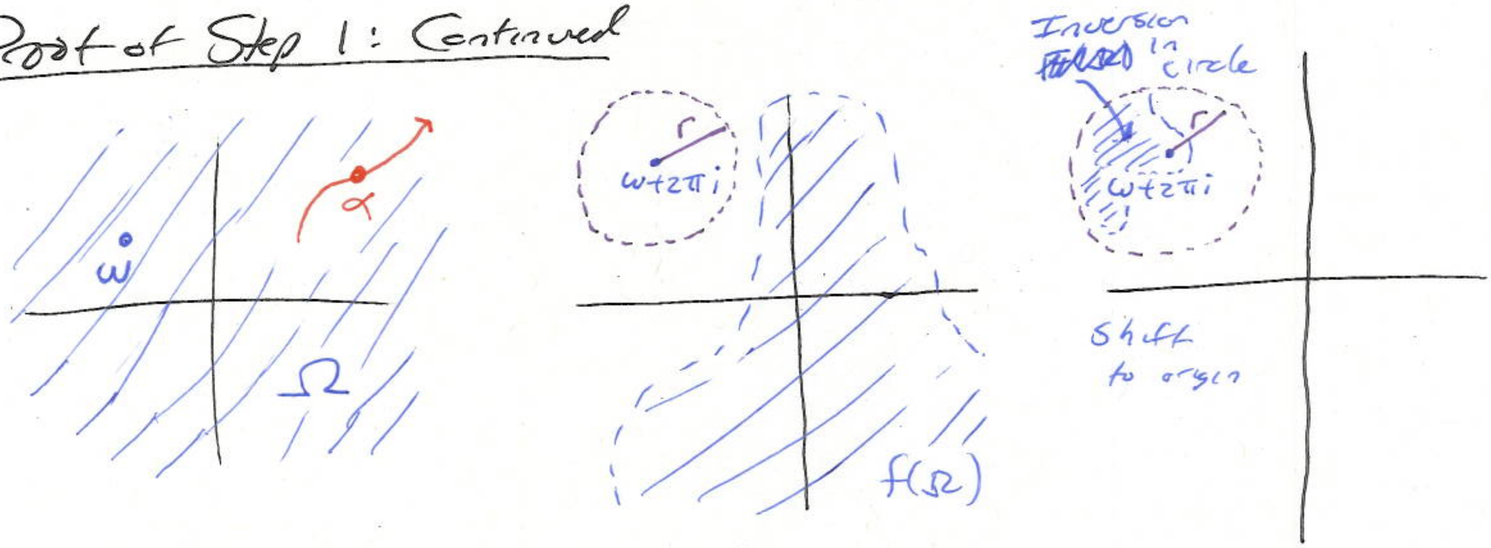
Step 3: Show f from Step 2 is conformal map from Ω to $\mathbb{D}$.

Proof of Step 1:

As Ω proper, $\exists \alpha \notin \Omega$ (as simply connected, infinitely many so not unique)
Note $z-\alpha$ never 0 on Ω, Ω simply connected
Thus $\exists$ holomorphic function "log" given by $f(z) = \log(z-\alpha)$
↳ essentially Thm 6.1 of Chapter 3, page 98
↳ have $e^{f(z)} = z - \alpha$ which implies f is injective
↳ if $f(z_1) = f(z_2)$ then $e^{f(z_1)} = e^{f(z_2)}$ or $z_1 - \alpha = z_2 - \alpha$
For $w \in \Omega$, $\forall z \in \Omega$ have $f(z) \neq f(w) + 2\pi i$ (choice of w not unique)
↳ if did by exponentiating get $z = w \to f(z) = f(w)$ contradiction
Claim: $\exists r$ st $f(\Omega) \notin B_r(w + 2\pi i)$
↳ if not, $\exists$ seq $\{z_n\}$ st $z_n \in \Omega$ and $f(z_n) \to f(w) + 2\pi i$
But then $e^{f(z_n)} \to e^{f(w)}$ or $z_n - \alpha \to w - \alpha$ or $z_n \to w$
Then by injectivity of f get $f(z_n) \to f(w)$
↳ contradiction as $f(z_n) \to f(w) + 2\pi i$

INVERT ABOUT THE CIRCLE / BALL $B_r(w+2\pi i)$

Proof of Step 1: Continued



Consider $F(z) = \dfrac{1}{f(z) - (f(w) + 2\pi i)}$

↳ as denom non-zero, well-defined

note denom is at least r in absolute value

Thus $F(\Omega) \subset B_{1/r}(w+2\pi i)$

↳ Now translate / rescale to map to unit disc

$$G(z) = r\, F(z - (f(w) + 2\pi i))$$

Thus G maps Ω conformally to subset $\mathbb{D}$.

Apply a fractional linear transform to G to get a map containing origin:

↳ let $\alpha \in \mathrm{Im}(G(\Omega))$. Then $H = \varphi_\alpha \circ G$

maps Ω conformally to a subset of $\mathbb{D}$ containing origin. ▨

Sec 3.3: Proof of Riemann Mapping Thm (cont)

Proof of Step 2:

By step one, can apply a conformal map and wlog assume $\Omega \subset \mathbb{D}$ and $0 \in \Omega$.

Consider $\mathcal{F} = \{ f: \Omega \to \mathbb{D}$ with f holo, injective and $f(0) = 0 \}$

↳ non-empty as $Id \in \mathcal{F}$

↳ uniformly bounded as all functions map into $\mathbb{D}$

↳ Claim $\exists$ fn in $\mathcal{F}$ maximizing abs value of its deriv at 0.

↳ Note $|f'(0)|$ uniformly bounded for $f \in \mathcal{F}$

↳ follows by Cauchy Ineq (Chap 2, Cor 4.3, Pg 48)

↳ $|f^{(n)}(z_0)| \leq \dfrac{n! \, \|f\|_{\partial B_R(z_0)}}{R^n}$

↳ note R is independent of f

$0 \in$ open Ω so $\exists R$ st $B_R(0) \subset \Omega$

Use $\|f\| \leq 1$

↳ Let $s' = \sup_{f \in \mathcal{F}} |f'(0)|$, choose seq $\{f_n\} \subset \mathcal{F}$

st $|f_n'(0)| \xrightarrow[n \to \infty]{} s'$. Note $s' \geq 1$ as $Id \in \mathcal{F}$.

By Montel's Thm, $\exists$ subseq converging uniformly on compacta to a holo f on Ω.

As $s' \geq 1$, f is non-const and thus injective (Prop 3.5, Pg 227)

By continuity $|f(z)| \leq 1$ for $z \in \Omega$, by Maximum Modulus Principle have $|f(z)| < 1$ for $z \in \Omega$ (f is non-constant).

Since $f(0) = 0$, conclude $f \in \mathcal{F}$ and $|f'(0)| = s'$

Sec 3.3: Proof of Riemann Mapping Thm (cont)

Proof of Step 3:

Consider f from Step 2: all that remains is to show $f: \Omega \to \mathbb{D}$ surjective

Proof by contradiction: assume f not surjective, will find "good" function with large deriv at 0

↳ let $\alpha \notin f(\Omega)$, consider $\psi_\alpha = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$.

As Ω simply connected, so too is $\mathcal{U} = (\psi_\alpha \circ f)(\Omega)$

↳ $0 \notin \mathcal{U}$, $\mathcal{U}$ simply connected $\Longrightarrow$ can define a square-root function (Chapter 3, Thm 6.1, pg 98)

Set $g(\omega) = e^{\frac{1}{2} \log \omega}$

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f, \quad h(\omega) = \omega^2$$

↳ Claim $F \in \mathcal{F}$

↳ F holo, $F(0) = 0$ (algebra)

↳ $F: \Omega \to \mathbb{D}$ as each fn is from Ω or $\mathbb{D}$ to $\mathbb{D}$

↳ F injective: clear for ψ_α, true by assump for f, by construction for g

↳ Algebra: $f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F$

↳ $\Phi: \mathbb{D} \to \mathbb{D}$ with $\Phi(0) = 0$

Φ is not injective as F is and h is not.

↳ By <u>Schwarz Lemma</u> $|\Phi'(0)| < 1$ ($<$ as not injective).

Note $f'(0) = \Phi'(0) F'(0)$ (chain rule) $\Longrightarrow |f'(0)| < |F'(0)|$

↳ Contradicts maximality of $|f'(0)|$, thus surjective.

Finally multiply f by complex # of size 1 to make $f'(0) > 0$

Sec 4: Schwarz-Christoffel Formula

Riemann mapping Thm is an existence proof

What can we say about nice regions?

Schwarz-Christoffel formula gives explicit soln to finding mapping for polygons.

See book for details.

Homework: #1, #3, #4, #5, #10, #12, #19, #21

Suggested: #11, #13, #14, #15, #16