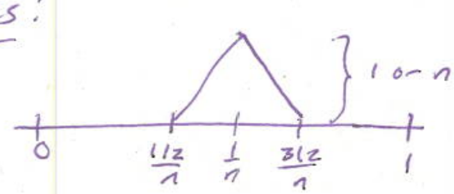
Normal: Family of holo fns st each seq has a subseq that conv uniformly on compact sets (limit need not be in family)

Unif Bounded on Compact Sets: $\mathcal{F}$ unif bounded on compact subsets of $\Omega$ if $\forall$ compact $K \subset \Omega$ $\exists B_K$ st $|f(z)| \le B$ $\forall f \in \mathcal{F}$ $\forall z \in K$

Equicont on Compact Set: Compact $K$: $\forall \varepsilon > 0$ $\exists \delta > 0$ st $\forall z, w \in K$ with $|z-w| < \delta$ then $\forall f \in \mathcal{F}$: $|f(z) - f(w)| < \varepsilon$

Study examples!

① $f_n(x)$




↳ Does this converge pointwise? Yes. Uniformly? No
If height is $n$, $\lim \int f_n \neq \int \lim f_n$

② $f_n(x) = \cos(2\pi\sqrt{n+x})$

↳ This has a subseq that conv unif on compact sets
but does every subseq have a unif conv subseq? Extra Credit

↳ is clearly equicont
↳ deriv bounded, MVT

Review Uniform Continuity: $\forall \varepsilon > 0$ $\exists \delta$ st $|x-y| < \delta \to |f(x) - f(y)| < \varepsilon$

Proof: Assume not: $x_n, y_n$ st $|x_n - y_n| < 1/n$, $|f(x_n) - f(y_n)| \ge \varepsilon$
$x_n \to x$, $y_n \to y$, by continuity $f(x_n) - f(y_n) \to f(x) - f(y) \geq \varepsilon$
as $|x_n - y_n| \to 0$ have $x = y$, contradicts $f(x) - f(y) \geq \varepsilon$

Exn $f(x) = x^2 - 4$ on $[0,3]$ vs $\frac{1}{x}$ on $(0,3)$

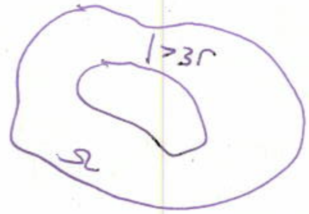
SUPPLEMENT 1

MONTEL'S THM: Assume $\mathcal{F}$ family of holo fns on $\Omega$ and uniformly bounded on compact subsets of $\Omega$. Then:

(1) $\mathcal{F}$ is equicont on every compact subset of $\Omega$.

(2) $\mathcal{F}$ is normal (Arzela-Ascoli: unif bounded + equicont implies normal)

Proof: (1)



$$\gamma = \partial B_{2r}(w)$$

$$f(z) - f(w) = \frac{1}{2\pi i} \oint_\gamma f(\zeta) \underbrace{\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right]}_{\frac{z-w}{(\zeta-z)(\zeta-w)}} dw$$

$$|f(z) - f(w)| \le \frac{1}{2\pi} \cdot \frac{2\pi r}{r^2} B\,|z-w|$$

Proof: (2) $\{f_n\} \subset \mathcal{F}$, Compact $K$, subseq $\{w_j\}$ dense in $\Omega$ (points $\mathbb{Q}$ coords)

Subseq $\{f_{n,1}\}$ st $f_{n,1}(w_1)$ converges

subseq $\{f_{n,2}\}$ st $f_{n,2}(w_2)$ converges

$\vdots$

$\{g_n = f_{n,n}\}$: Get $g_n(w_j)$ converges for all $j$ as $n \to \infty$

Now show conv unif on a compact $K$.

↳ $\forall \varepsilon\ \exists\ \delta$ (as in defn equicont) st $K \subset \bigcup_{j=1}^{J} B_\delta(w_j)$ PROVE

$\exists N$ st $n, m \geq N \to |g_m(w_j) - g_n(w_j)| < \varepsilon \ \forall j \in \{1, \dots, J\}$

↳ $z \in K \to z \in B_\delta(w_j)$ for some $j \le J$

Then $|g_n(z) - g_m(z)| \le |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| < 3\varepsilon$

Thus $\{g_n\}$ conv unif on $K$

Want unif conv on all compact subsets:

↳ Exhaustion $K_1 \subset K_2 \subset \cdots$

$\{g_{n,1}\}$ conv unif on $K_1$

$\{g_{n,2}\}$ subseq conv unif on $K_2$ ...

diagonal argument...

# UNIFORM CONTINUITY EXAMPLE

$f(x) = x^2 - 9$ on interval $[0,2]$

Goal: Given any $\varepsilon > 0$ find a $\delta$ st whenever $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Note $\delta$ is indep of $x$ and $y$

---

Analysis: Study $f(x) - f(y)$. By seeing how big it is, it suggest a $\delta$.

$$f(x) - f(y) = (x^2 - 9) - (y^2 - 9)$$
$$= x^2 - y^2$$
$$= (x-y)(x+y)$$

$$|f(x) - f(y)| = |x-y| \cdot |x+y|$$

Note $|x+y| \leq 2+2 = 4$ as $x, y \in [0,2]$

So $|f(x) - f(y)| \leq 4|x-y| \leq 4\delta$.

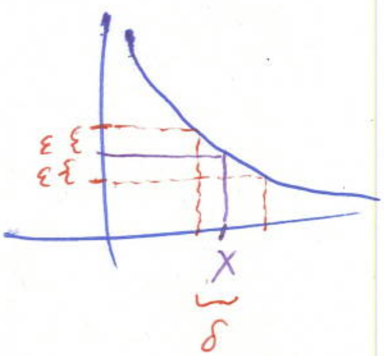
Take $\delta < \dfrac{\varepsilon}{5}$ and win.

CONTINUITY IS NOT NECC UNIF CONT

$f(x) = 1/x$ on $(0,1)$

Given $x$, $\varepsilon$ find $\delta$ st $x, y \in (0,1)$ and $|x-y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon$.

Analysis:




$f(x-\delta) = f(x)+\varepsilon$ In this special case - not in general

$$\frac{1}{x-\delta} = \frac{1}{x} + \varepsilon$$

$$\frac{1}{x-\delta} = \frac{1+\varepsilon x}{x} \rightarrow \frac{x}{1+\varepsilon x} - x = -\delta$$

$$\text{so } \delta = x\left(1 - \frac{1}{1+\varepsilon x}\right) = \frac{\varepsilon x^2}{1+\varepsilon x}$$

We've found a delta!

Unfortunately as $x \to 0$, $\delta \to 0$ since $\delta < \varepsilon x^2$. Thus we cannot find a $\delta$ that works for all $x$ for a fixed $\varepsilon$.

# TOPOLOGICAL PROOF OF CONT → UNIFORM CONTINUITY ON A COMPACT SET

Thm: f cont on a compact K then f is uniformly continuous

Proof: Given $\varepsilon > 0$, for each x find $\delta_x$ such that $\forall y$ st $|x-y| < \delta_x$ then $|f(x) - f(y)| < \frac{\varepsilon}{2}$.

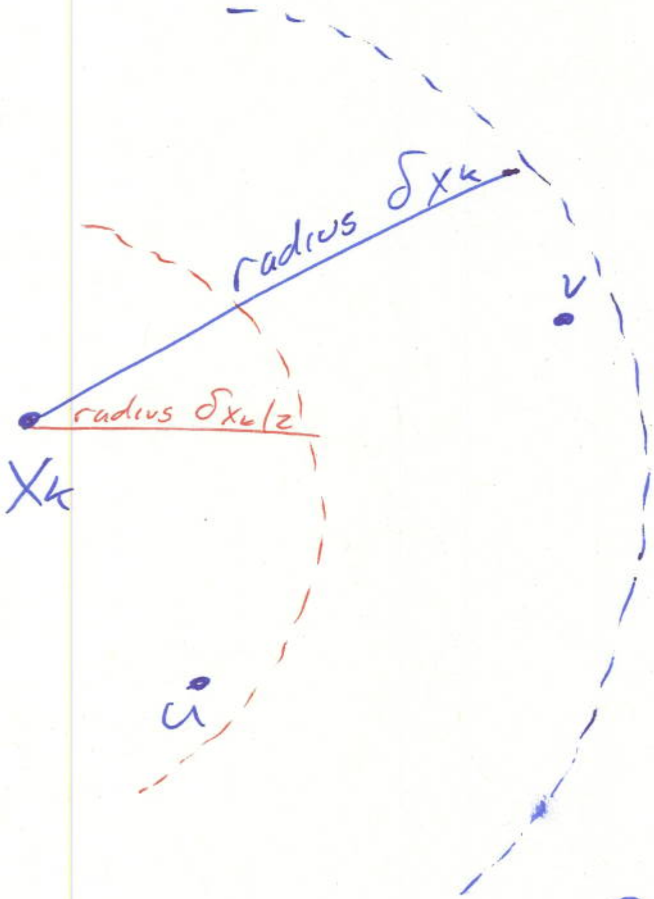
This gives us open sets $B_{\delta_x/2}(x)$, which are balls of radius $\delta_x/2$ about x.

$$K \subset \bigcup_{x \in K} B_{\delta_x/2}(x)$$

$$K \subset \bigcup_{k=1}^{n} B_{\delta_{x_k}/2}(x_k)$$

Compact: all open covers have finite sub-cover

Let $u, v \in K$ with $|u-v| < \delta$ with $\delta = \min_k (\delta_{x_k}/2)$




Have $u, v$ st $|u-v| < \delta = \min_k \delta x_k/2$

Know $K \subset \bigcup_k B_{\delta x_k/2}(X_k)$, so $u$ is in one of these balls, say $X_k$ one. Note $|X_k - u| < \frac{\delta x_k}{2}$, as $|u-v| < \frac{\delta x_k}{2}$ have (triangle ineq) $|X_k - v| < \delta x_k$.

So $|f(u) - f(v)| \leq |f(u) - f(X_k)| + |f(X_k) - f(v)|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Shows uniform continuity with our $\delta$ coming from the minimum of a finite set!