$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Functional Eq of $\zeta(s)$

$$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$$

$$\xi(s) = \tfrac{1}{2}s(s-1)\Gamma\left(\tfrac{s}{2}\right)\pi^{-s/2}\zeta(s) = \xi(1-s)$$

$\hookrightarrow$ Start $\int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x}\,dx = \Gamma(s/2)/n^s\pi^{s/2}$

$$\pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\omega(x)\,dx$$

$$\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x} \xrightarrow[x\to\infty]{} 0$$

$$\omega(1/x) = -\tfrac{1}{2} - \tfrac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\omega(x)\,dx$$

Simple Proof: $Re(s) > 0$:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx$$

In the interest of time didn't cover proof of analytic continuation to <u>ALL</u> $s \in \mathbb{C}$, only did $Re(s) > 0$. See Chapter 3 of my book "An Invitation to Modern Number Theory"

Number Theory: Sketch of Proof of Prime Number Theorem

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1-p^{-s})^{-1} \quad Re(s) > 1$$

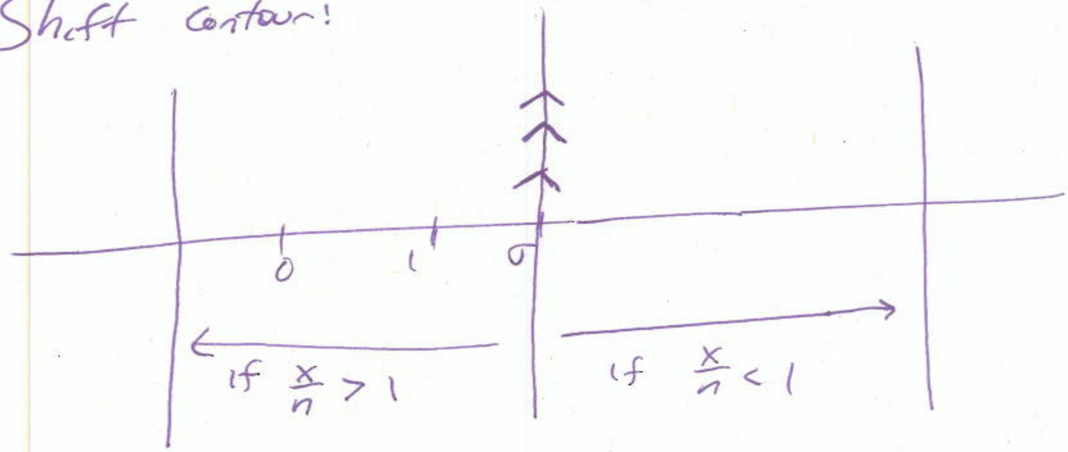
↳ unique factorization + geometric series

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds} \log \zeta(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \Lambda(n) = \begin{cases} \log p & n = p^k \text{ } p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

<u>Claim</u>: $Re(s) > 1$: $\displaystyle\int_{Re(s)=\sigma} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \begin{cases} 1 & n < x \\ 0 & n > x \end{cases} \quad \sigma > 1$

$s = \sigma + it$

<u>Proof</u>: Shift contour!



Say $\frac{x}{n} > 1$ then as $Re(s) \to -\infty$ have $\left|\left(\frac{x}{n}\right)^s\right| = e^{Re(s) \log \frac{x}{n}}$

As $Re(s) \to -\infty$ and $\log \frac{x}{n} > 0$, integral over this vertical piece tends to zero. Residue at $s=0$ is just $\left(\frac{x}{n}\right)^0 = 1$.

Similar proof for $\frac{x}{n} < 1$, shift to right and no poles.

Sketch of Proof of PNT

Left with

$$\int_{Re(s)=c>0} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{x^s}{s}\, ds$$

$$= \sum_{n=1}^{\infty} \Lambda(n) \int_{Re(s)=c>0} \left(\frac{x}{n}\right)^s \frac{ds}{s}$$

$$= \sum_{n \le x} \Lambda(n)$$

$$= \sum_{p \le x} \log p + \sum_{p^2 \le x} \log p + \cdots + \sum_{p^{\log_2 x} \le x} \log p$$

$$= \sum_{p \le x} \log p + \underbrace{\text{Error}\left(x^{\frac{1}{2}} \log x + x^{\frac{1}{3}} \log x + \cdots + x^{\frac{1}{\log_2 x}} x \right)}_{\text{at most } x^{\frac{1}{2}} \log x + x^{\frac{1}{3}} \log^2 x \ll x^{\frac{1}{2}} \log x}$$

Study with partial summation

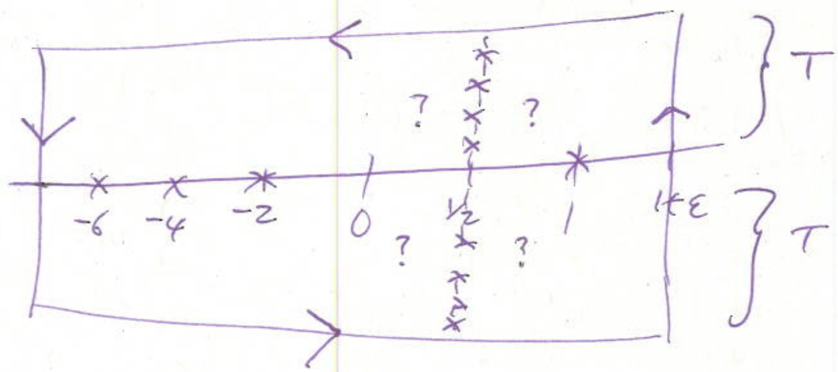
Clearly $\sum_{p \le x} \log p \le \pi(x) \log x$ with $\pi(x) = \#\{p : p \le x \text{ prime}\}$

Also $\sum_{p \le x} \log p \geq \sum_{x^{1-\varepsilon} \le p \le x} \log p \geq \left[(1-\varepsilon) \log x\right]\left[\pi(x) - \pi(x^{1-\varepsilon})\right]$

$$\geq \left[(1-\varepsilon) \log x\right]\left[\pi(x) - x^{1-\varepsilon}\right]$$

main term: $\pi(x) \cdot (1-\varepsilon) \log x$

Sketch of Proof of PNT



- Show integral of ζ'/ζ over the pieces "small"
- Use Residue Thm — need analytic continuation!
- Note $\zeta(s) \neq 0$ if $\mathrm{Re}(s) > 1$ (easy) or ≥ 1 (harder)

$$\text{Get } \int_{\mathrm{Re}(s)=1+\varepsilon} -\frac{\zeta'(s)}{\zeta(s)} x^s \frac{ds}{s} = \frac{x^1}{1} - \sum_{\substack{\rho \\ \zeta(\rho)=0 \\ |\mathrm{Im}(\rho)| \leq T}} \frac{x^\rho}{\rho}$$

ds/s

$$\text{Gives } \sum_{n \leq x} \Lambda(n) \cong x - \sum_\rho \frac{x^\rho}{\rho}$$

$$\pi(x) \log x \approx x$$

$$\Rightarrow \pi(x) \approx \frac{x}{\log x}$$