Introduction to Complex Dynamics

General situation: study the effect of iteration.

Imagine discrete time, want to know "future".

Often extreme dependence on initial conditions.

Examples

- 3X+1 Map: Soviet conspiracy to slow down US math

$$T(n) = \begin{cases} 3n+1 & n \text{ odd} \\ n/2 & n \text{ even} \end{cases}$$

$$\text{or } U(n) = \frac{3n+1}{2^k} \text{ with } 2^k \| 3n+1$$

$$(\text{means } 2^k | 3n+1,\ 2^{k+1} \nmid 3n+1)$$

Very Hard problem

Connected to lots of difficult problems.

Why does 27 have such a long path to 1

while 25, 29 "short"

Conj: $\forall n \ \exists m$ st $T^{(m)}(n) = 1$

Examples (cont)

- Weather: iterate, small changes lead to wildly different results exponentially fast.

  "Butterfly Effect": flap wings → tornado Kansas

  ↳ as a friend suggests: get that Butterfly!

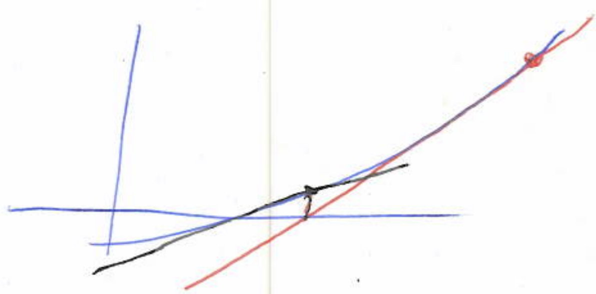
- Planetary Motion: Orbits of planets.

  Will Pluto stay around sun? Escape?

  ↳ becoming a planet again up to us.

- Newton's Method:

  Review of 1-variable: $f(x) = x^2 - a$, root $\sqrt{a}$



keep iterating and if f is "nice" approach the nearby root.

What happens in $\mathbb{C}$ and searching for complex roots? Look at Newton Fractal Pic

EXAMPLES (CONT)

• NEWTON Fractal (CONT)

Color the complex plane by what root the Newton's Method converges to for each seed.

↳ map $g(z) = z - \frac{p(z)}{p'(z)}$ where $p(z)$ is our initial polynomial. Given some point z_0, form sequence $z_0, g(z_0) = z_1, g(z_1) = g^2(z_0) = z_2, \ldots$ where by abuse of notation $g^2 = g \circ g$.

↳ For example $p(z) = z^3 - 1$, note how "Complicated" the coloring is, and how "structured/symmetric".

↳ Quick checks: color of 0?

Quick checks: rotational symmetries?

MANDELBROT SET

Mandelbrot set is all $c \in \mathbb{C}$ such that the orbit of 0 under the map $z_{n+1} = z_n^2 + c$ $(z_0 = 0)$ is bounded.

↳ Example: $c = 1 \rightarrow 0, 1, 2, 5, 26 \ldots$

$c = i \rightarrow 0, i, -1+i, -i, -1+i, -i, \ldots$ periodic

↳ Boundary is a fractal

PROPERTIES

- Bounded: Contained in closed disk $B_0(2)$
- Real Component: $[-2, 1/4]$
  relate to logistic map: $z \rightarrow \lambda z(1-z)$
  with $\lambda \in [1, 4]$
- Connected: have a conformal isomorphism b/w complement of Mandelbrot set and complement of closed unit disk.
- Self-similarity: see picture

Computation: Horner's Algorithm: $(((a_n x + a_{n-1})x + a_{n-2})x + \cdots$