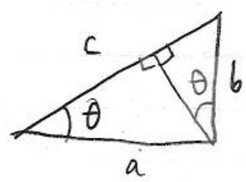
Math 302, Take 2 - Complex Analysis (w/ Steve Miller)

(*ask about textbook.)

* skim notes from Steve (available on website)

Definitions: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

New Pythagoras Proof:



There must exist some function $f(\theta)$ such that

$$A(\text{big}) = f(\theta)\, c^2.$$

(area proportional to c^2)

$\to f(\theta)\, c^2 = f(\theta)\, a^2 + f(\theta)\, b^2$.

$\to f(\theta) \neq 0$. ∎

Def'n of the derivative:

$\mathbb{R}$ Let $f : \mathbb{R} \to \mathbb{R}$. Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ is the derivative of f at x_0.

Alternately: $$\lim_{x\to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right] = 0.$$

$$\Rightarrow \lim_{x\to x_0} \left[\frac{f(x) - f(x_0) \overbrace{- f'(x_0)(x - x_0)}}{x - x_0} \right] = 0$$

reminiscent of 1st order Taylor expansion.
$\Rightarrow$ differentiable = can be approximated by a linear transformation.

In several variables: $$\lim_{\vec{x}\to \vec{x}_0} \left[\frac{f(\vec{x}) - f(\vec{x}_0) - (\nabla f)(\vec{x}_0)(x - x_0)}{\|\vec{x} - \vec{x}_0\|} \right] = 0.$$

$\underline{\mathbb{C}}$: Let $f: \mathbb{C} \to \mathbb{C}$. Then $f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$$= \lim_{h \to 0} \frac{f(z+h) - f(z)}{\underbrace{h}_{h_1 + i h_2}}$$

Can approach along any direction/curve.

<u>Differentiable functions examples:</u>

<u>ex1</u>: $f(z) = c$.

<u>2</u>: $f(z) = z$.

<u>3</u>: $f(z) =$ polynomial.

NOT example: $f(z) = \bar{z}$.

$$f'(z) = \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$$

if h is real valued: 1

if h is purely imaginary: -1.

$\Rightarrow$ Limit dne.!

$\therefore \bar{z}$ is not differentiable.

$\to$ <u>generally</u>: (intuitively) if $f(z)$ can be written purely in terms of z's, it will be complex differentiable.

$\to$ avoid $\bar{z}, x, y, r, \theta$, etc.

<u>Question</u>: how to understand complex differentiability?

$\to$ compare $f: \mathbb{C} \to \mathbb{C}$ to $f: \mathbb{R} \to \mathbb{R}$?

$\to$ compare $f: \mathbb{C} \to \mathbb{C}$ to $f: \mathbb{R}^2 \to \mathbb{R}^2$?

Statement	True for $f: \mathbb{R}^{(2)} \to \mathbb{R}^{(2)}$?	True for $f: \mathbb{C} \to \mathbb{C}$?
If diff'ble once, is ∞-ly diff'ble	NO eg. $f(x) = \int_0^x \lvert t \rvert dt$	YES
If ∞-ly diff'ble, equals Taylor expansion on open nbhd?	NO eg. $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$	YES
Bounded implies constant?	NO eg. $f(x) = \sin x$	YES

line integral over a closed curve is 0?	NO	YES

Note on $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

- ∞-ly differentiable, $f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0$.

  $\therefore$ Taylor series is 0!

  $\Rightarrow$ results in entire class of similar functions:

  eg. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ for $\sin x + \underbrace{1701 f(x)}_{\text{does not contribute to Taylor series.}}$

Series Expansions

Most important example: $e^x = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$(e^x)' = e^x.$$

exercise: Prove $e^2 e^3 = e^5$ (via series)

$\rightarrow$ use binomial expansion

$\Rightarrow$ When we plug in $e^{ix} = \sum_{n=0}^{\infty} \frac{(i^n x^n)}{n!}$ $\leftarrow$ regroups to get

$$= \cos\theta + i \sin\theta.$$

Thm 2.5: Let $\sum_{n=0}^{\infty} a_n z^n$, $\frac{1}{R} = \limsup_{n \to \infty} |a_n^{1/n}|$, $\quad 1/0 := \infty, \frac{1}{\infty} := 0$.

Then the series converges absolutely for $|z| < R$.

R is the radius of convergence, and $\{|z| < R\}$ is the disk of convergence.

Sketch pf: n big: $|a_n^{1/n}| < \frac{1}{R - \varepsilon}$ (assume $R \neq 0$).

Then $a_n z^n = (a_n^{1/n} z)^n$. Assuming $|z| < R$, say $|z| < R - 2\varepsilon$,

then $|a_n z^n| \leq \left|\frac{R - 2\varepsilon}{R - \varepsilon}\right|^n$ by comparison with geo. series with ratio $\frac{R - 2\varepsilon}{R - \varepsilon}$. ∎

Thm 2.6: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R.
Then $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$; in fact f is ∞ differentiable,
and we can interchange $\frac{d}{dz}$ and Σ.

Pf. Step 1. Series for f' conv.

Pf: $\limsup_{n\to\infty} |(n a_n)^{1/n}| = \limsup_{n\to\infty} \underbrace{|n^{1/n}|}_{\to 1} \underbrace{|a_n^{1/n}|}_{\to \frac{1}{R}}$. ∎

Step 2: $f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$.

Study $\lim_{h\to 0} \left(\frac{f(z_0+h) - f(z_0)}{h} - \sum_{n=0}^{\infty} n a_n z_0^{n-1} \right)$

write as $f(z) = \underbrace{S(z)}_{\text{up to } N} + \underbrace{E(z)}_{\text{tail}}$

$\to S'(z)$ is a finite sum.

$= \lim_{h\to 0} \left[\frac{S(z_0+h) - S(z_0)}{h} - \sum_{n=0}^{\infty} n a_n z_0^{n-1} + \frac{E(z_0+h) - E(z_0)}{h} \right]$

$- S'(z_0) + S'(z_0)$

(add zero)

1. $\Rightarrow$ choose N_1 s.t. $\left[\frac{S(\ ) - S(\ )}{h} - S'(z_0) \right]$ is small when h is small.

2. $\Rightarrow$ also, $\left[S'(z_0) - \sum_{n=0}^{\infty} n a_n z_0^n \right]$ is small bec. series converges uniformly! tail has to be small.

$\hookrightarrow$ choose N_2 for this.

3. factorize: $(z_0+h)^n - z_0^n = h\left(z_0^{n-1} + z_0^{n-2} h + \dots + z_0 h^{n-2} + h^{n-1} \right)$
$\leq h \cdot n \cdot \max(|z_0|, |h|)^{n-1}$.

So $\frac{E(z_0+h) - E(z_0)}{h} = \sum_{n=N+1}^{\infty} n^2 \cdot a_n \cdot \max(|z_0|, |h|)^{n-1}$, which converges unif'ly.
$\Rightarrow$ choose N_3. ∎

Example: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ Compute $\limsup_{n\to\infty} \left|\frac{1}{n!}\right|^{1/n} \ldots$

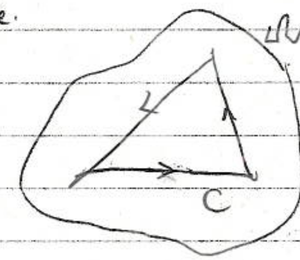
Cauchy-Riemann Eqns

- $f(x,y) = u(x,y) + i v(x,y)$

$$\rightarrow \boxed{u_x = v_y \,, \quad u_y = -v_x .}$$

Goursat's Thm: Let f holomorphic on Ω, C a triangle.

Then $\int_C f(z)\,dz = 0$.

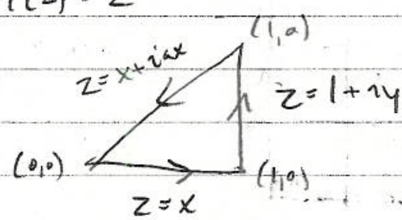


Pf: • Divide into similar triangles.

- use pigeonhole principle to zoom in
- use linear approx. to bound integral: $f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)$ leftover.

$\therefore = 0$. ∎

Example: $f(z) = z$



$$\int_C f\,dz = \int_0^1 x\,dx + \int_0^a (1+iy)\,i\,dy$$

$$+ \int_1^0 (x+iax)(dx+iadx)$$

$$= \cancel{\frac{1}{2}} + \cancel{ia} - \cancel{\frac{a^2}{2}} - \int_0^1 (x+a^2x)\,dx - \int_0^1 2iax\,dx$$

$$- \cancel{\frac{1}{2}} + \cancel{\frac{a^2}{2}} - \cancel{ia}.$$

$$= 0.$$

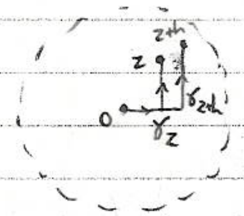
Recall: If f is holo in Ω and closed curve γ and f has primitive F, then $\oint_\gamma f\,dz = 0$.

Finding primitives

Thm: Let f holomorphic in a disk D. Then f has primitive.

Proof: WLOG $D = B_R(0)$.

Let $F(z) = \int_{\gamma_z} f(w)\,dw$

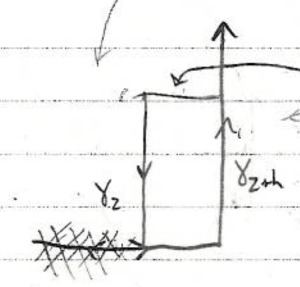


we don't know for sure that F is a primitive yet, because we had to define it relative to a specific path.

Goal: $F'(z) = f(z)$

i.e. F is a primitive for f.

So: look @ $\dfrac{F(z+h) - F(z)}{h} = \dfrac{\int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw}{h}$.



put in this segment, add & subtract.

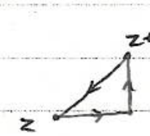
$\Rightarrow$ by Goursat's Thm, rectangle cancels:

$= 0$

traversed twice, cancelled out.

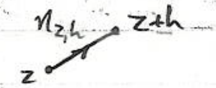
Leaves behind:

so add & subtract the hypotenuse



plus

$= 0$ by Goursat's Thm.

so $$\frac{F(z+h) - F(z)}{h} = \frac{\int_{\eta_{z,h}} f(w)\,dw}{h}$$, where 

Now, since f is cont's, $\exists$ a function ψ such that $f(w) = f(z) + \psi(w)$ and $\lim_{w \to z} \psi(w) = 0$.

$$\text{so: } = \frac{1}{h}\int_{\eta_{z,h}} f(z) + \psi(w)\,dw = \frac{f(z)}{h}\underbrace{\int_{\eta_{z,h}} dw}_{=h} + \frac{1}{h}\int \psi(w)\,dw$$

$$= f(z) + \frac{1}{h}\int \psi(w)\,dw.$$

$$* \text{ consider } \left|\frac{1}{h}\int_{\eta_{z,h}} \psi(w)\,dw\right| \le \frac{1}{|h|}\int_{\eta_{z,h}} |\psi(w)||dw|$$

$$\le \frac{\max_{w \in \eta_{z,h}} |\psi(w)|}{\cancel{|h|}} \cdot \cancel{|h|}$$ gets arbitrarily small as $h \to 0$.

Hence $F'(z) = f(z)$. ∎

Corollary: If f is holo on disk D, γ a closed curve then $\oint_\gamma f\,dz = 0$.

or, if $\gamma = \gamma_1 - \gamma_2$ , then $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$.

Proof: f has a primitive!

__Cauchy's Integral Formula__: Let f holo in open set Ω, γ closed curve.
Then $\int_\gamma f\,dz = 0$.

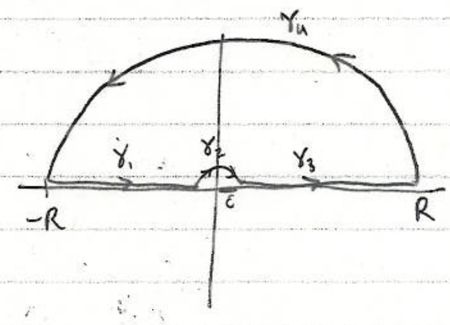
__Aside__: __Example 2__: $\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}$. Use $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

Note: $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$

So near origin $\frac{1-\cos x}{x^2} = \frac{1}{2} + \frac{x^2}{24} - \dots \approx \frac{1}{2}$.

__Evaluate__: $\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\,dx$, take real part @ the end.

let $f(z) = \frac{1-e^{iz}}{z^2}$.



By Cauchy's Thm,

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f\,dz = 0.$$

$\rightarrow$ Evaluate γ_4 first. Claim: as $R\to\infty$, $\int_{\gamma_4} f\,dz \to 0$.

Proof: $e^{iz} = e^{ix}e^{-y}$ gives __decay__ as R increases (i.e. y)

so $|e^{iz}| = e^{-y} \leq 1$.

$$\text{so } \left|\int_{\gamma_4} f\,dz\right| \leq \int \left|\frac{1-e^{iz}}{z^2}\right| dz \leq \int_{\gamma_4}\left|\frac{2}{z^2}\right| dz \leq \frac{2}{R^2}\cdot\pi R$$

$|1-e^{iz}| \leq 1-(-1) = 2$

$$= \frac{4\pi}{R} \to 0 \text{ as } R\to\infty.$$

$\rightarrow \gamma_1$ and γ_3: $$\int_{\gamma_1+\gamma_3} \frac{1-e^{iz}}{z^2}\,dz = \int_{\gamma_1+\gamma_3} \frac{1-e^{ix}}{x^2}\,dx = \int_{\gamma_1+\gamma_3} \frac{1-\cos x}{x^2}\,dx$$

$$= \int_{\varepsilon<|x|<R} \frac{1-\cos x}{x^2}\,dx.$$ Eventually let $R\to\infty$ and $\varepsilon\to 0$, get $\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx$.

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \cdots$$

$\underline{\gamma_2}$: Danger is: $\dfrac{1-e^{iz}}{z^2} = \dfrac{-i}{z} \underbrace{- \dfrac{i^2}{2} + \cdots}$

blows up of size $1/\varepsilon$ on a circle of radius ε. (but perimeter decreases.)

nice, convergent series Small when $|z|$ small.

So as $\varepsilon \to 0$ and $R \to \infty$, $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx + \lim_{\varepsilon \to 0} \int_{\gamma_2} \frac{1-e^{iz}}{z^2}\,dz = 0.$

$$\text{So} \quad \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = -\lim_{\varepsilon\to 0} \int_{\gamma_2} \frac{1-e^{iz}}{z^2}\,dz.$$

Consider: $\displaystyle\int_{\gamma_2} \frac{-i}{z}\,dz$. θ: π to 0.

$z = \varepsilon e^{i\theta}$. $dz = \varepsilon e^{i\theta} i\,d\theta$

$$\int_{\pi}^{0} \cancel{-i\varepsilon e^{-i\theta}} \cdot \cancel{i}\,d\theta = -\pi.$$

Now let $g(z) = \dfrac{1-e^{iz}}{z^2} - \left(\dfrac{-i}{z}\right) = -\displaystyle\sum_{n=2}^{\infty} \frac{(iz)^n}{n!\,z^2} = -\sum_{n=0}^{\infty} \frac{i^{n+2} z^n}{(n+2)!}$

size 1

Small, helps convergence. (denom $\geq$ denom of exponential)

→ So, for any small number η, can choose ε so that

$\left| \dfrac{1-e^{iz}}{z^2} - \left(\dfrac{-i}{z}\right) \right| \leq \eta$ inside and on semi-circle of radius ε.

Hence $\left| \displaystyle\int_{\gamma_2} \left(\frac{1-e^{iz}}{z^2} - \left(\frac{-i}{z}\right) \right) dz \right| \leq$ max value $*$ length of semicircle $= \eta \cdot \pi \varepsilon$.

converges to 0.

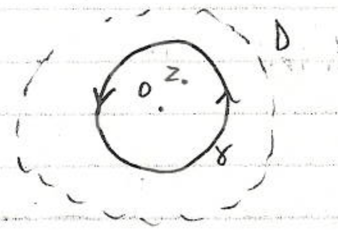
Substituting gives: $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} = -(-\pi) = \pi.$

Sanity check: ① good to see π
② real and positive

Cauchy's Integral Formula

Let f holo on a disk. Let γ be a circle in the disk.
Then $\forall z$ in the circle,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z).$$



- This doesn't vanish (like previous results), but $\frac{f(\zeta)}{\zeta - z}$ not holo at z.

- We are not using any information about values of $f(z)$ anywhere off of the curve γ!

Proof: 1. choose contour

- So $\int_\gamma \frac{f(\zeta)}{\zeta - z}\,dz = 0$ (holo).

- also, the two straight lines cancel each other (reversed orientations)

- So, letting the two circles be γ_R and γ_ε counter-clockwise, we have

(letting $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$) $$\oint_{\gamma_R} F(\zeta)\,d\zeta + \oint_{-\gamma_\varepsilon} F(\zeta)\,d\zeta = 0.$$

$$\text{So } \frac{1}{2\pi i}\oint_{\gamma_R} F(\zeta)\,d\zeta = \frac{1}{2\pi i}\oint_{\gamma_\varepsilon} F(\zeta)\,d\zeta.$$

$\Rightarrow$ Have to show $\frac{1}{2\pi i}\oint_{\gamma_\varepsilon} F(\zeta)\,d\zeta$ is $f(z)$.

$\Rightarrow$ rewrite $$F(\zeta) = \underbrace{\frac{f(\zeta) - f(z)}{\zeta - z}}_{\text{as } \zeta \to z\text{, this goes to } f'(z).} + \underbrace{\frac{f(z)}{\zeta - z}}_{\text{Constant.}}$$

Claim: $\frac{1}{2\pi i}\oint_{\gamma_\varepsilon} \frac{f(\zeta)-f(z)}{\zeta - z}\, d\zeta$ goes to 0 as $\varepsilon \to 0$.

$\Rightarrow$ quotient tends to $f'(z)$, say differs by at most 1.

$$\text{So } \left| \frac{1}{2\pi i}\oint_{\gamma_\varepsilon} \frac{f(\zeta)-f(z)}{\zeta-z}\, d\zeta \right| \leq \frac{1}{2\pi}\oint_{\gamma_\varepsilon} |f'(z)| + 1 \, d\zeta, \quad \varepsilon \text{ small.}$$

$$\leq \frac{\cancel{2\pi}\varepsilon}{\cancel{2\pi}} \cdot \underbrace{(|f'(z)|+1)}_{\text{constant}}.$$

$\therefore$ goes to 0.

Claim: $\frac{1}{2\pi i}\oint_{\gamma_\varepsilon} \frac{f(z)}{\zeta - z}\, d\zeta = \frac{f(z)}{2\pi i}\underbrace{\oint_{\gamma_\varepsilon} \frac{1}{\zeta - z}\, dz}_{} = \frac{f(z)}{2\pi i}\cdot 2\pi i.$

$$= \oint_{\text{unit circle}} \frac{1}{w}\, dw = 2\pi i.$$

$\blacksquare$

Consequences:

Thm: If f is holomorphic, then f is ∞-ly differentiable!

Proof: see book.

Sketch: $f(z) = \frac{1}{2\pi i}\oint_{\gamma_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ so $f'(z) = \frac{1}{2\pi i}\cdot \frac{d}{dz}\oint_{\gamma_R} \frac{f(\zeta)}{\zeta-z} dz$

$$\text{"}=\text{"} \frac{1}{2\pi i}\oint_{\gamma_R} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

$\hookrightarrow$ continue in this fashion:

$$f^{(n)}(z) = \frac{1}{2\pi i}\oint_{\gamma_R} \frac{n!\, f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta. \quad \blacksquare$$

Difficulty: exchange derivative and integral

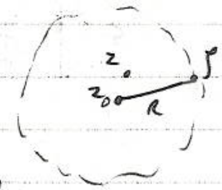
$\hookrightarrow$ use defn of derivative.

about point z_0

Thm: f holomorphic on a disk, γ closed curve in disk, z inside curve.
Then f equals its Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{with} \quad a_n = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Proof: $$f(z) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$



$$\zeta - z = \underbrace{(\zeta - z_0)}_{\text{want}} - \underbrace{(z - z_0)}_{\text{dist moved, smaller than } |z-\zeta|}$$

Fact: $\dfrac{|z-z_0|}{|\zeta - z_0|} < 1$. $\boxed{\text{think: geometric series}}$

So write $$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\zeta - z_0}\right)^n.$$

geometric series

So: $$f(z) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\zeta - z_0)^n}\, d\zeta.$$

- want to switch $\oint_{\gamma_R}$ and $\sum_{n=0}^{\infty}$

check this

- use Fubini Thm: if abs value is convergent (ie, finite)
  - → cont's function is bounded on the circle
  - → geometric series converges.

  } so everything is finite.

Thus $$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n.$$ ∎

$$\boxed{\text{So: in complex analysis, HOLOMORPHIC = ANALYTIC.}}$$

Cauchy's Inequalities: If f holo in Ω, γ closed circle, then for all z inside circle,

$$\left|f^{(n)}(z)\right| \leq \frac{n! \max_{w\in\gamma}|f(w)|}{R^n}$$

Pf: · f attains its max b/c γ a compact set

· use $\left|\oint_\gamma \frac{f(\xi)}{(\xi-z)^{n+1}} d\xi\right| \leq \oint_\gamma \left|\frac{f(\xi)}{(\xi-z)^{n+1}}\right| |d\xi| \leq \frac{\max_{w\in\gamma} f(w)}{R^{n+1}} \underbrace{\text{Length}(\gamma)}_{=2\pi R}$

$$\therefore f^{(n)}(z) = \frac{n!}{2\pi} \oint \quad d\xi \leq \frac{n! \max|f|}{R^n}. \quad \blacksquare$$

Liouville's Thm: If f is entire & bounded. Then f is constant!

Pf: Cauchy's Inequalities.

f holomorphic $\Rightarrow$ analytic, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n = \frac{f^{(n)}(0)}{n!}$.

· Since f is bounded, $|f(w)| \leq B$ for some $B \in \mathbb{R}$.

· by Cauchy ineq, $|f^{(n)}(0)| \leq \frac{n!}{R^n} B$ for any R.

$$\therefore f^{(n)}(0) = 0. \quad \blacksquare$$

Real Comparison:

① $f(x) = \sin x$

② $g(x) = \frac{1}{1+x^2}$.

FTA: $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, then $\exists z_1, \ldots, z_n$ s.t. $f(z_i) = 0$.

Proof: suffices to prove $\exists$ at least one root, for an arbitrary nonconstant polynomial.

So let $f(z)$ be of deg n and (1) not Constant

(2) never zero

→ cannot Liouville $f(z)$

→ try to Liouville $1/f(z)$. Claim: $g(z) = \frac{1}{f(z)}$ bounded.

↳ cont's function on compact set is bounded.

$\Rightarrow g(z)$ bdd inside and on any large circle.

Consider $f(z) = z^n$ (motivation).

if γ is a circle of radius $|R|$, then $|z|>R$ means $\left|\frac{1}{f(z)}\right| \leq \frac{1}{R^n}$.

<u>General case:</u> $f(z) = a_n z^n + \cdots + a_1 z + a_0$

$$= a_n z^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \frac{a_{n-2}}{a_n}\frac{1}{z^2} + \cdots + \frac{a_1}{a_n}\frac{1}{z^{n-1}} + \frac{a_0}{a_n}\frac{1}{z^n}\right).$$

want lower bound for $f(z)$.

So $$|f(z)| \geq |a_n||z|^n \left(1 - \frac{|a_{n-1}|}{|a_n|}\frac{1}{|z|} - \left|\frac{a_{n-2}}{a_n}\right|\frac{1}{|z|^2} - \cdots\right)$$

Let $a = \max_k \left|\frac{a_k}{a_n}\right|$.

$$\geq |a_n||z|^n\left(1 - \frac{a}{|z|} - \frac{a}{|z|^2} - \cdots\right)$$

$$\geq |a_n||z|^n\left(1 - \frac{na}{|z|}\right) \quad \text{assuming } |z| \geq 1.$$

$$\geq |a_n||z|^n \underbrace{\left(1 - \frac{na}{R}\right)}.$$

choose R so that $\frac{na}{R} < \frac{1}{2}$.

$\Rightarrow$ then $|f(z)| \geq \frac{|a_n|}{2} R^n$.

So $\frac{1}{|f(z)|} \leq \frac{2}{|a_n| R^n}$ for $|z| \geq R$, and we know f bounded inside circle

Know $\frac{1}{f(z)}$ is entire, bounded $\xrightarrow{\text{Liouville}}$ $\frac{1}{f(z)} = c$ for some $c \neq 0$.

$\therefore f(z) = \frac{1}{c}$ constant. ∎

Accumulation Points

Real motivation: $g(x) = x^3 \sin\left(\frac{1}{x}\right)$

$\hookrightarrow$ diff

$\hookrightarrow$ zeros at $\frac{1}{\pi n}$, $n \in \mathbb{Z} \setminus \{0\}$.

Big Thm: Let f holo on Ω, $f(z_n) = 0$ for a sequence $z_n \to z_0$ in Ω, then f is identically zero.

Application: if $f(z) = g(z)$ for $z = z_n \to z_0$ in Ω, then $f = g$.

Pf: Step 1: Show f is zero in some small disk about z_0

Step 2: Use topology to show zero in all of Ω. (Connected sets)

1. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. $\overbrace{f(z_0)}^{= a_0 =} = 0$ by continuity:

$\lim_{n\to\infty} f(z_n) = 0 = f\left(\lim_{n\to\infty} z_n\right)$
$= f(z_0)$.

WLOG $z_0 = 0$

Let N be smallest integer s.t. $a_N \neq 0$.

$$f(z) = a_N z^N + a_{N+1} z^{N+1} + \cdots$$

$$= a_N z^N \left(1 + \frac{a_{N+1}}{a_N} z^1 + \frac{a_{N+2}}{a_N} z^2 + \cdots\right)$$

$$= a_N z^N \left(1 + g_N(z)\right).$$

So $g_N(0) = 0$ since z divides each term.

By continuity, $|g(z)| \leq \frac{1}{2}$ (say, for all $|z| < \delta$.) so $|1 + g_N(z)| \geq \frac{1}{2}$ for $|z| < \delta$.

So $f(z) \neq 0$ except at $z = 0$ for all z with $|z| < \delta$

Contradicts $z_n \to z_0$ with $f(z_n) = 0$.

Assumption of finite N is false. Thus $a_N = 0$ for all N, $f(z) = 0$. ∎

Chapter 3

(1) Extending Cauchy's Theorem
(2) Picard's Thm and the ilk
(3) Riemann Mapping Thm:

Ω be any simply connected open set, not all of $\mathbb{C}$. Then $\exists$ holo map $f: \Omega \to \mathbb{D}$ (unit disk) that is a homeomorphism.

(4) Analytic continuation

$\hookrightarrow$ extend domain of definition of functions.

Language Lesson: $f(z_0) = 0$ means z_0 is a zero of $f(z)$.

$\rightarrow f(z) = (z-z_0)g(z)$.

$\Rightarrow z_0$ is a zero of order n if $f(z) = (z-z_0)^n g(z)$ and $g(z_0)$ exists and $\neq 0$.

$\Rightarrow z_0$ is a pole of order n if it is a zero of order n of $1/f(z)$.

If f has a pole of order n at z_0,

$$f(z_0) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + \underbrace{g(z)}_{\text{nice}}.$$

Most important is a_{-1}, $\text{Res}_{z_0}(f) = a_{-1}$.

Finding Residues

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{-1/2i}{z+i} + \frac{1/2i}{z-i}$$

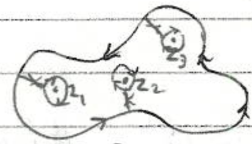
$\text{Res}_f(i) = ?$

$$\frac{1}{z+i} = \frac{1}{2i\left(1+\frac{z-i}{2i}\right)} = \frac{1}{2i}\sum_{n=0}^{\infty}\frac{(-1)^n(z-i)^n}{(2i)^n} = \frac{1}{2i}\left[1 + \frac{-(z-i)}{2i} + \frac{(-(z-i))^2}{(2i)^2} + \cdots\right]$$

So $f(z) = \frac{1}{2i}\frac{1}{z-i}$ + higher terms.

So $a_{-1} = \frac{1}{2i}$.

Claim: Let f holo in Ω except at $z_1, \dots, z_n$ which are inside closed curve γ.



$$\frac{1}{2\pi i} \oint_\gamma f(z)\,dz$$

insert circles to form curve Γ not enclosing the z_i's.

$$\oint_\Gamma f(z)\,dz = 0 = \oint_\gamma f(z)\,dz - \sum_{k=1}^{n} \oint_{\text{circle } k} f(z)\,dz$$

$\therefore$

Cauchy's Residue Thm: $\boxed{\frac{1}{2\pi i} \oint_\gamma f(z)\,dz = \sum_{k=1}^{n} Res_f(z_k)}$

where each z_k is inside the curve. (not on)

Example:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix} - e^{-ix}}{2ix}\,dx \quad \dots$$

integrate in upper half plane

integrate in lower half plane

Argument Principle

Let f be meromorphic in Ω. γ closed curve, f has no zeros or poles on the curve.

Then $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \#\{\text{zeros inside } \gamma\} - \#\{\text{poles inside } \gamma\}$.

(counting multiplicity)

Pf: Look @ series expansion about z_0:

$$f(z) = a_n(z-z_0)^n + \cdots$$

if $n=0$, $f(z_0)=a_n$
if $n>0$, $f(z_0)$ is a zero of order n
if $n<0$, $f(z_0)$ is a pole of order $-n$.

So write $f(z) = a_n(z-z_0)^n\left[1+g(z)\right]$ holo and $\lim_{z\to z_0} g(z) = 0$.
since $g(z) = b(z-z_0)+\cdots$

$$f'(z) = na_n(z-z_0)^{n-1}\left[1+g(z)\right] + a_n(z-z_0)^n g'(z).$$

$$\frac{f'(z)}{f(z)} = \frac{na_n(z-z_0)^{n-1}\left[1+g(z)\right] + a_n(z-z_0)^n g'(z)}{a_n(z-z_0)^n\left[1+g(z)\right]}$$

$$= \frac{n}{z-z_0} + \underbrace{\frac{g'(z)}{1+g(z)}}_{\text{holo @ } z_0 \text{ bec/ denominator is 1 here.}}$$

So, $\operatorname{Res}_{f'/f}(z_0) = n$.

Hence $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum \text{residues}$. ∎

More generally, for $h(z)$ holomorphic

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}h(z)\,dz = \sum_{\substack{z_j \text{ a zero} \\ \text{or a pole}}} \operatorname{ord}_f(z_j)\cdot h(z_j)$$

because locally $h(z) = h(z_0) + b(z-z_0) + \cdots$

Rouché's Thm

- f, g holo on and inside closed curve γ.
- $|f(z)| > |g(z)|$ on γ.

Then # zeros of f inside γ = # zeros of $f+g$ inside γ.

Pf: $\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz$ = # zeros since f has no poles.

Consider $f_t(z) = f(z) + t\,g(z)$, $t \in [0,1]$.

$$\begin{cases} f_0(z) = f(z) \\ f_1(z) = f(z) + g(z). \end{cases}$$

Observe: on γ, $|f(z)| > |g(z)| \geq 0$ on γ.

Note: similarly, for $t \in [0,1]$, $f_t(z) \neq 0$ on γ.

So, let $\text{NumZeroF}(t) = \frac{1}{2\pi i} \oint \frac{f_t'(z)}{f_t(z)}\, dz$ = # zeros f_t inside γ.

$$\begin{cases} \text{NumZerof}(0) = \text{\# zeros } f \text{ inside } \gamma \\ \text{NumZerof}(1) = \text{\# zeros } f+g \text{ inside } \gamma. \end{cases}$$

$\hookrightarrow$ NumZerof is $\underbrace{\text{integer-valued}}_{\text{by argument principle}}$ and $\underbrace{\text{continuous}}_{\text{bring in } \varepsilon, \delta, \dots \text{ book-keeping.}}$, therefore constant! ∎

Application: FTA

- $f+g = a_n z^n + \cdots + a_0$
- $f = a_n z^n \to n$ zeros
- $g = a_{n-1} z^{n-1} + \cdots + a_0$

$\Big\}$ examine $|f|$ and $|g|$.

Application: Open Mapping Thm (to be proved)

$\left.\begin{array}{l}\cdot\ f \text{ holo, nonconstant} \\ \cdot\ \mathcal{U} \text{ open}\end{array}\right\} \longmapsto f(\mathcal{U})$ is open

Application: Max Modulus Principle

- f is a non-constant holo fn.

Then: (1) if f holo in open set $\mathcal{U}$, then f does not attain a maximum on $\mathcal{U}$

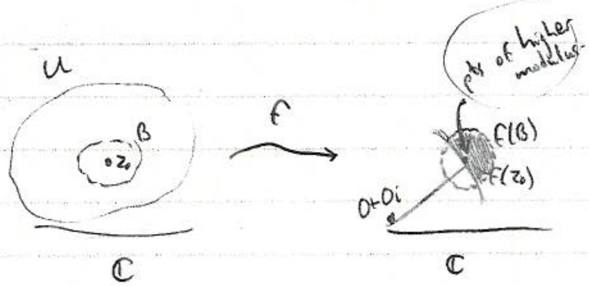
(2) f attains its max on boundary of $\mathcal{U}$ (requires f holo in $\mathcal{U}$ + f cont's on $\overline{\mathcal{U}}$) if $\overline{\mathcal{U}}$ is compact.

Pf: Claim: (1) + real analysis (topology) = (2)

$\Vert$ b/c
- maximum not on interior
- cont's function on compact set attains max.

(1): Use open mapping theorem:

Suppose f has a max at z_0:



and $f(B)$ must include points of higher modulus.

Infinite Products $\prod_{n=1}^{\infty} C_n \quad \prod_{n=1}^{\infty}(1+a_n)$.

Def'n: Infinite product is $\lim_{N\to\infty} \prod_{n=1}^{N}(1+a_n)$.

Key fact: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

Radius of convergence $= 1$.

ex: Consider: $a_n = \begin{cases} -1/2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$ so $1+a_n = \begin{cases} 1/2 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$

Fact: • convergence requires $a_n \to 0$.
(to a nonzero)

* Before Tayloring, assume $|a_n| \le 1/2$, so can use Taylor:

$$\log \prod_{n=1}^{N}(1+a_n) = \sum_{n=1}^{N} a_n - \frac{1}{2}\sum_{n=1}^{N} a_n^2 + \frac{1}{3}\sum_{n=1}^{\infty} a_n^3 - \cdots$$

Note: $|a_n| > |a_n|^2 > \cdots$ as $|a_n| < 1/2$.

Assume $\lim_{N\to\infty}$ Taylor exists:

1. $-\infty \Rightarrow$ if a_n are real, $\prod$ goes to zero.

2. FINITE $\to$ just exponentiate the limit.

3. $+\infty \Rightarrow$ if a_n are real, $\prod$ diverges to $+\infty$.

Continuity Stuck

Prop: If $\sum |a_n| < \infty$ then prod converges to a nonzero real number.

$\vdots$

Weierstrass Products

Thm: Let $\{a_n\}$ be a sequence that is "good". Then $\exists$ an entire function, not identically zero, $f(z)$ vanishing at the a_n and nowhere else,

and if $g(z)$ is any other function vanishing at just the a_n's, then $\exists$ an entire fn $h(z)$ s.t. $f(z) = g(z)e^{h(z)}$.

(1) Each value occurs in $\{a_n\}$ only finitely often

(2) No accumulation points

Candidates: say zeros at a_n.

Guess: $$z^{\mathrm{ord}_0(f)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

Try $a_n = n$, then $$\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right) \quad \xrightarrow{z \text{ negative}} \infty, \quad \xrightarrow[z \text{ +ve}]{} 0.$$

Goal: find convergence factors that don't introduce zeros.

Recall: $\prod_{n=1}^{\infty}(1+a_n)$ means $\lim_{N\to\infty} \prod_{n=1}^{N}(1+a_n)$

Modify: $$\lim_{N\to\infty} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)C_n(z)$$

$\hookrightarrow \exp(?_n(z))$.

Let $E_k(z) = e^{z + z^2/2 + \cdots + z^k/k}$

$$(1-z)\,E_k(z) = e^{\log(1-z) + z + \cdots + z^k/k}.$$

assume $|z| < 1/2$.

then $\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots$

so, $$\log(1-z) + z + z^2/2 + \cdots + z^k/k = -\sum_{k+1}^{\infty} \frac{z^l}{l}$$
$$= z^{k+1} \sum_{l=0}^{\infty} \frac{z^l}{l+1+k}$$
$$\le |z^{k+1}| \underbrace{\left|\sum_{l=0}^{\infty} z^l\right|}\ = \frac{|z^{k+1}|}{1-|z|} \le 2|z|^{k+1}.$$

$z < 1/2$

→ Freedom to choose k ("Free parameter" argument)

$k = f(n)$, maybe $f(n, a_n)$. not a function of z.

$$\left|\left(1 - \frac{z}{a_n}\right) E_k\left(\frac{z}{a_n}\right)\right| \le e^{2|z/a_n|^{k+1}}.$$

Study: $$z^{\mathrm{ord}_0(f)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) E_{k(n)}\left(\frac{z}{a_n}\right).$$

$$= z^{\mathrm{ord}_0(f)} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right) E_{k(n)}\left(\frac{z}{a_n}\right) \cdot \prod_{n=N+1}^{\infty} \left(1 - \frac{z}{a_n}\right) E_{k(n)}\left(\frac{z}{a_n}\right).$$

→ choose N so that $\forall n \ge N$, $\left|\frac{z}{a_n}\right| < 1/2$.

Want to show convergence:

1) no problem first N factors and $z^{\mathrm{ord}_0(f)}$.

2) $$\left|\prod_{n=N+1}^{\infty} \left(1 - \frac{z}{a_n}\right) E_{k(n)}\left(\frac{z}{a_n}\right)\right| \le \prod_{n=N+1}^{\infty} e^{2\left|\frac{z}{a_n}\right|^{k(n)+1}}$$

$$= e^{2\sum \left|\frac{z}{a_n}\right|^{k(n)+1}}$$ try $k(n) = n$, geo series!

$$= e^{2\frac{1 - \left|\frac{z}{a_n}\right|^{k+1}}{1 - R/a_n}} < \infty.$$

Strange/Ponder: . choice of $k(n)$ "wasteful", indep. of the a_n...
· could have refined our choice of $k(n)$.

Open Mapping Theorem - Rouché Proof

Pf: Let $z_0 \mapsto w_0 = f(z_0) \in f(\Omega)$. Must show: $\forall w$ close to w_0, $\exists z(w)$ s.t. $f(z(w)) = w$.

$\to$ study $g_w(z) = f(z) - w$, find a root.

easy: $g_{w_0}(z) = w_0$; use Rouché's Thm:

$$g_w(z) = \left(f(z) - w_0\right) + \left(w_0 - w\right)$$
$$F(z) \quad + \quad G(z)$$

· Clearly F has a root: z_0.
· Study $F(z)$ for z on a small circle near z_0, need
1) $|F(z)| > 0$ on circle
2) $|F(z)| > |G(z)|$ on circle. } Rouché.

Note $(2) \to (1)$.

Assume (1) false. Then for each circle (radius r), there is a z_r s.t. $f(z_r) = w_0$. Then z_0 is an accumulation point!
↳ So f is locally constant: contradiction.

$\Rightarrow f(z) \neq w_0$ on a circle about z_0 of radius δ.

So, $\exists\, \varepsilon > 0$ s.t. $\underbrace{|f(z) - w_0|}_{|F(z)|} > \varepsilon$ on circle (compact set + continuity) $|z - z_0| = \delta$.

So take $\underbrace{|w - w_0|}_{G(z)} < \varepsilon$, have $|F(z)| > \varepsilon > |G(z)|$. ∎

Chap 8: Conformal Mappings

Conformal map: • bijective
• holo

Def: $\mathcal{U}$ is conformally equivalent to V if $\exists$ conformal mapping $f: \mathcal{U} \to V$.

Equivalence relation: 1. reflexivity: clearly $\mathcal{U} \approx_c \mathcal{U}$.
2. Symmetry: show $\mathcal{U} \approx V$ iff $V \approx \mathcal{U}$.
* has non-trivial content!

Pf of (2): Sketch: f has an inverse (since it is bijective), g.
Show g is holo.

idea: 1-1 implies f' is never zero.
(note, $f(z) = z^n$ is not a $\mathbb{C}$-bijection if $n \geq 2$.)

• normalize, write $f(z) = z(1 + g(z))$.

• Compute formal inverse;
Know series are holo in region of convergence.

Book pf: Rouché's.

3. transitivity: $\mathcal{U} \approx V$, $V \approx W$, then $\mathcal{U} \approx W$.

Pf: Function composition.

Automorphism Groups

Key Prop: If $f: \mathcal{U} \to V$ is conformal, then f^{-1} is conformal.

$\therefore$ The automorphism form a group.

Schwarz Lemma: Let $f: \mathbb{D} \to \mathbb{D}$ be a conformal map that isn't constant and $f(0)=0$.

Then: (1) $|f(z)| \le |z| \quad \forall z \in \mathbb{D}$

(2) If $\exists z_0 \in \mathbb{D}$ s.t. $|f(z_0)| = |z_0|$ then $f(z) = e^{i\theta} z \quad \theta \in \mathbb{R}$ (rotation).

(3) $|f'(0)| \le 1$; if $|f'(0)| = 1$ then f is a rotation.

Classifying $Aut(\mathbb{D})$:

① All rotations belong

② Candidate: $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$. Note: $\left.\begin{array}{l}\psi_\alpha(\alpha) = 0 \\ \psi_\alpha(0) = \alpha\end{array}\right\}$ interchange α, 0.

- ψ_α is holo (only problem point is $z = \frac{1}{\bar{\alpha}} \notin \mathbb{D}$.)

- to see bijectiveness:
  - into by algebra, if $|z| < 1$ then $|\psi_\alpha(z)| < 1$.
  - onto bec. $|\psi(z)| = 1$ on boundary
  - injective.

Note: $\psi_\alpha \circ \psi_\alpha$ = identity

$0 \mapsto 0$

$\alpha \mapsto \alpha$.

③ Compositions of these

That's all!

Note: $\psi_\alpha \circ \psi_\beta = \psi_{\frac{\alpha - \beta}{1 - \bar{\alpha}\beta}}$ because $0 \xrightarrow{\psi_\beta} \beta \xrightarrow{\psi_\alpha} \frac{\alpha - \beta}{1 - \bar{\alpha}\beta}$

$\beta \xrightarrow{\psi_\beta} 0 \xrightarrow{\psi_\alpha} \alpha$

Thm: If $f \in Aut(\mathbb{D})$ then $\exists\, \theta \in \mathbb{R}$, $\alpha \in \mathbb{D}$ s.t. $f(z) = e^{i\theta} \dfrac{\alpha - z}{1 - \bar{\alpha} z}$.

So, $f = Rot_\theta \circ \psi_\alpha$.

Proof: Let $f(\alpha) = 0$. Study $g = f \circ \psi_\alpha$. Know $g(0) = f(\psi_\alpha(0))$
$= f(\alpha)$
$= 0$.

So, $g \in Aut(\mathbb{D})$ and $g(0) = 0$.

Note: g^{-1} also $\in Aut(\mathbb{D})$ and fixes 0.

By Schwarz Lemma 1, $|g(z)| \le |z|$ and $|g^{-1}(z)| \le |z|$

$$|w| \le |g(w)|.$$

So, $|g(w)| = |w|$ for all $w \in \mathbb{D}$!

So by Schwarz lemma 2, g is a rotation! $g(z) = f(\psi_\alpha(z))$.

$$= e^{i\theta} z.$$

Now look @ $g(\psi_\alpha(z)) = e^{i\theta}\psi_\alpha(z)$

$$= e^{i\theta}\frac{\alpha - z}{1-\bar{\alpha}z} = f(\underbrace{\psi_\alpha(\psi_\alpha(z))}_{=z}) = f(z).$$

Thus $f(z) = e^{i\theta}\dfrac{\alpha - z}{1 - \bar{\alpha} z}$. ∎

<u>Thm</u>: Let $f: \mathcal{U} \to \mathbb{D}$ be a conformal equiv map.
Then $Aut(\mathcal{U})$ is readily determined.

$$Aut(\mathcal{U}) = \{ f^{-1} \circ g \circ f : g \in Aut(\mathbb{D}) \}.$$

<u>Proof of Schwarz Lemma</u>:

(1) f holo $\mapsto$ f analytic; $f(0) = 0$.

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

$$\frac{f(z)}{z} = a_1 + a_2 z + \cdots$$ is holomorphic ($z = 0$ is a removable singularity for $\frac{f(z)}{z}$.)

Assume $|z| = r$

Then $\dfrac{|f(z)|}{|z|} < \dfrac{1}{r}$ because $|f(z)| \le 1$ and $|z| = r$.

So, by max modulus, $\left|\dfrac{f(z)}{z}\right| \le \dfrac{1}{r}$ for <u>all</u> $|z| \le r$.

Take limit as $r \to 1$: $\left|\dfrac{f(z)}{z}\right| \le 1$ for all $|z| \le 1$.

So, $|f(z)| \le |z| \;\; \forall z \in \mathbb{D}$.

(2) . Assume $\exists z_0 \in \mathbb{D}$ s.t. $|f(z_0)| = |z|$, ie. $\left|\frac{f(z_0)}{z_0}\right| = 1$.

$\therefore$ By Max-Modulus, since $z_0 \in \mathbb{D}$ (interior), $\frac{f(z)}{z}$ is constant

$$\therefore \quad f(z) = c \cdot z$$

must have $|c| = 1$, so $f(z) = e^{i\theta} z$.

(3) Let $g(z) = \frac{f(z)}{z} = a_1 + a_2 z + \cdots$

$g(0) = a_1 = \lim_{z\to 0} \frac{f(z)}{z}$. Know $|g(z)| \le 1$ by max modulus (from part (1).)

$$= \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = f'(0).$$

→ Suppose $|f'(0)| = 1 = |a_1| = |g(0)|$.

$\therefore$ g is constant by max modulus again, so $g(z) = c$, $|c| = 1$.

Thus $f(z) = e^{i\theta} z$. ▨ Schwarz lemma

4/Nov

Analysis review:

Def'n: A family F of fns is normal if every subseq has a convergent subsequence.

Def'n: A family F of fns is uniformly bounded on compact sets if $\forall$ compact $K \subset \Omega$ $\exists B_K$ such that $\forall f \in F$ $\forall z \in K$, $|f(z)| \le B_K$.

Def'n: A family F of fns equicontinuous on compact sets if $\forall$ compact $K \subset \Omega$ $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall f \in F$, $\forall z, w \in K$ with $|z-w| < \delta$, $|f(z) - f(w)| < \varepsilon$.

ie. "uniform" uniform continuity

Def'n: Exhaustion: Let Ω be an open set. An exhaustion is a sequence of compact subsets $\{K_\ell\}_{\ell=1}^{\infty}$ such that

(1) $K_\ell \subseteq \text{Interior}(K_{\ell+1})$

(2) For any compact $K \subset \Omega$, $\exists \ell$ such that $K \subseteq K_\ell$.

(3) Each $K_\ell \subset \Omega$ and $\bigcup_{\ell=1}^{\infty} K_\ell = \Omega$.

Thm: Let Ω be open. Then $\exists$ an exhaustion.

Pf: Case 1: Ω bounded.

Let $K_\ell = \left\{ z \in \Omega : d(z, \partial\Omega) \geq \frac{1}{\ell} \right\}$.

$\to$ As ℓ increases, K_ℓ cannot decrease.

K_ℓ is closed and bounded, therefore compact.

Case 2: Ω unbounded.

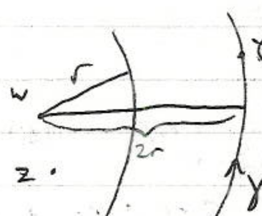
$\to$ set $\tilde{K}_\ell = K_\ell \cap B_\ell(0)$. ∎

Montel's Theorem: Let $\mathcal{F}$ be a family of holo functions on Ω open that is uniformly bounded on compact sets.

Then: (1) $\mathcal{F}$ is equicontinuous on any compact subset of Ω.

(2) $\mathcal{F}$ is normal (Arzela-Ascoli Thm).

Pf of (1):



- Compact $K \subset \Omega$

$\exists r$ such that $\text{dist}(K, \delta\Omega) > 3r$.

Must show $\forall \varepsilon\ \exists \delta_K$ such that if $|z-w| < \delta_K$

$$|f(z) - f(w)| < \varepsilon \quad \forall z \forall w \forall f \in \mathcal{F}.$$

⊛ assume $\delta_K < r$.

By Cauchy's Integral formula, $f(z) - f(w) = \frac{1}{2\pi i} \oint \left[\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right] d\zeta$

So $$|f(z) - f(w)| \leq \frac{1}{2\pi} \oint_{|\zeta - w| = 2r} |f(\zeta)| \cdot \frac{|z-w|}{|\zeta - z| \cdot |\zeta - w|}\, d\zeta$$

B_{K_0}: bound for $K_0 = K_{\text{fat}} = \{ z \in \Omega \mid d(z, K) \leq 2r \}$

$$\leq \frac{1}{2\pi} \underset{|f(\zeta)|}{B_{K_0}} \underset{|z-w|}{\delta_K} \cdot \underset{\text{length}}{2\pi \cdot 2r} \cdot \underset{\zeta - w}{\frac{1}{2r}} \cdot \underset{\zeta - z}{\frac{1}{r}} = \frac{\delta_K B_{K_0}}{r}.$$ So let $\delta_K < \underbrace{\frac{\varepsilon}{2\left(\frac{B_{K_0}}{r} + 1\right)}}_{\text{depends only on } K}$. ∎

11/ Nov

last technical lemma:

Let Ω proper open simply connected subset of $\mathbb{C}$.
Let $\mathcal{F}$ be a family of injective holo fns on Ω,
say $\mathcal{F} = \{f_n\}$. Assume $\lim_{n\to\infty} f_n(z)$ exists and is holo.
Then the limit is injective or constant.

Pf: uses Rouché's.

Thm (Riemann Mapping Thm):

Let Ω proper, open, simply connected subset of $\mathbb{C}$. Then $\exists!$ conformal equivalence $f: \Omega \to \mathbb{D}$ such that $f(z_0) = 0$, $\underbrace{f'(z_0)}_{\in\mathbb{R}} > 0$.

Sketch of pf: 1. Conformally map Ω into $\mathbb{D}$.
Key ingredient is the logarithm.
$\hookrightarrow$ requires simply connected & proper

e.g. $\alpha \notin \Omega$, look at $\log(z-\alpha)$.

2. $\Omega \subseteq \mathbb{D}$, look at maps $\Omega \to \mathbb{D}$ with $f(0) = 0$.
Find one with maximal $|f'(0)|$. Montel.

3. Observe that the above f is onto $\mathbb{D}$ (if not, contradicts maximality). Schwarz lemma.

Proof: As simply connected, log exists (Thm 6.1, Chap 3, page 98.)
Define $f(z) = \log(z-\alpha)$ for some $\alpha \notin \Omega$.
so that
$$e^{f(z)} = z - \alpha.$$
Then f is injective: if $f(z_1) = f(z_2)$, then $e^{f(z_1)} = e^{f(z_2)}$
$$z_1 - \alpha = z_2 - \alpha, \quad z_1 = z_2.$$
Take $w \in \Omega$, Claim: $\forall z \in \Omega$, $f(z) \neq f(w) + 2\pi i$.
Pf: if so, $e^{f(z)} = e^{f(w)+2\pi i} = e^{f(w)}$
$$z - \alpha = w - \alpha. \quad \blacksquare$$

→ By open mapping theorem, $\exists$ ball $B_r(f(z_0))$ that is in $f(\Omega)$.
→ then $B_r(f(z_0)) + 2\pi i$ is disjoint from $f(\Omega)$. $\blacksquare$

→ then invert about the disk: $g(z) = \dfrac{r}{z - z_0}$. Then $g \circ f: \Omega \to \mathbb{D}$ is conformal!

Step 2: $\Omega \subset \mathbb{D}$. Can assume $0 \in \Omega$ (if not, use automorphism of $\mathbb{D}$ that sends $\beta \rightleftarrows 0$.)

Look @ all holomorphic, 1-1 maps $f: \Omega \to \mathbb{D}$ s.t. $f(0) = 0$, $0 \in \Omega$.

Consider for each f, $f'(0)$.

Claim 1: The values $|f'(0)|$ are (uniformly) bounded in our family of maps.

(example: if $\Omega \subseteq B_{1/2}(0)$, $f(z) = 2z$, then $f'(0) = 2$.)

Pf of claim: Take open ball about zero, $B_R(0) \subseteq \Omega$. ($R = R_\Omega$.)

$$f'(0) = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{f(\zeta)}{(\zeta-0)^2}\, d\zeta. \quad |f(\zeta)| \leq 1 \text{ so } |f'(0)| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{1}{R^2} = \frac{1}{R}$$

So, since $|f'(0)|$ unif. bounded, $\exists$ a sequence f_n converging to supremum. (Montel's Thm.)

$$S = \sup |f'(0)|; \quad \lim_{n\to\infty} |f_n'(0)| \to S.$$

Let $f = \lim_{n\to\infty} f_n$.

Claim 2: This limit f is in our family.

(1) $f_n(0) = 0$, so $f(0) = 0$.

(2) by Technical Lemma, limit is injective or constant.

(3) The identity map is in our family, $S \geq 1$ (as $Id'(z) \equiv 1$.)

So f is not constant! $S > 0$!

Idea of step 3: Assume limit f is not onto. Say $\alpha \notin f(\Omega)$.

Apply Ψ_α, interchange $\alpha \rightleftarrows 0$.

→ Form a nice mess of functions, and see that you have a larger derivative. ∎

Branch cuts

Consider $\displaystyle\int_0^\infty \frac{dx}{x^3+1} \dots$

$\to$ look @ $f(z) = \dfrac{1}{z^3+1}$ on contour

note: γ_3 technically should avoid the origin

$$\int_{\gamma_1} f(z)\,dz = \int_0^R \frac{1}{x^3+1}\,dx$$

$z = e^{2\pi i/3} t$
$t : R \to 0$

$$\int_{\gamma_2} f(z)\,dz = \int_R^0 \frac{1}{t^3+1} e^{2\pi i/3}\,dt = -e^{2\pi i/3}\int_0^R \frac{1}{t^3+1}\,dt.$$

$$\int_{\gamma_3} f(z)\,dz \leq \frac{2\pi}{3}\,\frac{R}{R^3-1} \to 0 \text{ as } R\to\infty.$$

Residue @ $e^{\pi i/3}$: $\dfrac{\cancel{(z-e^{\pi i/3})}}{(z+1)\cancel{(z-e^{\pi i/3})}(z-e^{-i\pi/3})} = \dfrac{1}{\underbrace{(1+e^{\pi i/3})}_{\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)}\underbrace{(e^{\pi i/3}-e^{-\pi i/3})}_{=2i\sin\pi/3=\sqrt{3}i}}$

$\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)^2 = \frac{9}{4} - 2i\frac{3\sqrt{3}}{4} - \frac{3}{4}$
$= \frac{3}{2} - \frac{3\sqrt{3}}{2}i.$

$$\frac{\frac{3}{2}-\frac{\sqrt{3}}{2}i}{\sqrt{3}i\underbrace{\left(\frac{9}{4}+\frac{3}{4}\right)}_{3}} = \frac{1}{2\sqrt{3}i} - \frac{1}{6}.$$
$$= -\frac{1}{6} - i\frac{1}{2\sqrt{3}}.$$

So, $\underbrace{\left(1-e^{2\pi i/3}\right)}_{\left[1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]}\displaystyle\int_0^R \frac{1}{t^3+1}\,dt = -\frac{1}{6} - i\frac{1}{2\sqrt{3}}$

$\cdot\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)$

$\frac{3}{2}+\frac{\sqrt{3}}{2}i$

$$3\int_0^R \frac{1}{t^3+1}\,dt = -\frac{1}{4} + \frac{\sqrt{3}}{12}i - i\frac{\sqrt{3}}{4} - \frac{1}{4}.$$