Calculus Review Problems for Math 341 (Probability)

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Chapter 1

Calculus Review Problems

Calculus is an essential tool in probability and statistics. These questions are designed to ensure that you have a sufficient mastery of the subject for a typical course. We first list several review problems, and then give detailed solutions to some of them. We urge the reader who is rusty in their calculus to do many of the problems below. Even if you are comfortable solving all these problems, we still recommend you look at both the solutions and the additional comments. We discuss various techniques to solve problems like this; some of these techniques may not have been covered in your course. Further, for some of the problems we discuss *why* we chose to attack it one way as opposed to another, analyzing why some approaches work and others fail.

For the convenience of the reader, we collect some standard calculus results.

- Derivatives of Standard Functions

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
(x^n)' &= nx^{n-1}\\
(\sin x)' &= \cos x\\
(\cos x)' &= -\sin x\\
(e^x)' &= e^x\\
(b^x)' &= (\log_e b)b^x\\
(\log_e x)' &= \frac{1}{x}\\
(\log_b x)' &= \frac{1}{\log_e b}\frac{1}{x}
\end{aligned}$$

- Useful Rules

Sum Rule:	$h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$
Constant Rule:	$h(x) = af(x)$	$h'(x) = af'(x)$
Product Rule:	$h(x) = f(x)g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule:	$h(x) = \frac{f(x)}{g(x)}$	$h'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain Rule:	$h(x) = g(f(x))$	$h'(x) = g'(f(x)) \cdot f'(x)$
	$h(x) = (f(x))^n$	$h'(x) = n(f(x))^{n-1} \cdot f'(x)$
Multiple Rule:	$h(x) = f(ax)$	$h'(x) = af'(ax)$
Reciprocal Rule:	$h(x) = f(x)^{-1}$	$h'(x) = -f'(x)f(x)^{-2}$

1.1 Problems

1.1.1 Derivatives (one variable)

Question 1.1.1 *Find the derivative of* $f(x) = 4x^5 + 3x^2 + x^{1/3}$.

Question 1.1.2 *Find the derivative of* $f(x) = (x^4 + 3x^2 + 8)\cos x$.

Question 1.1.3 *Find the derivative of* $f(x) = \log(1 - x^2)$.

Question 1.1.4 *Find the derivative of* $\log(4x) - \log(2x)$.

Question 1.1.5 *Find the derivative of* $e^{-x^2/2} = \exp(-x^2/2)$.

Question 1.1.6 *Find the second derivative of* $e^{-x^2/2} = \exp(-x^2/2)$.

Question 1.1.7 *Find the derivative of* $e^{x^8}\cos(3x^4) = \exp(x^8)\cos(3x^4)$.

Question 1.1.8 *Find the derivative of the function* $f(x) = 4x + \sqrt{2}\cos(x)$ *and then use it to find the tangent line to the curve* $y = f(x)$ *at* $x = \pi/4$. *Use the tangent line to approximate* $f(x)$ *when* $x = \frac{\pi}{4} + .01$.

Question 1.1.9 *Find the second derivative of* $f(x) = \ln x + \sqrt{162}$.

Question 1.1.10 *Find the maximum value of* $x^4e^{-x} = x^4\exp(-x)$ *when* $x \geq 0$.

Question 1.1.11 *Find the critical points of* $f(x) = 4x^3 + 3x^2$, *and decide whether each is a maximum, a minimum, or a point of inflection.*

Question 1.1.12 *Find the derivative of* $(x^2 - 1)/(x - 1)$.

Question 1.1.13 *Find the derivative of the function* $f(x) = \sqrt[3]{(5x-2)^2} = (5x-2)^{2/3}$.

Question 1.1.14 *Find the points on the graph of* $f(x) = \frac{1}{3}x^3 + x^2 - x - 1$ *where the slope is* (*a*) -1, (*b*) 2, *and* (*c*) 0.

Question 1.1.15 *Find the second derivative of* $f(x) = (x^4 + 3x^2 + 8)\cos x$.

1.1.2 Taylor Series (one variable)

Question 1.1.16 *Find the first five terms of the Taylor series for* $f(x) = x^8 + x^4 + 3$ *at* $x = 0$.

Question 1.1.17 *Find the first three terms of the Taylor series for* $f(x) = x^8 + x^4 + 3$ *at* $x = 1$.

Question 1.1.18 *Find the first three terms of the Taylor series for* $f(x) = \cos(5x)$ *at* $x = 0$.

Question 1.1.19 *Find the first five terms of the Taylor series for* $f(x) = \cos^3(5x)$ *at* $x = 0$.

Question 1.1.20 *Find the first two terms of the Taylor series for* $f(x) = e^x$ *at* $x = 0$.

Question 1.1.21 *Find the first six terms of the Taylor series for* $f(x) = e^{x^8} = \exp(x^8)$ *at* $x = 0$.

Question 1.1.22 *Find the first four terms of the Taylor series for* $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \exp(-x^2/2)/\sqrt{2\pi}$ *at* $x = 0$.

Question 1.1.23 *Find the first three terms of the Taylor series for* $f(x) = \sqrt{x}$ *at* $x = \frac{1}{3}$.

Question 1.1.24 *Find the first three terms of the Taylor series for* $f(x) = (1+x)^{1/3}$ *at* $x = \frac{1}{2}$.

Question 1.1.25 *Find the first three terms of the Taylor series for* $f(x) = x \log x$ *at* $x = 1$.

Question 1.1.26 *Find the first three terms of the Taylor series for* $f(x) = \log(1+x)$ *at* $x = 0$.

Question 1.1.27 *Find the first three terms of the Taylor series for* $f(x) = \log(1-x)$ *at* $x = 1$.

Question 1.1.28 *Find the first two terms of the Taylor series for* $f(x) = \log((1-x) \cdot e^x) =$
$\log((1-x) \cdot \exp(x))$ *at* $x = 0$.

Question 1.1.29 *Find the first three terms of the Taylor series for* $f(x) = \cos(x) \log(1+x)$ *at* $x = 0$.

Question 1.1.30 *Find the first two terms of the Taylor series for* $f(x) = \log(1+2x)$ *at* $x = 0$.

1.1.3 Integrals (one variable)

Question 1.1.31 *Find the following integral:* $\int_0^1 (x^4 + x^2 + 1)dx$.

Question 1.1.32 *Find the following integral:* $\int_0^1 (x^2 + 2x + 1)dx$.

Question 1.1.33 *Find the following integral:* $\int_0^1 (x^2 + 2x + 1)^2 dx$.

Question 1.1.34 *Find the following integral:* $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)dx$.

Question 1.1.35 *Find the following integral:* $\int_{-4}^{4} (x^3 + 6x^2 - 2x - 3)dx$.

Question 1.1.36 *Find the following integral:* $\int_0^1 \frac{x}{1+x^2}\, dx$.

Question 1.1.37 *Find the following integral:* $\int_0^3 (x^3 + 3x)^8\ (x^2 + 1)dx$.

Question 1.1.38 *Find the following integral:* $\int_0^2 x \cos(3x^2)dx$.

Question 1.1.39 *Find the following integral:* $\int_0^\infty xe^{-x^2/4}dx$.

Question 1.1.40 *Find the following integral:* $\int_a^b x^3 e^{-x^2/2}dx$.

Question 1.1.41 *Let*

$$f(x) \;=\; \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{\infty} f(t)f(x-t)dt$.

1.1.4 Derivatives (several variables)

Question 1.1.42 *Let* $f(x) = x^2 y + e^x + \sin(xy)$. *Find* $\partial f/\partial x$ *and* $\partial f/\partial y$.

Question 1.1.43 *Let*

$$f(x;\mu,\sigma) \;=\; \frac{\exp(-(x-\mu)^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}}. \qquad (1.1.1)$$

Find $\partial f/\partial \mu$ *and* $\partial f/\partial \sigma$.

Question 1.1.44 *Find* $\partial f/\partial x$ *and* $\partial f/\partial y$ *for the function* $f(x,y) = xe^{x^2+y^2} = x\exp(x^2+y^2)$.

Question 1.1.45 *Find* $\partial f/\partial x$ *and* $\partial f/\partial y$ *for the function* $f(x,y) = e^{xy} - \log(x^2+y^2)$.

Question 1.1.46 *Find* $\partial f/\partial x$ *and* $\partial f/\partial y$ *for the function* $f(x,y,t) = 5t^4 - 4t^5 \cos(t \sin t)$.

1.1.5 Integrals (several variables)

Question 1.1.47 *Find*

$$\int_{x=0}^{2}\int_{y=0}^{3} 5(x^2y+xy^2+2)dxdy.$$

Question 1.1.48 *Find*

$$\int_{x=0}^{6}\int_{y=0}^{5} xe^{-xy}dxdy.$$

Question 1.1.49 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{1} x^m y^n dxdy,$$

where $m, n > 0$.

Question 1.1.50 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{x} xydydx.$$

Question 1.1.51 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{x} ye^{-xy}dydx.$$

Question 1.1.52 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{1} (x^2+2xy+y\sqrt{x})dxdy.$$

Question 1.1.53 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{1} (ax+by+c)dxdy.$$

The following double integral is significantly harder than the ones considered above. We chose to give it as there are a variety of approaches which lead to its evaluation, each emphasizing a nice technique.

Question 1.1.54 *Prove*

$$\int_{y=0}^{1}\int_{x=0}^{1} n(1-xy)^{n-1}dxdy \;=\; 1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}.$$

1.2 Solutions

1.2.1 Derivatives (one variable)

Question 1.2.1 *Find the derivative of* $f(x) = 4x^5 + 3x^2 + x^{1/3}$.

Solution: We use the **sum** and **constant rules**, as well as the **power rule** (which says the derivative of x^n is nx^{n-1}. This yields $f'(x) = 20x^4 + 6x + \frac{1}{3}x^{-2/3}$. □

Question 1.2.2 *Find the derivative of* $f(x) = (x^4 + 3x^2 + 8)\cos x$.

Solution: In problems like this, it helps to write down what rule we are going to use. We have a product of two functions, and thus it is natural to use the **product rule**: the derivative of $A(x)B(x)$ is $A'(x)B(x) + A(x)B'(x)$. The easiest way to avoid making an algebra error is to write all the steps down; while this is time-consuming and boring, it does cut down on the mistakes. Thus, we note

$$A(x) = x^4 + 3x^2 + 8, \quad A'(x) = 4x^3 + 6x \tag{1.2.1}$$

and

$$B(x) = \cos x, \quad B'(x) = -\sin x. \tag{1.2.2}$$

Therefore $f'(x) = A'(x)B(x) + A(x)B'(x)$ with A, A', B, B' as above; as we have written everything out in full detail, we need only substitute to find

$$f'(x) = (4x^3 + 6x)\cos x - (x^4 + 3x^2 + 8)\sin x. \tag{1.2.3}$$

□

Question 1.2.3 *Find the derivative of* $f(x) = \log(1 - x^2)$.

Solution: This problem requires the **chain rule**. A good way to detect the chain rule is to read the problem aloud. We are finding the derivative of the logarithm ***of*** $1 - x^2$; the ***of*** almost always means a chain rule. If $f(x) = g(h(x))$ then $f'(x) = g'(h(x))h'(x)$. We must identify the functions g and h which we compose to get $\log(1-x^2)$. Usually what follows the ***of*** is our $h(x)$, and this problem is no exception. We see we may write

$$f(x) = \log(1 - x^2) = g(h(x)),$$

with

$$g(x) = \log x, \quad h(x) = 1 - x^2.$$

Recall the derivative of the natural logarithm function is the one-over function; in other words, $\log'(x) = 1/x$. Taking derivatives yields

$$g'(x) = \frac{1}{x}, \quad g'(h(x)) = \frac{1}{h(x)} = \frac{1}{1 - x^2}$$

and

$$h'(x) \;=\; -2x,$$

or

$$f'(x) \;=\; -\frac{2x}{1-x^2}.$$

□

Important Note: One of the most common mistakes in chain rule problems is evaluating the outer function at the wrong place. Note that even though initially we calculate $g'(x)$, it is $g'(h(x))$ that appears in the answer. This shouldn't be surprising. Imagine $f(x) = \sqrt{4-x}$; we may write this as $f(x) = g(h(x))$ with $g(x) = \sqrt{x}$ and $h(x) = 4 - x$. Note $g(-5)$ does not make sense; this is $\sqrt{-5}$, and we should only take square-roots of non-negative numbers. If $g(-5)$ doesn't even make sense, how could $g'(-5)$? The reason this is not a problem is that we do not care about $g(-5)$, but rather $g(h(-5))$; as $h(-5) = 9$, we see $g(h(-5)) = 3$.

Question 1.2.4 *Find the derivative of* $\log(4x) - \log(2x)$.

Solution: One way to evaluate this is to use the difference rule and then compute the derivative of $\log(cx)$ with $c = 4$ and $c = 2$. We can do this by either using the chain rule or the **multiple rule** (the derivative of $f(cx)$ is $cf'(cx)$). A better approach is to simplify the problem: as $\log A - \log B = \log(A/B)$, our problem is to find the derivative of $\log(4x/2x) = \log 2$. The **constant rule** says the derivative of any constant is zero; note $\log 2$ is a constant, approximately .69. Thus the derivative of this function is zero. □

Important Note: For the problem above, note how much faster it is to do some algebra first before differentiating. It is frequently a good idea to spend a few moments mulling over a problem and thinking about the best way to attack it. Often a little inspection suggests a way to rewrite the algebra to greatly simplify the computations.

Question 1.2.5 *Find the derivative of* $e^{-x^2/2} = \exp(-x^2/2)$.

Solution: This is another chain rule; the answer is $-x\exp(-x^2/2)$, and uses the fact that the derivative of e^x is e^x. □

Question 1.2.6 *Find the second derivative of* $e^{-x^2/2} = \exp(-x^2/2)$.

Solution: To find the second derivative, we just take the derivative of the first derivative. The first derivative (by the previous problem) is $-x\exp(-x^2/2)$. We now use the product rule with $f(x) = -x$ and $g(x) = \exp(-x^2/2)$. The answer is $-\exp(-x^2/2) + x^2\exp(-x^2/2)$. □

Question 1.2.7 *Find the derivative of* $e^{x^8}\cos(3x^4) = \exp(x^8)\cos(3x^4)$.

Solution: When there are several rules to be used, it is important that we figure out the right order. There is clearly going to be a power rule, as we have terms such as x^8 and x^4. There will be a chain rule, as we have cosine ***of*** $3x^4$; there will also be a product. Which rule do we use first? We have to ask: is the entire expression a product of one function? As the answer is no, the product rule isn't used first. Similarly we can't write our function as $f(g(x))$, so we don't use the chain rule first. We *can* write it as $f(x)g(x)$, with $f(x) = \exp(x^8)$ and $g(x) = \cos(3x^4)$. Thus by the product rule our derivative is

$$f'(x)g(x) + f(x)g'(x) \;=\; f'(x)\cos(3x^4) + \exp(x^8)g'(x);$$

to complete the problem we must compute $f'(x)$ and $g'(x)$. We use the chain rule for each, and find

$$f'(x) \;=\; 8x^7\exp(x^8), \quad g'(x) \;=\; -12x^3\sin(3x^4);$$

substituting these in yields the answer. □

Question 1.2.8 *Find the derivative of the function* $f(x) = 4x + \sqrt{2}\cos(x)$ *and then use it to find the tangent line to the curve* $y = f(x)$ *at* $x = \pi/4$. *Use the tangent line to approximate* $f(x)$ *when* $x = \frac{\pi}{4} + .01$.

Solution: The derivative is $f'(x) = 4 - \sqrt{2}\sin(x)$; while we could use the product rule for the second term, it is faster to just note that $\sqrt{2}$ is a constant and the derivative of $cg(x)$ is $cg'(x)$. The **tangent line** is the best linear approximation to our function at that point. The slope of the tangent line is given by the derivative at that point; this is one of the most important interpretations of the derivative. We thus have three pieces of information: we are at the point $(\pi/4, f(\pi/4)$ and the derivative (ie, the instantaneous rate of change) is $f'(\pi/4)$. We can thus find the line going through this point with this slope by using, not surprisingly, the point slope form:

$$y - y_1 \;=\; m(x - x_1).$$

Here

$$(x_1, y_1) \;=\; (\pi/4, f(\pi/4)) \;=\; (\pi/4, \pi - 1), \quad f'(\pi/4) \;=\; 3.$$

Thus

$$y - (\pi - 1) \;=\; 3(x - \pi/4) \quad \text{or} \quad y \;=\; (\pi - 1) + 3(x - \pi/4).$$

When $x = \pi/4 + .01$, this gives $f(\pi/4 + .01) \approx (\pi - 1) + .03 = \pi - .97$. Note $\pi - .97$ is about 2.17159, while $f(\pi/4 + .01)$ is about 4.17154. This is terrific agreement; our approximation is basically accurate to about four decimal places! In general, when we evaluate $f(x_0 + h)$ using the tangent line method, the error is on the order of h^2; for this problem $h = .01$ so we expect to be accurate to about .0001. □

Question 1.2.9 *Find the second derivative of* $f(x) = \ln x + \sqrt{162}$.

Solution: Remember the derivative of any constant is zero, so we see $f'(x) = 1/x$ and thus $f''(x) = -1/x^2$. □

Question 1.2.10 *Find the maximum value of* $x^4e^{-x} = x^4 \exp(-x)$ *when* $x \geq 0$.

Solution: To find the maximum (or minimum) value of a function, we must do two things: find the **critical points** (the places where the first derivative vanishes) and find the end points. We then evaluate our function at all these points and see where it is largest (or smallest). The first derivative is

$$4x^3 \exp(-x) - x^4 \exp(-x) \ = \ x^3 \exp(-x)\,(4-x)\,.$$

Thus the critical points are $x = 0$ and $x = 4$. We only have one end point, but note that as $x \to \infty$ our function very rapidly decays to zero (as exponential decay is faster than polynomial growth). Our function is 0 when $x = 0$ but $254e^{-4} \approx 4.7$ when $x = 4$. Comparing these points, we see the maximum is when $x = 4$. □

Question 1.2.11 *Find the critical points of* $f(x) = 4x^3 + 3x^2$*, and decide whether each is a maximum, a minimum, or a point of inflection.*

Solution: From the previous problem, we know the critical points are where the first derivative vanishes. In this case, $f'(x) = 12x^2 - 6x = 6x(2x-1)$, giving critical points of 0 and 1/2. There are several ways to determine if we have a maximum or minimum at a critical point. If we can take two derivatives, the **second derivative test** is a great way to proceed. It says that if $f'(a) = 0$ and $f''(a) > 0$, then we have a local minimum, while if $f'(a) = 0$ and $f''(a) < 0$ then we have a local maximum. For us, $f''(x) = 24x - 6$; thus $f''(0) = -6$, which tells us 0 is a local maximum, while $f''(1/2) = 6 > 0$, which tells us 1/2 is a local minimum. An **inflection point** is where the second derivative vanishes; this corresponds to the shape of the curve changing (from concave up to concave down, for instance). It is quite unusual for a maximum or minimum to also be an inflection point; as the second derivative is non-zero at each point, neither point is an inflection point. □

Important Note: A good way to remember the second derivative test is to look at the polynomials x^2 and $-x^2$. Both have critical points at 0, but the first has second derivative of 2 while the second has a second derivative of -2. The first is an up parabola, and clearly the vertex $x = 0$ is a minimum; the other is a down parabola, and clearly the vertex $x = 0$ is a maximum. Note the second derivative test is silent in the case when $f'(a) = 0$ and $f''(a) = 0$. There is a third and even a fourth derivative test.... □

Question 1.2.12 *Find the derivative of* $(x^2 - 1)/(x - 1)$.

Solution: We use the **quotient rule**: if $f(x) = g(x)/h(x)$ then

$$f'(x) \ = \ \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}.$$

For us

$$h(x) = x^2 - 1, \quad h'(x) = 2xc$$

and

$$g(x) = x - 1, \quad g'(x) = 1.$$

We just substitute in, and find

$$f'(x) = \frac{2x(x-1) - (x^2-1)1}{(x-1)^2} = \frac{x^2 - 2x + 1}{(x-1)^2} = 1.$$

□

Important Note: The above problem could have been done a lot faster if, as suggested above, we spent a moment thinking about algebra first. Such a pause might have allowed us to see that the numerator factors as $(x-1)(x+1)$; the $x - 1$ cancels with the denominator, and we get $f(x) = x + 1$. This is a *much* easier function to differentiate; the answer is clearly 1. Another way to do this problem is to avoid the quotient rule and use the product rule, by writing the function as $(x^2 - 1) \cdot (x - 1)^{-1}$.

Question 1.2.13 *Find the derivative of the function* $f(x) = \sqrt[3]{(5x-2)^2} = (5x-2)^{2/3}$.

Solution: This is an example of the **generalized power rule**: if $f(x) = g(x)^r$ then $f'(x) = rg(x)^{r-1}g'(x)$. Here $g(x) = 5x - 2$ and $r = 2/3$. Thus $g'(x) = 5$, $r - 1 = -1/3$, and the answer is $f'(x) = \frac{2}{3}(5x-2)^{-1/3} \cdot 5$.qed

Important Note: One of the most common mistakes in using the generalized power rule is forgetting the $g'(x)$ at the end. One reason this is so frequently omitted is the special case: if $f(x) = x^r$ then $f'(x) = rx^{r-1}$; however, we could write this as $f'(x) = rx^{r-1}x' = rx^{r-1}1$. Thus there is a $g'(x)$ term even in this case, but as it is 1 it is easy to forget about it when we generalize.

Question 1.2.14 *Find the points on the graph of* $f(x) = \frac{1}{3}x^3 + x^2 - x - 1$ *where the slope is* (a) -1, (b) 2, *and* (c) 0.

Solution: The first derivative gives the slope, so we must find where the first derivative equals -1, 2 and 0. Well, $f'(x) = x^2 + 2x - 1$. So for (a) we must solve $x^2 + 2x - 1 = -1$, or $x^2 + 2x = 0$; there are two solutions, $x = 0$ and $x = -2$. We can see this by factoring: $x^2 + 2x = 0$ is the same as $x(x+2) = 0$, and the only way the product can vanish is if one of the factors vanish. Thus either $x = 0$ or $x + 2 = 0$. For (b), a similar analysis gives $x^2 + 2x - 3 = 0$; this factors as $(x+3)(x-1) = 0$, so the solutions are $x = -3$ and $x = 1$. For (c), we have $x^2 + 2x - 1 = 0$. This does not factor nicely, so we must use the **quadratic formula**. Recall the quadratic formula says that if $ax^2 + bx + c = 0$ then $x = (-b \pm \sqrt{b^2 - 4ac})/2a$. In our case, we find the roots are $(-2 \pm \sqrt{4+4})/2$. We can simplify this with some algebra and find the roots are $-1 \pm \sqrt{2}$. □

Question 1.2.15 *Find the second derivative of* $f(x) = (x^4 + 3x^2 + 8)\cos x$.

Solution: This is another product rule problem; the answer is

$$(4x^3 + 6x)\cos x - (x^4 + 3x^2 + 8)\sin x.$$

□

1.2.2 Taylor Series (one variable)

Recall that the **Taylor series** of degree n for a function f at a point x_0 is given by

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

where $f^{(k)}$ denotes the k^{th} derivative of f. We can write this more compactly with summation notation as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where $f^{(0)}$ is just f. In many cases the point x_0 is 0, and the formulas simplify a bit to

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\ x^k \ = \ f(0) + f'(0)x + \frac{f''(0)}{2!}\ x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\ x^n.$$

The reason Taylor series are so useful is that they allow us to understand the behavior of a complicated function near a point by understanding the behavior of a related polynomial near that point; the higher the degree of our approximating polynomial, the smaller the error in our approximation. Fortunately, for many applications a first order Taylor series (ie, just using the first derivative) does a very good job. This is also called the **tangent line** method, as we are replacing a complicated function with its tangent line.

One thing which can be a little confusing is that there are $n + 1$ terms in a Taylor series of degree n; the problem is we start with the zeroth term, the value of the function at the point of interest. You should never be impressed if someone tells you the Taylor series at x_0 agrees with the function at x_0 – this is forced to hold from the definition! The reason is all the $(x - x_0)^k$ terms vanish, and we are left with $f(x_0)$, so of course the two will agree. Taylor series are only useful when they are close to the original function for x close to x_0.

Question 1.2.16 *Find the first five terms of the Taylor series for* $f(x) = x^8 + x^4 + 3$ *at* $x = 0$.

Solution: To find the first five terms requires evaluating the function and its first four derivatives:

$$\begin{aligned} f(0) &= 3 \\ f'(x) &= 8x^7 + 4x^3 \quad \Rightarrow \quad f'(0) \ = \ 0 \\ f''(x) &= 56x^6 + 12x^2 \quad \Rightarrow \quad f''(0) \ = \ 0 \\ f'''(x) &= 336x^5 + 24x \quad \Rightarrow \quad f'''(0) \ = \ 0 \\ f^{(4)}(x) &= 1680x^4 + 24 \quad \Rightarrow \quad f^{(4)} \ = \ 24. \end{aligned}$$

Therefore the first five terms of the Taylor series are

$$f(0) + f'(0)x + \cdots + \frac{f^{(4)}(0)}{4!}\, x^4 \ = \ 3 + \frac{24}{4!}\, x^4 \ = \ 3 + x^4.$$

This answer shouldn't be surprising as we can view our function as $f(x) = 3 + x^4 + x^8$; thus our function is presented in such a way that it's easy to see its Taylor series about 0. If we wanted the first six terms of its Taylor series expansion about 0, the answer would be the same. We won't see anything new until we look at the degree 8 Taylor series (ie, the first nine terms), at which point the x^8 term appears. □

Question 1.2.17 *Find the first three terms of the Taylor series for* $f(x) = x^8 + x^4 + 3$ *at* $x = 1$.

Solution: We can find the expansion by taking the derivatives and evaluating at 1 and not 0. We have

$$\begin{aligned} f(x) &= x^8 + x^4 + 3 \quad \Rightarrow \quad f(1) \ = \ 5 \\ f'(x) &= 8x^7 + 4x^3 \quad \Rightarrow \quad f'(1) \ = \ 12 \\ f''(x) &= 56x^6 + 12x^2 \quad \Rightarrow \quad f''(1) \ = \ 68. \end{aligned}$$

Therefore the first three terms gives

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \ = \ 5 + 12(x-1) + 34(x-1)^2.$$

□

Important Note: Another way to do this problem is one of my favorite tricks, namely converting a Taylor expansion about one point to another. We write x as $(x-1)+1$; we have just added zero, which is one of the most powerful tricks in mathematics. We then have

$$x^8 + x^4 + 3 \ = \ ((x-1)+1)^8 + ((x-1)+1)^4 + 3;$$

we can expand each term by using the Binomial Theorem, and after some algebra we'll find the same answer as before. For example, $((x-1)+1)^4$ equals

$$\binom{4}{0}(x-1)^4 1^0 + \binom{4}{1}(x-1)^3 1^1 + \binom{4}{2}(x-1)^2 1^2 + \binom{4}{3}(x-1)^1 1^3 + \binom{4}{4}(x-1)^0 1^5.$$

In this instance, it is not a good idea to use this trick, as this makes the problem more complicated rather than easier; however, there are situations where this trick does make life easier, and thus it is worth seeing. We'll see another trick in the next problem (and this time it *will* simplify things).

Question 1.2.18 *Find the first three terms of the Taylor series for* $f(x) = \cos(5x)$ *at* $x = 0$.

Solution: The standard way to solve this is to take derivatives and evaluate. We have

$$\begin{array}{rclcrcl} f(x) & = & \cos(5x) & \Rightarrow & f(0) & = & 1 \\ f'(x) & = & -5\sin(5x) & \Rightarrow & f'(0) & = & 0 \\ f''(x) & = & -25\cos(5x) & \Rightarrow & f''(0) & = & -25. \end{array}$$

Thus the answer is

$$f(0) + f'(0)x + \frac{f''(0)}{2}\,x^2 \;=\; 1 - \frac{25}{2}x^2.$$

□

Important Note: We discuss a faster way of doing this problem. This method assumes we know the Taylor series expansion of a related function, $g(u) = \cos(u)$. This is one of the three standard Taylor series expansions one sees in calculus (the others being the expansions for $\sin(u)$ and $\exp(u)$; a good course also does $\log(1 \pm u)$). Recall

$$\cos(u) \;=\; 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{(2k)!}.$$

If we replace u with $5x$, we get the Taylor series expansion for $\cos(5x)$:

$$\cos(5x) \;=\; 1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!} - \frac{(5x)^6}{6!} + \cdots .$$

As we only want the first three terms, we stop at the x^2 term, and find it is $1 - 25x^2/2$. The answer is the same as before, but this seems much faster. Is it? At first it seems like we avoided having to take derivatives. We haven't; the point is we took the derivatives years ago in Calculus when we found the Taylor series expansion for $\cos(u)$. We now use that. We see the advantage of being able to recall previous results – we can frequently modify them (with very little effort) to cover a new situation; however, we can of course only do this if we remember the old results!

Question 1.2.19 *Find the first five terms of the Taylor series for* $f(x) = \cos^3(5x)$ *at* $x = 0$.

Solution: Doing (a lot of!) differentiation and algebra leads to

$$1 - \frac{75}{2}\,x^2 + \frac{4375}{8}\,x^4 - \frac{190625}{48}\,x^6;$$

we calculated more terms than needed because of the comment below. Note that $f'(x) = -15\cos^2(5x)\sin(5x)$. To calculate $f''(x)$ involves a product and a power rule, and we can see that it gets worse and worse the higher derivative we need! It is worth doing all these derivatives to appreciate the alternate approach given below. □

Important Note: There is a faster way to do this problem. From the previous exercise, we know

$$\cos(5x) \;=\; 1 - \frac{25}{2}\,x^2 + \text{terms of size } x^3 \text{ or higher}.$$

Thus to find the first five terms is equivalent to just finding the coefficients up to x^4. Unfortunately our expansion is just a tad too crude; we only kept up to x^2, and we need to have up to x^4. So, let's spend a little more time and compute the Taylor series of $\cos(5x)$ of degree 4: that is

$$1 - \frac{25}{2}\,x^2 + \frac{625}{24}\,x^4.$$

If we cube this, we'll get the first six terms in the Taylor series of $\cos^3(5x)$. In other words, we'll have the degree 5 expansion, and all our terms will be correct up to the x^6 term. The reason is when we cube, the only way we can get a term of degree 5 or less is covered. Thus we need to compute

$$\left(1 - \frac{25}{2}\,x^2 + \frac{625}{24}\,x^4\right)^3;$$

however, as we only care about the terms of x^5 or lower, we can drop a lot of terms in the product. For instance, one of the factors is the x^4 term; if it hits another x^4 term or an x^2 it will give an x^6 or higher term, which we don't care about. Thus, taking the cube but only keeping terms like x^5 or lower degree, we get

$$1 + \binom{3}{1}1^2\left(-\frac{25}{2}\,x^2\right) + \binom{3}{2}1\left(-\frac{25}{2}\,x^2\right)^2 + \binom{3}{1}1^2\left(\frac{625}{24}\,x^4\right).$$

After doing a little algebra, we find the same answer as before.

So, was it worth it? To each his own, but again the advantage of this method is we reduce much our problem to something we've already done. If we wanted to do the first seven terms of the Taylor series, we would just have to keep a bit more, and expand the original function $\cos(5x)$ a bit further. As mentioned above, to truly appreciate the power of this method you should do the problem the long way (ie, the standard way).

Question 1.2.20 *Find the first two terms of the Taylor series for $f(x) = e^x$ at $x = 0$.*

Solution: This is merely the first two terms of one of the most important Taylor series of all, the Taylor series of e^x. As $f'(x) = e^x$, we see $f^{(n)}(x) = e^x$ for all n. Thus the answer is

$$f(0) + f'(0)x \;=\; 1 + x.$$

More generally, the full Taylor series is

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

□

Question 1.2.21 *Find the first six terms of the Taylor series for* $f(x) = e^{x^8} = \exp(x^8)$ *at* $x = 0$*.*

Solution: The first way to solve this is to keep taking derivatives using the chain rule. Very quickly we see how tedious this is, as $f'(x) = 8x^7\exp(x^8)$, $f''(x) = 64x^{14}\exp(x^8) + 56x^6\exp(x^8)$, and of course the higher derivatives become even more complicated. We use the faster idea mentioned above. We know

$$e^u \;=\; 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \;=\; \sum_{n=0}^{\infty} \frac{u^n}{n!},$$

so replacing u with x^8 gives

$$e^{x^8} \;=\; 1 + x^8 + \frac{(x^8)^2}{2!} + \cdots.$$

As we only want the first six terms, the highest term is x^5. Thus the answer is just 1 – we would only have the x^8 term if we wanted at least the first nine terms! For this problem, we see how much better this approach is; knowing the first two terms of the Taylor series expansion of e^u suffice to get the first six terms of e^{x^8}. This is magnitudes easier than calculating all those derivatives. Again, we see the advantage of being able to recall previous results.

Question 1.2.22 *Find the first four terms of the Taylor series for* $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \exp(-x^2/2)/\sqrt{2\pi}$ *at* $x = 0$*.*

Solution: The answer is

$$\frac{1}{2\pi} - \frac{x^2}{4\pi}.$$

We can do this by the standard method of differentiating, or we can take the Taylor series expansion of e^u and replace u with $-x^2/2$. □

Question 1.2.23 *Find the first three terms of the Taylor series for* $f(x) = \sqrt{x}$ *at* $x = \frac{1}{3}$*.*

Solution: If $f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$. Evaluating at 1/3 gives

$$\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2}\left(x - \frac{1}{3}\right) - \frac{3\sqrt{3}}{8}\left(x - \frac{1}{3}\right)^2.$$

□

Question 1.2.24 *Find the first three terms of the Taylor series for* $f(x) = (1+x)^{1/3}$ *at* $x = \frac{1}{2}$.

Solution: Doing a lot of differentiation and algebra yields

$$\left(\frac{3}{2}\right)^{1/3} + \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\left(x - \frac{1}{2}\right) - \frac{2}{27}\left(\frac{2}{3}\right)^{2/3}\left(x - \frac{1}{2}\right)^2.$$

□

Question 1.2.25 *Find the first three terms of the Taylor series for* $f(x) = x \log x$ *at* $x = 1$.

Solution: One way is to take derivatives in the standard manner and evaluate; this gives

$$(x-1) + \frac{(x-1)^2}{2}.$$

□

Important Note: Another way to do this problem involves two tricks we've mentioned before. The first is we need to know the series expansion of $\log(x)$ about $x = 1$. One of the the most important Taylor series expansions, which is often done in a Calculus class, is

$$\log(1+u) \;=\; u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{k+1}u^k}{k}.$$

We then write

$$x \log x \;=\; ((x-1)+1) \cdot \log\left(1 + (x-1)\right);$$

we can now grab the Taylor series from

$$((x-1)+1) \cdot \left((x-1) - \frac{(x-1)^2}{2}\right) \;=\; (x-1) + \frac{(x-1)^2}{2} + \cdots.$$

Question 1.2.26 *Find the first three terms of the Taylor series for* $f(x) = \log(1+x)$ *at* $x = 0$.

Question 1.2.27 *Find the first three terms of the Taylor series for* $f(x) = \log(1-x)$ *at* $x = 1$.

Solution: The expansion for $\log(1-x)$ is often covered in a Calculus class; equivalently, it can be found from $\log(1+u)$ by replacing u with $-x$. We find

$$\log(1-x) \;=\; -\left(x + \frac{x}{2} + \frac{x}{3} + \cdots\right) \;=\; -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

For this problem, we get $x + \frac{x^2}{2}$. □

Question 1.2.28 *Find the first two terms of the Taylor series for* $f(x) =$
$\log((1-x)\cdot e^x) =$
$\log((1-x)\cdot \exp(x))$ *at* $x = 0$*.*

Solution: Taking derivatives and doing the algebra, we see the answer is just zero! The first term that has a non-zero coefficient is the x^2 term, which comes in as $-x^2/2$. A better way of doing this is to simplify the expression before taking the derivative. As the logarithm of a product is the sum of the logarithms, we have $\log((1-x)\cdot e^x)$ equals $\log(1-x) + \log e^x$. But $\log e^x = x$, and $\log(1-x) = -x - x^2/2 - \cdots$. Adding the two expansions gives $-x^2/2 - \cdots$, which means that the first two terms of the Taylor series vanish. □

Question 1.2.29 *Find the first three terms of the Taylor series for* $f(x) =$ $\cos(x)\log(1+x)$ *at* $x = 0$*.*

Solution: Taking derivatives and doing the algebra gives $x - x^2/2$. □

Important Note: A better way of doing this is to take the Taylor series expansions of each piece and then multiply them together. We need only take enough terms of each piece so that we are sure that we get the terms of order x^2 and lower correct. Thus

$$\cos(x)\log(1+x) \;=\; \left(1 - \frac{x^2}{2} + \cdots\right)\cdot\left(x - \frac{x^2}{2} + \cdots\right) \;=\; x - \frac{x^2}{2} + \cdots .$$

Question 1.2.30 *Find the first two terms of the Taylor series for* $f(x) =$ $\log(1+2x)$ *at* $x = 0$*.*

Solution: The fastest way to do this is to take the Taylor series of $\log(1+u)$ and replace u with $2x$, giving $2x$. □

1.2.3 Integrals (one variable)

Question 1.2.31 *Find the following integral:* $\int_0^1 (x^4 + x^2 + 1)dx$*.*

Solution: We use the integral of a sum is the sum of the integrals, and the integral of x^n is $x^{n+1}/(n+1)$ (so long as $n \neq -1$; if $n = -1$ then the integral is $\log x$). Thus the answer is

$$\begin{aligned}
\int_0^1 (x^4 + x^2 + 1)dx &= \int_0^1 x^4 dx + \int_0^1 x^2 dx + \int_0^1 dx \\
&= \left.\frac{x^5}{5}\right|_0^1 + \left.\frac{x^3}{3}\right|_0^1 + \left.x\right|_0^1 \\
&= \frac{1}{5} + \frac{1}{3} + 1.
\end{aligned}$$

□

Question 1.2.32 *Find the following integral:* $\int_0^1 (x^2+2x+1)dx$.

Solution: We can solve this as we did the above problem, integrating term by term, or we can note that the integrand x^2+2x+1 is just $(x+1)^2$. Thus

$$\int_0^1 (x^2+2x+1)dx \;=\; \int_0^1 (x+1)^2 dx \;=\; \left.\frac{(x+1)^3}{3}\right|_0^1 \;=\; \frac{8}{3}-\frac{1}{3} \;=\; \frac{7}{3}.$$

□

Question 1.2.33 *Find the following integral:* $\int_0^1 (x^2+2x+1)^2 dx$.

Solution: We use the power rule:

$$\int_0^1 (x^2+2x+1)^2 dx \;=\; \left.\frac{(x^2+2x+1)^3}{3}\right|_0^1 \;=\; \frac{64}{3}-\frac{1}{3} \;=\; 21.$$

□

Question 1.2.34 *Find the following integral:* $\int_{-\pi/2}^{\pi/2} (\sin^3 x\cos x+\sin x\cos x)dx$.

Solution: We use the integral version of the power rule: $\int g(x)^n g'(x)dx = g(x)^{n+1}/(n+1)$. As $\sin' x = \cos x$, we have

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x\cos x+\sin x\cos x)dx \quad=\quad \left.\frac{\sin^4 x}{4}\right|_{-\pi/2}^{\pi/2} + \left.\frac{\sin^2 x}{2}\right|_{-\pi/2}^{\pi/2} \;=\; 0.$$

An alternate way to write the computations is with u-substitution. We show this for the first integral. If we let $u=\sin x$ then $du/dx=\cos x$ or $du = \cos x dx$; also, if x runs from $-\pi/2$ to $\pi/2$ then u runs from -1 to 1. Hence

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x\cos x dx \;=\; \int_{-1}^{1} u^3 du \;=\; \left.\frac{u^4}{4}\right|_{-1}^{1} \;=\; 0.$$

Question 1.2.35 *Find the following integral:* $\int_{-4}^{4} (x^3+6x^2-2x-3)dx$.

Solution: The indefinite integral is $-3x-x^2+2x^3+\frac{1}{4}x^4$; evaluating at the endpoints gives 232. □

Question 1.2.36 *Find the following integral:* $\int_0^1 \frac{x}{1+x^2}\, dx$.

Solution: This is another example of u substitution. Let $u=x^2$. Then $du/dx = 2xdx$ so $xdx = du/2$; as x runs from 0 to 1 we have u runs from 0 to 1 as well. Thus

$$\int_0^1 \frac{x}{1+x^2}\, dx \;=\; \int_0^1 \frac{1}{1+u}\frac{du}{2} \;=\; \log(1+u)\Big|_0^1 \;=\; \log 2.$$

□

Important Note: In the above problem, it is very important that the range of integration was from 0 to 1 and not from -1 to 1. Why? If we tried to do u-substitution in that case, we would say $u = x^2$ so when $x = -1$ we have $u = 1$, and also when $x = 1$ we get $u = 1$. In other words, the range of the u-integration is from 1 to 1! Any integral over a point is just zero. What went wrong? The problem is the function x^2 is not one-to-one on the interval $[-1, 1]$; in other words, different values of x are mapped to the same value of u. When we do u-substitution, it is essential that to each x there is one and only one u (and vice-versa).

Question 1.2.37 *Find the following integral:* $\int_0^3 (x^3 + 3x)^8\ (x^2 + 1)dx$.

Solution: There are several ways to do this problem. The slowest (but it will work!) is to expand the integrand and write it as a massive polynomial. The fastest is to let $u = x^3 + 3x$ and use u-substitution. Note that $du/dx = 3x^2 + 3 = 3(x^2 + 1)$, and thus our $\int (x^3 + 3x)^8 (x^2 + 1)dx$ becomes $\int u^8 du/3$. After some algebra we obtain $36^9/27$. □

Important Note: We need to use u-substitution and not the product rule here, as the product rule does not give a nice answer for $\int f(x)g(x)dx$, but only for $\int [f'(x)g(x) + f(x)g'(x)]\, dx$.

Question 1.2.38 *Find the following integral:* $\int_0^2 x\cos(3x^2)dx$.

Solution: This is another u-substitution; this is a very important technique in probability. We let $u = 3x^2$ so $du/dx = 6x$ or $xdx = du/6$. Thus

$$int_0^2 x\cos(3x^2)dx \;=\; \int_0^{27} \frac{\cos u du}{6} \;=\; \left.\frac{\sin u}{6}\right|_0^{27} \;=\; \frac{\sin 27}{6}.$$

□

Question 1.2.39 *Find the following integral:* $\int_0^\infty xe^{-x^2/4}dx$.

Solution: Surprise – another u-substitution! This time it is $u = x^2/4$ so $du/dx = x/2$ or $xdx = 2du$. We find

$$\int_0^\infty xe^{-x^2/4}dx \;=\; \int_0^\infty 2e^{-u}du \;=\; \left.-2e^{-u}\right|_0^\infty \;=\; 2.$$

□

Important Note: This integral is very important; it is basically how one calculates the mean of a normal distribution (except that it doesn't integrate to 1, this would be a normal distribution with mean 0 and variance 2.

Question 1.2.40 *Find the following integral:* $\int_a^b x^3 e^{-x^2/2}dx$.

Solution: We finally have an integral where we do not proceed by u-substitution. For this one, we integrate by parts. The formula is

$$\int_A^b u dv \ = \ u(x)v(x)\Big|_a^b - \int_a^b v du.$$

The explanation below is quite long because we want to highlight how to approach problems involving integration by parts. It is well worth the time to analyze approaches that work as well as those that do not, and see why some fail and others work. This is a great way to build intuition, which will be essential when you have to evaluate new integrals.

The difficulty in integrating by parts is figuring out what we should take for $u(x)$ and $v(x)$. The integrand is $x^3 e^{-x^2/2}$. There are several natural choices. Two obvious ones are to either take $u(x) = x^3$ and $dv = e^{-x^2/4}dx$, or to take $u(x) = e^{-x^2/2}$ and $dv = x^3 dx$. The first guess fails miserable, but it is illuminating to see why it fails. The second guess works but is a little involved. After analyzing these two cases we'll discuss another choice of u and dv that works quite well for problems like these.

In the first guess, it is easy to find du, which is just $du = 3x^2 dx$. While this looks promising, we're in trouble when we get to the dv term. There we have $dv = e^{-x^2/2}dx$, which requires us to find a function whose derivative is $e^{-x^2/2}$. Sadly, there is no elementary function that works!

What about the other idea? For the second guess, the $dv = x^3 dx$ is no problem; it leads to $v(x) = x^4/4$. Then the $u(x) = e^{-x^2/2}$ term gives $du = -xe^{-x^2/2}$. This will work, but it will be a tad cumbersome. We get

$$\int_a^b x^3 e^{-x^2/2} dx \quad = \quad e^{-x^2/2}\frac{x^4}{4}\Big|_a^b + \frac{1}{4}\int_a^b x^3 e^{-x^2/2} dx.$$

Thus after integrating by parts we are still left with a tough integral. Amazingly, however, this is the *same* integral as we started with, except multiplied by a factor of 1/4. It is essential that it is multiplied by something other than 1; the reason is we can subtract it from both sides, and find

$$\frac{3}{4}\int_a^b x^3 e^{-x^2/2} dx \ = \ e^{-x^2/2}\frac{x^4}{4}\Big|_a^b,$$

or, multiplying both sides by 4/3, we can solve for our original, unknown integral! This is another example of the **bring it over** method, which we saw in **ADD REF**.

The second method works, and involves a truly elegant trick. If we call our original integral I, we found $I = C + \frac{1}{4}I$ where C is some computable constant. This led to $\frac{3}{4}I = C$ or $I = \frac{4}{3}C$. It's nice, but will we *always* be lucky enough to get exactly our unknown integral back? If not, this trick will fail. Thus, it is worth seeing another approach to this problem. Let's analyze what went wrong in our first attempt. There we had $dv = e^{-x^2/2}dx$; the trouble was we couldn't find a nice integral (or anti-derivative) of $e^{-x^2/2}$. What if we took $dv = e^{-x^2/2}x dx$? The presence of the extra factor of x means we *can* find an anti-derivative, and we get $v = -e^{-x^2/2}$. This means we now take $u(x) = x^2$ instead of x^3, but this is fine as du is readily seen to be $du = 2xdx$. To recap, our choices are

$$u(x) \;=\; x^2 \quad \text{and} \quad du \;=\; 2xdx$$

and

$$dv \;=\; e^{-x^2/2}xdx \quad \text{and} \quad v = -e^{-x^2/2}.$$

This yields

$$\int_a^b x^3 e^{-x^2/2}dx \quad = \quad -x^2e^{-x^2/2}\Big|_a^b + 2\int_a^b xe^{-x^2/2}dx.$$

While we have not solved the problem, the remaining integral can easily be done by u-substitution (in fact, a simple variant of this was done in the previous problem). We leave it as a *very* good exercise for the reader to check and make sure these two methods give the same final answer. □

Question 1.2.41 *Let*

$$f(x) \;=\; \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{\infty} f(t)f(x-t)dt$.

Solution: This integral is significantly harder to evaluate than all the others we have looked at. The reason is that the function f is not one of the standard functions we've seen. The easiest way to attack problems like this is to break the problem up into cases. Note that the integrand $f(t)f(x-t)$ is zero unless both t and $x-t$ are in $[0,1]$. In particular, if $x > 2$ or $x < 0$, then at least one of these two expressions is not in $[0,1]$. For example, we must have $t \in [0,1]$. This means that that $x-1 \le x-t \le x$; for this to lie in $[0,1]$, we must have $x \ge 0$ and $x-1 \le 1$, which translates to $0 \le x \le 2$. For each such x we now do the integral directly. The answer turns out to depend on whether or not $0 \le x \le 1$ or $1 \le x \le 2$. Let's do the first case. If $0 \le x \le 1$, then $x-t \in [0,1]$ forces $t \in [0,x]$. Thus for $x \in [0,1]$,

$$\int_{-\infty}^{\infty} f(t)f(x-t)dt \;=\; \int_0^x dt \;=\; x.$$

If now $1 \le x \le 2$ then $x-t \in [0,1]$ implies $t \in [x-1,x]$; however, we must also have $t \in [0,1]$, so these two conditions restrict us to $t \in [x-1,1]$, and now we get

$$\int_{-\infty}^{\infty} f(t)f(x-t)dt \;=\; \int_{x-1}^{1} dt \;=\; 2-x.$$

To recap, the answer is x if $0 \le x \le 1$, $2-x$ if $1 \le x \le 2$, and 0 otherwise. □

Important Note: There is a nice probabilistic interpretation of the above integral. It is the convolution of f with itself. If f is the density of the uniform distribution on $[0,1]$, this represents the probability distribution for the sum of two uniform distributions on $[0,1]$.

1.2.4 Derivatives (several variables)

We quickly review some of the basics of partial derivatives. If we have a function of several variables (say $x_1, \dots, x_n$ for definiteness, the **partial derivative with respect to** x_i means we treat all the variables but x_i as constant, and use the standard techniques from calculus to take the derivative with respect to x_i. Explicitly, the partial derivative with respect to x_i at the point $(a_1, \dots, a_n)$ of the function f is

$$\frac{\partial f}{\partial x_i}(a_1, \dots, a_n) \ = \ \lim_{h \to 0} \frac{f(a_1, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_n)}{h}.$$

We often denote this by f_{x_i}. If we have the partial derivative of a partial derivative, we denote it as

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \ = \ \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \ = \ f_{x_i x_j}.$$

A very important result states that if all the first and second order partial derivatives of a function f exist and are continuous at a, then the order of the derivatives do not matter; in other words, $\partial^2 f / \partial x_i \partial x_j = \partial^2 f / \partial x_j \partial x_i$.

Question 1.2.42 *Let* $f(x) = x^2 y + e^x + \sin(xy)$*. Find* $\partial f / \partial x$ *and* $\partial f / \partial y$*.*

Solution: To find $\partial f / \partial x$, we treat y as a constant and differentiate with respect to x. We find

$$\frac{\partial f}{\partial x} \ = \ 2xy + e^x + y \cos(xy),$$

where the last piece involves using the chain rule (or the multiple rule) on $\sin(xy)$, remembering that y is a constant. Similarly we find

$$\frac{\partial f}{\partial y} \ = \ x^2 + x \cos(xy).$$

□

Question 1.2.43 *Let*

$$f(x; \mu, \sigma) \ = \ \frac{\exp(-(x-\mu)^2 / 2\sigma^2)}{\sqrt{2\pi\sigma^2}}. \tag{1.2.4}$$

Find $\partial f / \partial \mu$ *and* $\partial f / \partial \sigma$*.*

Solution: Treating σ as a constant, we find

$$\begin{aligned} \frac{\partial f}{\partial \mu} \ &= \ \frac{\exp(-(x-\mu)^2 / 2\sigma^2)}{\sqrt{2\pi\sigma^2}} \cdot \left(-\frac{1}{2\sigma^2} \cdot 2(x-\mu) \cdot (-1)\right) \\ &= \ \frac{(x-\mu)\exp(-(x-\mu)^2 / 2\sigma^2)}{\sigma^2 \sqrt{2\pi\sigma^2}}. \end{aligned}$$

A more involved computation, involving the quotient rule, gives

$$\frac{\partial f}{\partial \sigma} \ = \ -\frac{\exp(-(x-\mu)^2 / 2\sigma^2)}{\sqrt{2\pi}\sigma^2} + \frac{(x-\mu)^2 \exp(-(x-\mu)^2 / 2\sigma^2)}{\sqrt{2\pi}\sigma^4}.$$

□

Question 1.2.44 *Find $\partial f/\partial x$ and $\partial f/\partial y$ for the function $f(x,y) = xe^{x^2+y^2} = x\exp(x^2+y^2)$.*

Solution: A straightforward computation gives

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^{x^2+y^2} + 2x^2 e^{x^2+y^2}\\ \frac{\partial f}{\partial y} &= 2xye^{x^2+y^2}.\end{aligned}$$

□

Question 1.2.45 *Find $\partial f/\partial x$ and $\partial f/\partial y$ for the function $f(x,y) = e^{xy} - \log(x^2+y^2)$.*

Solution: Holding y constant, we see

$$\frac{\partial f}{\partial x} = ye^{xy} - \frac{2x}{x^2+y^2},$$

while holding x constant gives

$$\frac{\partial f}{\partial y} = xe^{xy} - \frac{2y}{x^2+y^2}.$$

□

Important Note: There is actually no need to find $\partial f/\partial y$ once we know $\partial f/\partial x$. The reason is that our original function is symmetric in x and y: $f(x,y) = f(y,x)$. This means we can interchange the roles of x and y. Thus if we know $\partial f/\partial x = g(x,y)$, then by symmetry $\partial f/\partial y = g(y,x)$. Using **symmetry** is a powerful technique to simplify computations.

Question 1.2.46 *Find $\partial f/\partial x$ and $\partial f/\partial y$ for the function $f(x,y,t) = 5t^4 - 4t^5\cos(t\sin t)$.*

Solution: It would be a long calculation if we were asked to find $\partial f/\partial t$; however, as $f(x,y,t)$ only depends on t, the partials with respect to x and y vanish. If we had wanted $\partial f/\partial t$, the answer is

$$20t^3 - 20t^4\cos(t\sin(t)) + 4t^5(t\cos(t) + \sin(t))\sin(t\sin(t)).$$

□

1.2.5 Integrals (several variables)

One of the most important theorems in multivariable integration is **Fubini's theorem**. One version states that if either

$$\int_a^b \left[\int_c^d |f(x,y)|dy\right] dx \quad \text{or} \quad \int_c^d \left[\int_a^b |f(x,y)|dx\right] dy$$

is finite, then

$$\int_a^b \left[\int_c^d f(x,y)dy\right] dx \;=\; \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

There are numerous problems where one integral is easier than the other, and thus you should always consider switching the order of integration. Whenever both integrals exist and are equal, we write

$$\int_a^b \int_c^d f(x,y)dxdy$$

to denote either integral. When it isn't clear which integration bounds refer to which variable, we can be a bit more explicit and write

$$\int_{x=a}^b \int_{y=c}^d f(x,y)dxdx.$$

Question 1.2.47 *Find*

$$\int_{x=0}^2 \int_{y=0}^3 5(x^2y + xy^2 + 2)dxdy.$$

Solution: We do the y-integration first and then the x-integration. We get

$$\begin{aligned}
\int_{x=0}^2 \int_{y=0}^3 5(x^2y + xy^2 + 2)dxdy \quad &= \quad 5\int_{x=0}^2 \left[x^2\frac{y^2}{2} + x\frac{y^3}{3} + 2y\right]_0^3 dx \\
&= \quad 5\int_0^2 \left[\frac{9}{2}x^2 + 9x + 6\right] dx \\
&= \quad 5\left[\frac{3x^3}{3} + \frac{9x^2}{2} + 6x\right]_0^2 \;=\; 190.
\end{aligned}$$

For this problem, it doesn't really matter if we do the x or the y-integration first; both lead to equally easy computations. □

Question 1.2.48 *Find*

$$\int_{x=0}^6 \int_{y=0}^5 xe^{-xy}dxdy.$$

Solution: Unlike the previous problem, for this problem life is much easier if we integrate in the correct order. Note the integrand is xe^{-xy}. We need to find an anti-derivative for this with respect to x or with respect to y. It is hard to find a function whose partial derivative with respect to x equals this, though it can be done. (The answer turns out to be $-(1+xy)\ e^{-xy}\ /y^2$, which should be a non-obvious guess.) If we try to find an anti-derivative with respect to y, however, things are much nicer as we have the factor of x.

We thus do the y-integration first, as the factor of x makes things very nice (ie, u-substitution). We find

$$\begin{aligned}
\int_{x=0}^{6}\int_{y=0}^{5} xe^{-xy}dxdy &= \int_{x=0}^{6}\left[-e^{-xy}\right]_0^5 dx \\
&= \int_0^6 \left(1-e^{-5x}\right)dx \\
&= x\Big|_0^6 + \frac{1}{5}e^{-5x}\Big|_0^6 \\
&= 6+\frac{e^{-30}}{5}-\frac{1}{5} = \frac{29}{5}-\frac{1}{5e^{30}}.
\end{aligned}$$

□

Question 1.2.49 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{1} x^m y^n dxdy,$$

where $m,n>0$.

Solution: The fact that $m,n>0$ means that the integrand is continuous and we can integrate in either order; further, we don't have to worry about an anti-derivative being a logarithm as none of the exponents are -1. We might as well do the y-integration first, and we see

$$\begin{aligned}
\int_{x=0}^{1}\int_{y=0}^{1} x^m y^n dxdy &= \int_{x=0}^{1}\left[x^m\frac{y^{n+1}}{n+1}\right]_0^1 dx \\
&= \frac{1}{n+1}\int_0^1 x^m dx \\
&= \frac{1}{n+1}\frac{x^{m+1}}{m+1}\Big|_0^1 = \frac{1}{(m+1)(n+1)}.
\end{aligned}$$

□

Important Note: For problems like the above, if we are integrating a function $f(x,y)$ which factors as $f(x,y)=g(x)h(y)$ *and* the region of integration is a rectangle whose boundary is independent of x and y, then we have

$$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)dxdy = \int_a^b g(x)dx\int_c^d h(y)dy.$$

We will be fortunate enough to have integrals like this when we are looking at independent random variables.

Question 1.2.50 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{x} xydydx.$$

Solution: For this problem, the region of integration is no longer a nice, simple rectangle. It turns out to be a triangle **ADD PICTURE?**, and clearly the order of integration matters. Why? Well, as it is written we must do the y-integration first as its bounds depend on x. If we wanted to do the x-integration first, we would have to change the bounds of integration. The triangle can also be written in the following manner: for each fixed $y \in [0,1]$, x ranges from y to 1. Thus

$$\int_{x=0}^{1}\int_{y=0}^{x} xydydx \;=\; \int_{y=0}^{1}\int_{x=y}^{1} xydxdy.$$

We compute both double integrals, and see they are equal. The first is

$$\int_0^1 \left.\frac{xy^2}{2}\right|_0^x dx \;=\; \int_0^1 \frac{x^3}{2}dx \;=\; \left.\frac{x^4}{8}\right|_0^1 \;=\; \frac{1}{8},$$

while the second method gives

$$\int_0^1 \left.\frac{x^2y}{2}\right|_y^1 dy \;=\; \int_0^1 \left(\frac{y}{2}-\frac{y^3}{2}\right) dy \;=\; \left.\frac{y^2}{4}\right|_0^1 - \left.\frac{y^4}{8}\right|_0^1 \;=\; frac14 - \frac{1}{8} \;=\; \frac{1}{8}.$$

For this problem it is not a good idea to change the order of integration; however, we'll see in the next problem the utility of changing orders. In other words, here it is not worth the time or effort needed to figure out how to write the region when we switch orders, but the function in the next problem is harder to integrate in the given order, and there the effort will be amply repaid. □

Question 1.2.51 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{x} ye^{-xy}dydx.$$

Solution: For this problem, it would be quite painful to integrate in the given order. The reason is that we would need to find a function whose partial with respect to y is ye^{-xy}; this can be done, but the answer of $-(1+xy)\,e^{-xy}\,/x^2$ is not obvious or easily seen. If, however, we switch the order of integration, then we just need a function whose derivative with respect to x is ye^{-xy}, and the answer to this problem is easily seen to be $-e^{-xy}$. Thus, unlike the previous problem, it is well worth the effort to switch orders of integration. In the previous problem we showed the region can be parametrized by for a fixed $y \in [0,1]$, x runs from y to 1. We find

$$\begin{aligned}
\int_{x=0}^{1}\int_{y=0}^{x} ye^{-xy}dydx &= \int_{y=0}^{1}\left[\int_{x=y}^{1} ye^{-xy}dy\right]dx \\
&= \int_{0}^{1} -e^{-xy}\Big|_{y}^{1} dy \\
&= \int_{0}^{1}\left(-e^{-y}+e^{-y^2}\right)dy \\
&= e^{-y}\Big|_{0}^{1} + \int_{0}^{1} e^{-y^2}dy.
\end{aligned}$$

At this point, we're in trouble – there is no nice, closed form expression for the anti-derivative of e^{-y^2} or a nice, closed form expression for the integral of this function from 0 to 1. (The solution involves a new function you may not have seen, the **error function**, often denoted by erf or Erf.) We chose to do a problem like this where there is no nice, closed form answer at the end of the day because, sadly, most probability problems are like this! It is actually quite unusual to be able to evaluate a two-dimensional integral and get a nice answer, unless we stick to very simple regions and very nice functions. □

Question 1.2.52 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{1}(x^2+2xy+y\sqrt{x})dxdy.$$

Solution: This problem is solved by using the same techniques as before. The answer turns out to be 7/6. If we do the y-integration first we get $x^2+x+\sqrt{x}/2$, which is then readily integrated. □

Question 1.2.53 *Find*

$$\int_{x=0}^{1}\int_{y=0}^{1}(ax+by+c)dxdy.$$

Solution: The answer is $a/2+b/2+c$. □

The following double integral is significantly harder than the ones considered above. We chose to give it as there are a variety of approaches which lead to its evaluation, each emphasizing a nice technique.

Question 1.2.54 *Prove*

$$\int_{y=0}^{1}\int_{x=0}^{1} n(1-xy)^{n-1}dxdy \;=\; 1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}.$$

Solution: We give several solutions. We first integrate with respect to x. Note

$$\int_{x=0}^{1} n(1-xy)^{n-1}dx \;=\; \frac{n}{y}\int_{x=0}^{1}(1-xy)^{n-1}y dx \;=\; \frac{n}{y}\left(1-(1-y)^n\right); \quad (1.2.5)$$

we do not need to worry about convergence issues when $y = 0$ (note that $1-(1-y)^n = O(y)$, where the $O(y)$ means some error at most a constant times y). Thus we are reduced to finding

$$\int_0^1 \frac{1-(1-y)^n}{y}\, dy.$$

The integral in (**??**) is easily evaluated by induction. We do one or two steps, as the pattern is clear and the induction straightforward. Write

$$(1-y)^n \;=\; (1-y)(1-y)^{n-1} \;=\; (1-y)^{n-1} - y(1-y)^{n-1}.$$

Thus our integral is just

$$\begin{aligned}\int_0^1 & \frac{1-(1-y)^{n-1}+y(1-y)^{n-1}}{y}\, dy \\ &= \int_0^1 \frac{1-(1-y)^{n-1}}{y}\, dy + \int_0^1 (1-y)^{n-1}\, dy \\ &= \int_0^1 \frac{1-(1-y)^{n-1}}{y}\, dy + \frac{1}{n}.\end{aligned}$$

By a straightforward induction, we have

$$\int_0^1 \frac{1-(1-y)^{n-1}}{y}\, dy \;=\; \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{1}.$$

Thus the original integral is just

$$\int_{y=0}^{1}\int_{x=0}^{1} n(1-xy)^{n-1}dxdy \;=\; \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1} \;\approx\; \log n + \gamma,$$

where $\gamma \approx .5772$ is the Euler-Mascheroni constant.

We provide an alternate way to evaluate the integral in (**??**). We use the finite geometric series formula:

$$1 + r + r^2 + \cdots + r^{n-1} \;=\; \frac{1-r^n}{1-r};$$

Taking $r = 1-y$ yields

$$1 + (1-y) + (1-y)^2 + \cdots + (1-y)^{n-1} \;=\; \frac{1-(1-y)^n}{y}.$$

We integrate the above with respect to y from 0 to 1. As trivially $\int_0^1 (1-y)^{k-1}dy = 1/k$, we have

$$\int_0^1 \frac{1-(1-y)^n}{y}\, dy \;=\; \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \;\approx\; \log n + \gamma,$$

providing another method to evaluate the integral.

We give yet another solution. We want to show

$$\int_0^1 \int_0^1 n(1-xy)^{n-1} dxdy \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n};$$

by induction it suffices to show

$$D(n+1) \;=\; \int_0^1 \int_0^1 (n+1)(1-xy)^n dxdy - \int_0^1 \int_0^1 n(1-xy)^{n-1} dxdy \;=\; \frac{1}{n+1}$$

(instead of induction one could also proceed by using telescoping series). This approach, of course, is similar to the other methods above. We use the Binomial Theorem to expand the two integrands and then integrate with respect to x and y. We have (the algebra is elementary, and was done in one of the author's head while feeding a bottle to a sixth month old!)

$$D(n+1) \;=\; \binom{n}{0}\frac{1}{1} - \binom{n}{1}\frac{1}{2} + \binom{n}{2}\frac{1}{3} - \binom{n}{3}\frac{1}{4} + \cdots + (-1)^n \binom{n}{n}\frac{1}{n+1};$$

note, however, that the right hand side above is the same as

$$\int_0^1 (1-t)^n dt$$

(just use the Binomial Theorem again and integrate term by term). This integral is easily seen to be $1/(n+1)$, which implies that $D(n+1) = 1/(n+1)$, or

$$\int_0^1 \int_0^1 n(1-xy)^{n-1} dxdy \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$