SEVERAL COMPLEX VARIABLES

• HOLOMORPHIC FNS IN $\mathbb{C}^N$

Need to figure out generalizations

Unit disk $B_r(a) = \{z \in \mathbb{C} : |z-a| < r\}$

becomes the polydisc centered at $a = (a_1, \dots, a_N)$

with polyradius $r = (r_1, \dots, r_N)$ $(r_i > 0)$

given by $\Delta(a,r) = \{(z_1, \dots, z_N) \in \mathbb{C}^N : |z_j - a_j| < r_j\}$

↳ closed polydisc $\bar{\Delta}(a,r)$

↳ Cartesian product of $\Delta(a_i, r_i) = B_{r_i}(a_i)$

POSSIBLE DEFNS OF HOLOMORPHIC

- holomorphic in each variable separately
- continuous as a fn of several variables and holomorphic in each variable separately
- multivariable power series converging in a nbhood of each point.

$\boxed{\text{THM: THE THREE DEFNS ABOVE ARE EQUIVALENT}}$

THM: CAUCHY'S INTEGRAL FORMULA

f defined in nbhood U of $\bar{\Delta}(a,r)$ and holomorphic in each variable in U. Then

$$f(z) = f(z_1,\dots,z_n) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n-a_n|=r_n} \cdots \int_{|\zeta_1-a_1|=r_1} \frac{f(\zeta_1,\dots,\zeta_n)\,d\zeta_1\cdots d\zeta_n}{(\zeta_1-z_1)\cdots(\zeta_n-z_n)}$$

Sketch of proof

Keep applying 1-dim Cauchy formula!

OSGOOD'S LEMMA

f as above and bounded on any compact set in U. Then for each polydisc $\bar{\Delta}(a,s)\subset U$ the power series

$$\sum_{i_1=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} c_{i_1 i_2\cdots i_n}(z_1-a_1)^{i_1}\cdots(z_n-a_n)^{i_n}$$

converges uniformly to f on $\bar{\Delta}(a,s)$ with

$$c_{i_1 i_2\cdots i_n} = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n-a_n|=r_n} \cdots \int_{|\zeta_1-a_1|=r_1} \frac{f(\zeta_1,\dots,\zeta_n)\,d\zeta_1\cdots d\zeta_n}{(\zeta_1-a_1)^{i_1+1}\cdots(\zeta_n-a_n)^{i_n+1}}$$

Sketch of proof

Same as in 1-dim: power series expansions

$$\frac{1}{\zeta_j - z_j} = \frac{1}{\zeta_j - a_j - (z_j-a_j)} = \frac{1}{\zeta_j-a_j}\,\frac{1}{1-\frac{z_j-a_j}{\zeta_j-a_j}}$$

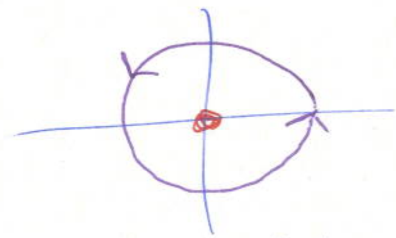
This is why have +1

So far, a lot of the subject of several complex variables looks like one complex variable. What changes? Geometry!

Real line

no way around

Plane / $\mathbb{C}$



can go around, but not contractible to a point

In higher dimensions, can "lift" path up and contract curve to a point. Might recall some of these differences in Multivariable Calculus in going from $\mathbb{R}^2$ to $\mathbb{R}^3$

DEFN: DOMAIN OF HOLOMORPHY

Open set $U \subset \mathbb{C}^n$ is a domain of holomorphy if $\exists$ holomorphic fn on U st $\forall w \in \partial U$ (boundary of U) and each polyradius r, there is no holomorphic function on $\Delta(w,r)$ which equals f on a component of $\Delta(w,r) \cap U$: IE, cannot get a holomorphic extension across part of boundary of U.

Domains of Holomorphy in $\mathbb{C}$

Consider the punctured unit disk $\mathbb{D}^*$.

Is this a domain of holomorphy?

Yes - simply look at $f(z) = 1/z$ on $\mathbb{D}^*$

↳ Clearly cannot extend holomorphically to $\mathbb{D}$.

Exercise: Thm: If U is an open set in $\mathbb{C}$, then there is a fn f that is holo on U and cannot be extended to be holo on any larger open set.

↳ Special case $U = \mathbb{D}^*$ done above

↳ What about annulus $\{z : a < |z| < b\}$ for $0 < a < b < \infty$?

DOMAINS OF HOLOMORPHY IN $\mathbb{C}^n$

THM (NOT MOST GENERAL)

Let $\Delta(0,r)$ be an open polydisc and K any compact set in $\Delta(0,r)$ that does not separate $\Delta(0,r)$ (Think $K = \{0\}$). Then any function holomorphic on $\Delta(0,r) - K$ extends to be holo on all of $\Delta(0,r)$!

Sketch of Sketch of Proof

Choose a good contour "missing" action – can do as have higher dimensions to work with.

Question: What about $f(z_1, z_2) = 1/z_1$?

Answer! This function is not holo in $\Delta(0,r) - \{0\}$

↳ note there is a "plane" of poles, namely

$z_1 = 0$, z_2 arbitrary.