MATH 302: COMPLEX ANALYSIS

COURSE MECHANICS : NOTE WILL NOT COVER ALL PROOFS IN CLASS

↳ Weekly HW
- ↳ mix computation and theory
- ↳ everyone must LaTeX a soln key (can be after graded)
  - ↳ can do in group

↳ Take home midterm / final / project

↳ Goals:
- ↳ Prepare for grad school
- ↳ see how to approach adv math
- ↳ gain expertise in proofs, master techniques
- ↳ learn complex analysis

↳ Project
- ↳ very rich subject, cannot cover all in a semester
- ↳ alone or in a group of 2 delve deeper into part of course / additional material
  - ↳ can give a talk, write it up, do problems...

↳ Permanent Office hours

↳ Pre-req Material
- ↳ Real Analysis: if taken abstract ok so long as study and quickly comfortable with some real analysis

Quick Introduction to Complex Analysis

Numbers:

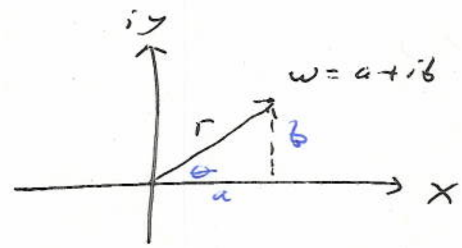
↳ $\mathbb{N}$ natural #s ~~integers~~: $\{0, 1, 2, \ldots\}$ or $\{1, 2, \ldots\}$

$\mathbb{Z}$ integers: $\{\ldots, -1, 0, 1, \ldots\}$ (zahl = number in German)

$\mathbb{Q}$ rationals: $\{p/q : p, q \in \mathbb{Z},\ q \neq 0\}$ (quotient)

$\mathbb{R}$ reals

$\mathbb{C}$ complex: $\{x + iy : x, y \in \mathbb{R}\}$, $i^2 = -1$



General Properties

$z = x + iy$, $w = a + ib$, $i^2 = -1$

↳ can write $z = re^{i\theta}$ with $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$

$\bar{z} = x - iy$ (complex conjugate)

$z + w = (x + a) + i(y + b)$, $zw = (xa - yb) + i(xb + ya)$

$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$

↳ Extra credit: show cannot order the complex numbers (ie, no generalization of $<$ from $\mathbb{R}$)

↳ $|z + w| \leq |z| + |w|$ (Triangle inequality)

↳ Commutativity: $z + w = w + z$

Associativity: $(z + w) + q = z + (w + q)$

Distributive: $z(w + q) = zw + zq$

↳ See quaternions and octonions for generalization

What's the Big Deal: Real vs Complex Analysis

- Real Analysis

↳ 1 variable: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

or $f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} \longleftrightarrow \lim\limits_{x \to x_0} \dfrac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} = 0$

↳ 2 variable: f is differentiable at $\vec{x}_0$ if

$$\lim_{\vec{x} \to \vec{x}_0} \frac{|f(\vec{x}) - f(\vec{x}_0) - (\nabla f)(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)|}{\| \vec{x} - \vec{x}_0 \|} = 0$$

↳ Note it is possible for $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ to exist without having f differentiable; if, however, partials $\partial f/\partial x$ and $\partial f/\partial y$ are continuous then f is differentiable

↳ Extra credit: find a function f st partials exist but f is not differentiable

↳ Note limit must exist along any path, not just along coordinate axes! or

↳ If you know Green's Thm/ Stokes' Thm: review!

- Complex Differentiable: $f: \mathbb{C} \to \mathbb{C}$ is complex diff if $\lim\limits_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$ exists for all paths.

REAL VS COMPLEX (CONTINUED)

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function and let also $g: \mathbb{C} \to \mathbb{C}$ be a complex differentiable function, and let $h: \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function.

PROPERTY	(Real diff) TRUE FOR f	(Complex diff) TRUE FOR g
function is infinitely differentiable	NO	YES
function equals its Taylor Series in some neighborhood of point where differentiable	NO	YES
the function may be bounded without being identically constant	YES	NO
The line integral of our function along a closed curve must be zero	NO*	YES

New behavior! Shame cannot visualize $g: \mathbb{C} \to \mathbb{C}$ as "4-dimensional"
↳ 2nd point important in probability. 4th point is Green/Stokes' Thm

* For h not f

REVIEW: CONVERGENCE

Seq $\{z_n\}_{n=0}^{\infty}$ converges to a limit L if $\lim_{n\to\infty} |z_n - L| = 0$

Should be able to prove the following:

(1) L is unique

(2) $\{z_n\}_{n=0}^{\infty}$ converges to L if and only if (iff) $\{Re(z_n)\}_{n=0}^{\infty}$ converges to $Re(L)$ and $\{Im(z_n)\}_{n=0}^{\infty}$ converges to $Im(L)$.

Hint: The triangle inequality is useful: $|\alpha \pm \beta| \leq |\alpha| + |\beta|$

NOTE: THIS IS PRE-REQ MATERIAL FOR THE CLASS. YOU SHOULD BE ABLE TO DO THESE PROBLEMS AND WRITE UP "WELL" AND "RIGOROUSLY". I'M HAPPY TO LOOK AT YOUR WRITE-UP *ANYTIME*

Cauchy sequence: $\{z_n\}$ is a Cauchy sequence (or said to be Cauchy) if $\forall \varepsilon > 0 \ \exists N$ st $\forall n, m > N$ have $|z_n - z_m| < \varepsilon$.

KEY FACT (BOLZANO - WEIERSTRASS) EVERY CAUCHY SEQ CONVERGES IN $\mathbb{R}$ / IN $\mathbb{C}$.

Exercise: Determine which sequences are Cauchy?

(1) $z_n = (-1)^n$ (2) $z_n = (-1)^n/n$ (3) $z_n = 2^n$

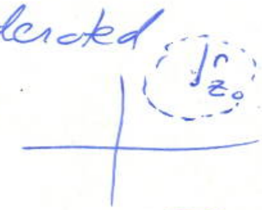
(4) $z_n = \cos(n)$ (in radians)

YOU SHOULD BE ABLE TO DO THESE PROBLEMS

REVIEW: POINT SET TOPOLOGY

This is pages 5 - 7 of our book

- Ball (or disk or disc) of radius r about z_0, denoted $B_r(z_0)$ or $D_r(z_0)$, is $\{z : |z - z_0| < r\}$



- Closed disk, $\overline{D}_r(z_0)$, is $\{z : |z - z_0| \le r\}$

- Unit disk centered at origin: denote $\mathbb{D}$, is $\{z : |z| < 1\}$

- Interior point: z is an interior point of Ω if $D_r(z_0) \subset \Omega$

- Open Set: Ω open if $\forall z_0 \in \Omega$ $\exists r$ (depending possibly on z_0) such that $D_r(z_0) \subset \Omega$.

- Closed Set: Ω is closed if complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open

- Limit point: z is a limit point of Ω if $\exists$ seq $\{z_n\}_{n=0}^{\infty}$ such that each $z_n \in \Omega$ and $z_n \to z$

  ↳ Set is closed iff it contains all its limit points

- Closure: Closure of Ω, denoted $\overline{\Omega}$, is $\Omega \cup$ limit points of Ω.
- Boundary: Boundary of Ω, denoted $\partial\Omega$, is $\overline{\Omega} - \Omega$, ie, all points not in the interior.

- Bounded: Ω is bounded if $\exists M$ st $\forall z \in \Omega$, $|z| < M$

  ↳ if bounded, diameter of Ω is $\sup\limits_{w,z \in \Omega} |z - w|$

- Compact: Ω compact if Ω is closed and bounded

REVIEW: POINT SET TOPOLOGY (CONT)

Open Cover: An open cover of Ω is a family of open sets $\{U_\alpha\}$ (not necc countable family) such that $\Omega \subset \bigcup_\alpha U_\alpha$

↳ Set is countable if 1-1 and onto map b/w it and $\{1, \ldots, n\}$ for some n or $\mathbb{N}$.

↳ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}^3$ countable

↳ $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{C}$ uncountable

> ↳ See Cantor's diagonalization argument useful in "finding" transcendental numbers
> See the Continuum Hypotheses and my mathematical grand father, Paul Cohen

THM: $\Omega \subset \mathbb{C}$ compact iff every seq $\{z_n\} \subset \Omega$ has a subseq that converges to a point in Ω.

THM: $\Omega \subset \mathbb{C}$ compact iff every open covering of Ω has a finite sub-covering.

THM: If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a seq of non-empty compact sets in $\mathbb{C}$ st $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$ Then $\exists$ a unique ($\exists!$) point w st $w \in \Omega_n$ for all n.

↳ use in proving Goursat's Thm

REVIEW: POINT SET TOPOLOGY

Connected Set: Ω is connected if cannot write $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1, Ω_2 two disjoint open sets if Ω is open or closed sets if Ω closed

↳ Call a connected open set a region.

Practice Exercises: You should be able to do all the exercises below. I'm happy to discuss your approaches, offer hints, look at solutions....

(1) Prove $D_r(z_0)$ is open for any $z_0 \in \mathbb{C}$ and any $r > 0$.

↳ Hint: show you may assume $z_0 = 0$ and $r = 1$, so suffices to do for unit disk.

(2) Show $\left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \right\}$ is closed

(3) Let $f(x) = \begin{cases} \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

Let $\Omega = \{ (x, f(x)) : x \in \mathbb{R} \}$.

Is Ω open? Closed?

Hint: a set is closed iff it contains all limit points

(4) Is the set from problem (2) bounded? Compact?

PRACTICE PROBLEMS (CONT)

(5) Consider all circles with centers with both coordinates rational and with a rational radius; thus if $\alpha = (x_0, y_0, r)$ then $x_0, y_0, r \in \mathbb{Q}$. We have the unit disk $\mathbb{D} \subset \bigcup_{\substack{\alpha \text{ with} \\ \text{rational} \\ \text{entries}}} D_r(x_0, y_0)$ with $\alpha = (x_0, y_0, r)$;

Thus these disks give an open cover of $\mathbb{D}$. Is there a finite sub-cover?

<u>Hint</u>: If you only have finitely many open sets, how close can you get to points <u>very</u> close to the boundary of the unit disk?

(6) Prove the unit square is compact in $\mathbb{R}^2$.

(7) Let $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ be a seq of non-empty compact sets such that $\lim_{n\to\infty} \operatorname{diam}(\Omega_n) = 1$. Prove or disprove: There is either an interval of length at least $1/4$ or a disk of radius at least $1/4$ that is in Ω_n for all n.

Review: Sequences and Series

Consider infinite sum $\sum_{n=0}^{\infty} a_n x^n$

Question: for what x does it converge?
What can we say about convergence

Recall Tests for Convergence

Comparison Test

If $|b_n| \leq a_n$ and $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} b_n$ converges,
if $b_n \geq a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges

↳ Exercise: Show only need $|b_n| \leq a_n$ for $n \geq N$
or $b_n \geq a_n \geq 0$ for $n \geq N$

Ratio Test

If $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, then $\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges if } |\rho| < 1 \\ \text{diverges if } |\rho| > 1 \\ \text{no info if } |\rho| = 1 \end{cases}$

↳ Proof uses Geometric Series and Comparison Test

↳ Exercise: Show $\sum_{n=0}^{\infty} n^2/3^n$ converges

Root Test

If $\rho = \lim_{n\to\infty} |a_n|^{1/n}$ exists, then $\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges if } |\rho| < 1 \\ \text{diverges if } |\rho| > 1 \\ \text{no info if } \rho = 1 \end{cases}$

Review: Sequences and Series

Thm 2.5 (Pg 15) Set $1/0 = \infty$ and $1/\infty = 0$. Let R satisfy $1/R = \limsup |a_n|^{1/n}$. Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$ and diverges for $|z| > R$.

↳ Call R the radius of convergence, and the region $|z| < R$ the disc of convergence.

↳ See page 15 for a proof.

Exercise: Find the radius of convergence for

$$(1) \sum_{n=0}^{\infty} z^n/n! \qquad (2) \sum_{n=0}^{\infty} \frac{n^2 z^n}{n!}$$

Most important examples

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

↳ note $e^{z+w} = e^z e^w$ is a Theorem, not immediate!

↳ means $\displaystyle\sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{l=0}^{\infty} \frac{z^l}{l!} \sum_{m=0}^{\infty} \frac{w^m}{m!}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\log(1-x) = -\left(\cancel{z} + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\right) \quad \text{for } |x| < 1$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{for } |x| < 1$$

SEC 2: FUNCTIONS IN THE COMPLEX PLANE

(2.1) CONTINUOUS FUNCTIONS

This section reviews real analysis

$\Omega \subset \mathbb{C}$, $f: \Omega \to \mathbb{C}$ is <u>continuous</u> at $z_0 \in \Omega$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$

↳ f is <u>continuous on Ω</u> if continuous at all $z_0 \in \Omega$

<u>Thm 2.1 (Page 8)</u> A continuous function on a compact set Ω is bounded and attains its maximum and minimum on Ω.

↳ You should be able to generalize the real analysis proof for this. This is a very good <u>EXERCISE</u>.

(2.2) HOLOMORPHIC FNS

Key definition for the course:

> <u>COMPLEX DIFFERENTIABILITY</u>
>
> A complex valued function f on an open $\Omega \subset \mathbb{C}$ is holomorphic (ie, complex differentiable) at $z_0 \in \Omega$ if $\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$ exists, in which case we denote the limit $f'(z_0)$.

VERY STRONG CONDITION AS CAN DO <u>ANY</u> PATH

(2.1) Holomorphic Functions

Examples

$f(z) = z$: $f'(z_0) = \lim_{h\to 0} \frac{z_0 + h - z_0}{h} = \lim_{h \to 0} 1 = 1$

$$f(z) = z^2: \ f'(z_0) = \lim_{h\to 0} \frac{(z_0+h)^2 - z_0^2}{h}$$
$$= \lim_{h\to 0} \frac{z_0^2 + 2z_0 h + h^2 - z_0^2}{h}$$
$$= \lim_{h \to 0} (2z_0 + h) = 2z_0$$

More generally, any finite poly $\sum_{n=0}^{N} a_n z^n$

$$f(z) = \bar{z}: \ f'(z_0) = \lim_{h\to 0} \frac{\overline{z_0 + h} - \bar{z}_0}{h}$$
$$= \lim_{h \to 0} \bar{h}/h \quad \text{Does not exist!}$$

↳ take h in $\mathbb{R}$ and get 1

take h in $i\mathbb{R}$ and get -1

Standard Properties

- f, g holomorphic $\Rightarrow (f+g)' = f' + g'$, $(fg)' = f'g + fg'$ and if $g(z_0) \neq 0$ then $(f/g)' = \frac{f'g - fg'}{g^2}$
- $f: \Omega \to U$ holo and $g: U \to \mathbb{C}$ holo, then $g\circ f$ holo and $(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0)$

CAUCHY-RIEMANN EQS

Write $f: \Omega \to \mathbb{C}$ as $f(z) = u(x,y) + i v(x,y)$, $z = x+iy$

f holo $\Rightarrow$ $f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$ for any path

↳ take path h real: $h = (h_1, h_2)$ or $h_1 + i h_2$ so on this path $h_2 = 0$. Get

$$f'(z_0) = \lim_{\substack{h_1 \to 0 \\ h_1 \text{ real}}} \frac{f(z_0 + h_1) - f(z_0)}{h}$$

write $f(x,y)$ for $f(z)$: abuse of notation

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0+h_1, y_0) - f(x_0, y_0)}{h_1} = \frac{\partial f}{\partial x}(z_0)$$

↳ Now take path $h = i h_2$ and find

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{i h_2} = \frac{1}{i} \frac{\partial f}{\partial y}(z_0)$$

Thus $\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}$

As $f(x,y) = u(x,y) + i v(x,y)$, find

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equate real and imaginary parts

CAUCHY-RIEMANN EQS

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann Eq (cont)

Define $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Prop 2.3: f holo $\Longrightarrow$ $\partial f/\partial \bar{z} = 0$ and $f'(z_0) = \frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}$

Important Converse (See Page 13 for Proof)

Thm 2.4: $f = u + iv$ complex valued fn defined on open $\Omega \subset \mathbb{C}$. If u, v cont diff and satisfy the Cauchy-Riemann Eqs on Ω then f is holo on Ω and $f'(z) = \partial f/\partial z$.

Connection with Stokes' Thm

Say $\vec{F}(x,y) = (P(x,y), Q(x,y))$ and have

Then $$\iint_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_{\partial\Omega}\vec{F}\cdot d\vec{s} = \int_{\partial\Omega} P\,dx + Q\,dy$$

Say f is holomorphic, so $f = u + iv$

By Cauchy-Riemann Eqs: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$dz = dx + i\,dy$ $\quad f = u + iv$ so

$f\,dz = (u+iv)(dx + i\,dy) = (u\,dx - v\,dy) + i(v\,dx + u\,dy)$

Apply Stokes' (actually Green's) Thm to each

~~CONNECTION WITH STOKES' THM~~

$$\int_{\partial\Omega} f\,dz = \int_{\partial\Omega} (u+iv)(dx+i\,dy)$$

$$= \int_{\partial\Omega} u\,dx - v\,dy + i\int_{\partial\Omega} v\,dx + u\,dy$$

$$= \iint_{\Omega} \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_{\Omega}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

↳ $= 0$ by Cauchy-Riemann Eqs

Give Additional, optional lecture on proof of Stokes' Thm

Thus "expect" $\int_{\partial\Omega} f\,dz = 0$ for holomorphic f!

↳ first instance that complex differentiability is very special

(2.3) POWER SERIES

Thm 2.6: Power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holo fn in the disc of convergence. Deriv of f is also a power series and is obtained by differentiating term by term: $f'(z) = \sum_{n=0}^{\infty} n a_n z^n$ and has the same radius of conv

↳ Corollary: applying above again and again, find a power series is infinitely diff in its disc of convergence.

(2.3) Power Series (cont)

Proof of Theorem

- Clearly ~~all~~ the term by term derivative has same radius of convergence as $\lim_{n\to\infty} n^{1/n} = 1$

  ↳ <u>Exercise:</u> Prove this!

  Hint: argue equivalent to proving $\lim_{n\to\infty} \log(n^{1/n}) = 0$

- Must show $f'(z)$ is $\sum_{n=0}^{\infty} n a_n z^{n-1} := g(z)$

  Key input: deriv of finite sum is sum of derivs

  Let R be radius of conv, assume $|z_0| < r < R$.

  Write $f(z) = \sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n := S_N(z) + E_N(z)$

  Let h be so small that $|z_0 + h| < r$. Then

$$\frac{f(z_0+h) - f(z_0)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1}$$

$$= \frac{S_N(z_0+h) - S_N(z_0)}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1} + \frac{E_N(z_0+h) - E_N(z_0)}{h}$$

$$= \left[\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right] + \left[S_N'(z_0) - \sum_{n=0}^{\infty} n a_n z^{n-1}\right]$$

$$+ \left[\frac{E_N(z_0+h) - E_N(z_0)}{h}\right]$$

↳ Added zero: important technique

↳ Three epsilon proof

POWER SERIES (CONT)

Use $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

Gives $|(z_0+h)^n - z_0^n| \leq h\, n \max(|z_0+h|, |h|)^{n-1} \leq h\, n\, r^{n-1}$

Thus $\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}$

$\hookrightarrow$ tail of conv series, tends to 0

Choose N_1 st $\forall N \geq N_1$ less than $\varepsilon/3$

As $\lim_{N\to\infty} S_N'(z_0) = \sum_{n=0}^{\infty} n a_n z^{n-1}$, $\exists N_2$ st $\forall N \geq N_2$ less than $\varepsilon/3$,

ie, $\left| S_N'(z_0) - \sum_{n=0}^{\infty} n a_n z^{n-1} \right| < \varepsilon/3$.

Choose $N \geq \max(N_1, N_2)$

Last term easiest: by continuity/prop of deriv of <u>finite</u> poly,

have $\left|\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right| < \varepsilon$.

DEF: ANALYTIC

f on open $\Omega \subset \mathbb{C}$ is analytic at $z_0 \in \Omega$ if there is an open neighborhood of z_0 st $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for all z in the nbhood. If analytic at each $z_0 \in \Omega$ say f is analytic on Ω.

SEC 3: INTEGRATION ALONG A CURVE

Parametrized curve: maps $[a,b] \subset \mathbb{R}$ to $\mathbb{C}$, often write as $z(t)$

Smooth if $z'(t)$ exists and is continuous, and $z'(t) \neq 0$

Piecewise smooth if smooth on subintervals (finitely many) making up $[a,b]$

↳ Note: at endpoints derivs are one-sided

Equivalent parametrizations: $z, \tilde{z}$ equiv if $\exists$ cont diff bijection st:

(1) $z : [a,b] \to \mathbb{C}$ $\quad \tilde{z} : [c,d] \to \mathbb{C}$

(2) $t : [c,d] \to [a,b]$ cont diff bijection with

(i) $t'(s) > 0$ (orientation preserving)

(ii) $\tilde{z}(s) = z(t(s))$

↳ If $t'(s) < 0$ then orientation reversing

↳ family of all equiv param determines a curve γ, let γ^- be obtained by reversing orientation.

Example: Circle radius r centered at z_0

- $z(t) = z_0 + re^{it}$ $\quad t \in [0, 2\pi]$ "positive orientation" or "counterclockwise"

- $z(t) = z_0 + re^{-it}$ $\quad t \in [0, 2\pi]$ "negative orientation" or "clockwise"

Sec 3: Integration along a curve

<u>Integral of f along γ</u> is $\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

where z is any parametrization of γ.

↳ must show well-defined

↳ Change of variable formula shows two equiv param give same answer.

Length of curve: $\int_a^b |z'(t)|\,dt = \text{length}(\gamma)$

Standard Properties

- $\int_\gamma (\alpha f + \beta g)\,dz = \alpha \int_\gamma f\,dz + \beta \int_\gamma g\,dz$
- $\int_\gamma f\,dz = -\int_{\gamma^-} f\,dz$
- $\left|\int_\gamma f\,dz\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$

Say F is a <u>primitive</u> for f on Ω if $F'(z) = f(z)$. If γ is a curve in Ω then $\int_\gamma f(z)\,dz = F(z(b)) - F(z(a))$

↳ Proof: Chain rule, Fund Thm of Calc

Note that if $\cancel{F}$ f has a primitive F, then if γ is a closed curve (so $z(b) = z(a)$) then $\int_\gamma f(z)\,dz = 0$.

Sec 3: Integration Along a Curve

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series that conv in a disc of radius R. Claim $F(z) = \sum_{n=0}^{\infty} \frac{a_n z^{n+1}}{n+1}$ is a primitive, and has same radius of convergence. Implies that if γ is a closed curve in Ω then $\int_\gamma f\,dz = 0$.

Later will see that if f is differentiable then f is infinitely differentiable and given by a power series, ie, f is analytic. Leads to the important:

$\boxed{\text{THM: } f \text{ is holomorphic iff } f \text{ is analytic}}$

Homework Problems

#1 abcd	#17
#3	#24
#13	#25a, #25b
#16abc	

Important: Extra Credit: #4

Additional: #5, #7, #16d, #18, #23, #26