$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$

$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$

$\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s) = \xi(1-s)$

$\hookrightarrow$ Start $\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx = \Gamma(s/2)/n^s\pi^{s/2}$

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\omega(x)dx$$

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} \xrightarrow[x\to\infty]{} 0$$

$$\omega(1/x) = -\tfrac{1}{2} - \tfrac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})\,\omega(x)\,dx$$

Simple Proof: $Re(s)>0$:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx$$

Method of Stationary Phase

Laplace's Method: Pg 323

Consider $\int_a^b e^{-s\Phi(x)} \cdot \psi(x)\, dx$ where phase $\Phi(x)$ is real and Φ, ψ infinitely differentiable (for convenience). Suppose Φ has a minimum at $x_0 \in (a,b)$, so $\Phi'(x_0) = 0$, and also $\Phi''(x_0) > 0$ in $[a,b]$. For $s \to \infty$ have

$$\int_a^b e^{-s\Phi(x)} \psi(x)\, dx = e^{-s\Phi(x_0)} \left[\frac{A}{s^{1/2}} + O\left(\frac{1}{s}\right) \right]$$

with $A = \sqrt{2\pi}\, \psi(x_0) / \sqrt{\Phi''(x_0)}$

Step 1: Wlog $\Phi(x_0) = 0$ in proof (study $\Phi(x) - \Phi(x_0)$)

Step 2: Taylor: $\Phi(x) = \frac{1}{2}\Phi''(x_0)\,(x - x_0)^2\, \varphi(x)$ with $\varphi(x_0) = 1 + O(x - x_0)$ as $x \to x_0$

Step 3: Change vars: $x \mapsto y = (x - x_0)\, \varphi(x)^{1/2}$

↳ Note $dy/dx|_{x_0} = 1$ so $dx/dy = 1 + O(y)$ as $y \to 0$

↳ have $\psi(x) = \tilde{\psi}(y) = \psi(x_0) + O(y)$ as $y \to 0$

Step 4: $[a', b'] \ni x_0$ Then, with $\alpha < 0 < \beta$

$$\int_{a'}^{b'} e^{-s\Phi(x)} \psi(x)\, dx = \psi(x_0) \int_\alpha^\beta e^{-s\frac{\Phi''(x_0)}{2} y^2}\, dy + O\left(\int_\alpha^\beta e^{-s\frac{\Phi''(x_0)}{2} y^2} |y|\, dy \right)$$

Step 5: Different than book:

Use $\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, dy = \sqrt{2\pi} \cdot \sigma$

with $\sigma^2 = 1/s\Phi''(x_0)$, "small" error with $-\alpha = \beta = \infty$

Method of Stationary Phase

Application: Stirling's Formula

See power of continuation of factorial to understand it at <u>integer</u> values. Study

$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx = n!$ (so $s = n+1$)

$\Gamma(s+1) = \int_0^\infty x^s e^{-x} dx$

$= \int_0^\infty e^{s\log x} \cdot e^{-x} dx$

$= \int_0^\infty e^{-s(-\log x)} e^{-x} dx$ $\quad \left.\begin{array}{l}\Phi(x) = -\log x \\ \psi(x) = e^{-x}\end{array}\right\}$ doesn't work

<u>Better</u>

$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}$

$= \int_0^\infty e^{-x + s\log x} \frac{dx}{x}$ $\quad$ Change var: $x \mapsto xs$ so $\frac{dx}{x} \mapsto \frac{dx}{x}$

$= \int_0^\infty e^{-sx + s\log sx} \frac{dx}{x}$

$= e^{s\log s} e^{-s} \int_0^\infty e^{-s\Phi(x)} \frac{dx}{x}$ $\quad$ with $\Phi(x) = x - 1 - \log x$

Why this change?

$\Phi(1) = \Phi'(1) = 0$

$\Phi''(x) = 1/x^2 > 0$ so $\Phi''(x_0) = 1$ as $x_0 = 1$

$\psi(x) = 1/x$ so $\psi(x_0) = 1$ as $x_0 = 1$

$\hookrightarrow$ In theorem $A = \sqrt{2\pi} \cdot 1/\sqrt{1} = \sqrt{2\pi}$

$\Gamma(s) \sim e^{s\log s} e^{-s} \cdot e^{-s\Phi(1)} \cdot \frac{A}{s^{1/2}}$

$\Gamma(s) \sim s^s e^{-s} \sqrt{\frac{2\pi}{s}} \, (1 + \text{small})$

$\Gamma(n+1) = n\Gamma(n) \approx n^n e^{-n} \sqrt{2\pi n} \, (1 + \text{small})$