Sequences

(2.2.1) **Sequence.** A sequence is a function whose domain is \( \mathbb{N} \).

(2.2.3) **Convergence of a Sequence.** A sequence \((a_n)\) converges to a real number \(a\) if, for every \( \varepsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that whenever \( n \geq N \), it follows that \( |a_n - a| < \varepsilon \).

(2.2.3b) **Convergence of a Sequence, Topological Characterization.** A sequence \((a_n)\) converges to a if, given any \( \varepsilon \)-neighborhood \( V(a) \) of \( a \), there exists a point in the sequence after which all of the terms are in \( V(a) \) \( \Leftrightarrow \) every \( V(a) \) contains all but a finite number of terms of \((a_n)\).

(2.2.7) **Uniqueness of Limits.** The limit of a sequence, when it exists, must be unique.

(2.2.9) **Divergence.** A sequence that does not converge is said to diverge.

(2.3.1) **Bounded.** A sequence \((x_n)\) is bounded if \( \exists M > 0 \) such that \( |x_n| < M \) for all \( n \in \mathbb{N} \).

(2.3.2) Every convergent sequence is bounded.

(2.3.3) **Algebraic Limit Theorem.** Let \( \lim a_n = a \) and \( \lim b_n = b \). Then, (i) \( \lim(c a_n) = ca \) for all \( c \in \mathbb{R} \); (ii) \( \lim(a_n + b_n) = a + b \); (iii) \( \lim(a_n b_n) = ab \); (iv) \( \lim(a_n/b_n) = a/b \) provided \( b \neq 0 \).

(2.3.4) **Order Limit Theorem.** Assume \( \lim a_n = a \) and \( \lim b_n = b \). Then, (i) if \( a_n \geq 0 \) for all \( n \in \mathbb{N} \), then \( a \geq 0 \); (ii) if \( a_n \geq b_n \) for every \( n \in \mathbb{N} \), then \( a \geq b \); (iii) if there exists \( c \in \mathbb{R} \) for which \( c \leq a_n \) for all \( n \in \mathbb{N} \), then \( c \leq a \).

(2.4.3) **Convergence of a Series.** Let \((b_n)\) be a sequence, and define the corresponding sequence of partial sums \((s_m)\) of the series \( \sum b_n \), where \( s_m = b_1 + b_2 + \ldots + b_m \). The series \( \sum b_n \) converges to \( B \) if the sequence \((s_m)\) converges to \( B \). Thus, \( \sum b_n = B \).

(2.4.6) **Cauchy Condensation Test.** Suppose \((b_n)\) is decreasing and satisfies \( b_n \geq 0 \) for all \( n \in \mathbb{N} \). Then, the series \( \sum b_n \) converges if and only if the series \( \sum 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + \ldots \) converges.

(2.4.7) The series \( \sum 1/n^p \) converges if and only if \( p > 1 \).

(2.5.1) **Subsequences.** Let \((a_n)\) be a sequence of real numbers, and let \( n_1 < n_2 < n_3 < \ldots \) be an increasing sequence of natural numbers. Then the sequence \((a_{n_1}, a_{n_2}, a_{n_3}, \ldots)\) is called a subsequence of \((a_n)\) and is denoted by \((a_{n_k})\), where \( k \in \mathbb{N} \) indexes the subsequence.

(2.5.2) Subsequence of a convergent sequence converge to the same limit as the original sequence.

(2.5.5) **Bolzano-Weierstrass Theorem.** Every bounded sequence contains a convergent subsequence.
(2.6.1) **Cauchy Sequence.** A sequence \((a_n)\) is called a Cauchy sequence if, for every \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\) such that whenever \(m, n \geq N\), it follows that \(|a_n - a_m| < \epsilon\).

(2.6.2) Every convergent sequence is a Cauchy sequence.

(2.6.3) Cauchy sequences are bounded.

(2.6.4) **Cauchy Criterion.** A sequence converges if and only if it is a Cauchy sequence.

**Series**

(2.7.1) **Algebraic Limit Theorem for Series.** If \(\sum a_k = A\) and \(\sum b_k = B\), then (i) \(\sum ca_k = cA\) for all \(c \in \mathbb{R}\) and (ii) \(\sum (a_k + b_k) = A + B\).

(2.7.2) **Cauchy Criterion for Series.** The series \(\sum a_k\) converges if and only if, given \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\) such that whenever \(n > m \geq N\), it follows that \(|a_{m+1} + a_{m+2} + \ldots + a_n| < \epsilon\).

(2.7.3) If the series \(\sum a_k\) converges, then the sequence \((a_k)\) converges to 0.

(2.7.4) **Comparison Test.** Assume \((a_k)\) and \((b_k)\) are sequences satisfying \(0 \leq a_k \leq b_k \ \forall \ k \in \mathbb{N}\). Then, (i) if \(\sum b_k\) converges, then \(\sum a_k\) converges and (ii) if \(\sum a_k\) diverges, then \(\sum b_k\) diverges.

(2.7.6) **Absolute Convergence Test.** If the series \(\sum |a_n|\) converges, then \(\sum a_n\) converges as well.

(2.7.7) **Alternating Series Test.** Let \((a_n)\) be a sequence satisfying (i) \(a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots\) and (ii) \((a_n)\) converges to 0. Then, the alternating series \(\sum (-1)^{n+1}a_n\) converges.

**Topology of The Reals**

(3.2.1) **Open.** A set \(O\) is open if for all points \(a \in O\), there exists a \(V(a) \subseteq O\).

(3.2.3) (i) The union of an arbitrary collection of open sets is open. (ii) The intersection of a finite collection of open sets is open.

(3.2.4) **Limit Point.** A point \(x\) is a limit point of a set \(A\) if every \(V(x)\) intersects the set \(A\) at some point other than \(x\).

(3.2.5) A point \(x\) is a limit point of a set \(A\) if and only if \(x = \lim a_n\) for some sequence \((a_n)\) contained in \(A\) satisfying \(a_n \neq x\) for all \(n \in \mathbb{N}\).

(3.2.6) **Isolated Point.** A point \(a \in A\) is an isolated point of \(A\) if it is not a limit point of \(A\).
(3.2.7) **Closed.** A set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.

(3.2.8) A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in $F$ has a limit that is also an element of $F$.

(3.2.10) **Density of $\mathbb{Q}$ in $\mathbb{R}$.** For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to $y$.

(3.2.11) **Closure.** Given a set $A \subseteq \mathbb{R}$, let $L$ be the set of all limit points of $A$. The closure of $A$ is defined to be $\text{Cl}(A) = A \cup L$.

(3.2.12) For any $A \subseteq \mathbb{R}$, the closure of $A$ is a closed set and is the smallest closed set containing $A$.

(3.2.13) A set $O$ is open if and only if $O^c$ is closed.

(3.2.14) (i) The union of a finite collection of closed sets is closed. (ii) The intersection of an arbitrary collection of closed sets is closed.

(3.3.1) **Compactness.** A set $K \subseteq \mathbb{R}$ is compact if every sequence in $K$ has a subsequence that converges to a limit that is also in $K$.

(3.3.3) **Bounded.** A set $A \subseteq \mathbb{R}$ is bounded if there exists $M > 0$ such that $|a| < M$ for all $a \in A$.

(3.3.4) **Characterization of Compactness in $\mathbb{R}$.** A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

(3.3.5) If $K_1 \supseteq K_2 \supseteq K_3 \supseteq ...$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap K_n$ is not empty.

(3.3.6) **Open Cover.** Let $A \subseteq \mathbb{R}$. An open cover for $A$ is a (possibly infinite) collection of open sets $\{O_d : d \in D\}$ whose union contains the set $A$. A finite subcover is a finite subcollection of open sets from the original open cover whose union still manages to completely contain $A$.

(3.3.8) **Heine-Borel Theorem.** Let $K$ be a subset of $\mathbb{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others: (i) $K$ is compact; (ii) $K$ is closed and bounded; (iii) Every open cover for $K$ has a finite subcover.

**Functional Limits**

(4.2.1) **Functional Limit.** Let $f$ be defined on $A$, and let $c$ be the limit point of $A$. Then, $\lim_{x \to c} f(x) = L$ provided that for all $\epsilon > 0$, $\exists \delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $|f(x) - L| < \epsilon$. 

(4.2.1B) **Functional Limit - Topological Characterization.** Let c be the limit point of the domain of f. We say \( \lim_{x \to c} f(x) = L \) provided that, for every \( V(L) \) of L, there exists a \( V(c) \) such that for all \( x \in V(c) \) it follows that \( f(x) \in V(L) \).

(4.2.3) **Sequential Criterion for Functional Limits.** Given a function \( f \) defined on \( A \) and a limit point \( c \) of \( A \), then \( \lim_{x \to c} f(x) = L \iff \) for all sequences \( (x_n) \subseteq A \) satisfying \( x_n \neq x \) and \( (x_n) \to c \), then \( f(x_n) \to L \).

(4.2.4) **Algebraic Limit Theorem for Functional Limits.** Let \( f \) and \( g \) be functions defined on domain \( A \subseteq \mathbb{R} \), and assume that \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \) for some limit point \( c \in A \). Then, (i) \( \lim_{x \to c} k f(x) = k L \) for all \( k \in \mathbb{R} \), (ii) \( \lim_{x \to c} (f(x) + g(x)) = L + M \), \( \lim_{x \to c} (f(x))g(x) = LM \) and (iv) \( \lim (f(x)/g(x)) = L/M \), provided \( M \neq 0 \).

(4.2.5) **Divergence Criterion for Functional Limits.** Let \( f \) be a function defined on \( A \), and \( c \) be a limit point of \( A \). If there exists two sequences \( (x_n), (y_n) \) with \( x_n \neq c \) and \( y_n \neq c \) and \( \lim_{x \to c} x_n = \lim_{x \to c} y_n = c \) but \( \lim_{x \to c} f(x_n) \neq \lim_{x \to c} f(y_n) \), then \( \lim f(x) \) does not exist.

(4.3.1) **Continuity.** A function \( f \) is continuous at a point \( c \in A \) if, for all \( \delta > 0 \), there exists a \( \epsilon > 0 \) such that whenever \( |x - c| < \delta \) it follows that \( |f(x) - f(c)| < \epsilon \). If \( f \) is continuous at every point in the domain \( A \), then \( f \) is continuous on \( A \).

(4.3.2) **Characterizations of Continuity.** Let \( f \), defined on \( A \), and \( c \in A \). The function \( f \) is continuous at \( c \) if and only if any one of the following conditions is met: (i) For all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( |x - c| < \delta \implies |f(x) - f(c)| < \epsilon \); (ii) For all \( V(f(c)) \), there exists a \( V(c) \) such that \( x \in V(c) \implies f(x) \in V(f(c)) \); (iii) If \( (x_n) \to c \), then \( f(x_n) \to f(c) \); If \( c \) is a limit point of \( A \), then the above conditions are equivalent to (iv) \( \lim_{x \to c} f(x) = f(c) \).

(4.3.3) **Criterion for Discontinuity.** Let \( f \), defined on \( A \), and \( c \in A \) be a limit point of \( A \). If there exists a sequence \( (x_n) \subseteq A \) where \( (x_n) \to c \) but such that \( f(x_n) \) does not converge to \( f(c) \), we may conclude that \( f \) is not continuous at \( c \).

(4.3.4) **Algebraic Continuity Theorem.** Assume \( f, g \) defined on \( A \), continuous at a point \( c \in A \). Then, (i) \( k f(x) \) is continuous at \( c \forall k \in \mathbb{R} \); (ii) \( f(x) + g(x) \) is continuous at \( c \); (iii) \( f(x)g(x) \) is continuous at \( c \); and (iv) \( f(x)/g(x) \) is continuous at \( c \), provided the quotient is defined.

(*) All polynomials are continuous on \( \mathbb{R} \).

(4.3.9) **Compositions of Continuous Functions.** Given \( f \) defined on \( A \) and \( g \) defined on \( B \), and assume the range \( f(A) = \{ f(x) : x \in A \} \) is contained in the domain \( B \) so that the composition \( g \cdot f(x) = g(f(x)) \) is defined on \( A \). If \( f \) is continuous at \( c \in A \), and \( g \) is continuous at \( f(c) \in B \), then \( g(f(x)) \) is continuous at \( c \).

(4.4.1) **Preservation of Compact Sets.** Let \( f \) defined on \( A \) be continuous on \( A \). If \( K \subseteq A \) is compact, then \( f(K) \) is compact as well.
(4.4.2) **Extreme Value Theorem.** If \( f \), defined on \( K \) compact, is continuous on \( K \subseteq \mathbb{R} \), then \( f \) attains a maximum and a minimum value. In other words, there exists \( x_0, x_1 \in K \) such that \( f(x_0) \leq f(x) \leq f(x_1) \) for all \( x \in K \).

(4.4.4) **Uniform Continuity.** A function \( f \) defined on \( A \) is uniformly continuous on \( A \) if for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( x, y \in A \), \( |x - y| < \delta \) implies \( |f(x) - f(y)| < \varepsilon \).

(4.4.5) **Sequential Criterion for Absence of Uniform Continuity.** A function defined on \( A \) fails to be uniformly continuous on \( A \) if and only if there exists a particular \( \varepsilon_0 > 0 \) and two sequences \((x_n), (y_n)\) in \( A \) satisfying \( |x_n - y_n| \to 0 \) but \( |f(x_n) - f(y_n)| \geq \varepsilon_0 \).

(4.4.7) **Uniform Continuity on Compact Sets.** A function that is continuous on a compact set \( K \) is uniformly continuous on \( K \).

(5.5.1) **Intermediate Value Theorem.** Let \( f \) be defined on \([a, b]\) be continuous. If \( L \) is a real number satisfying \( f(a) < L < f(b) \) or \( f(a) > L > f(b) \), then there exists a point \( c \in (a, b) \) such that \( f(c) = L \).

(5.5.3) **Intermediate Value Property.** A function \( f \) has the intermediate value property on an interval \([a, b]\) if for all \( x < y \) in \([a, b]\) and all \( L \) between \( f(x) \) and \( f(y) \), it is always possible to find a point \( c \in (x, y) \) where \( f(c) = L \).

**Sequences of Functions**

(6.2.1) **Pointwise Convergence:** For each \( n \in N \), let \( f_n \) be a function defined on set \( A \subseteq \mathbb{R} \). The sequence of functions converges pointwise on \( A \) to a function \( f \) if, (1) for all \( x \in A \), the sequence of real numbers \( f_n(x) \) converges to \( f(x) \) \( \iff \) for every \( \epsilon > 0 \) and \( x \in A \), there exists an \( N \) such that \( |f_n(x) - f(x)| < \epsilon \quad \forall n \geq N \).

(6.2.3) **Uniform Convergence:** Let \( (f_n) \) be a sequence of functions defined on a set \( A \subseteq \mathbb{R} \). Then, \( (f_n) \) converges uniformly on \( A \) to a limit function \( f \) defined on \( A \) if, for every \( \epsilon > 0 \), there exists an \( N \in N \) such that \( |f_n(x) - f(x)| < \epsilon \quad \text{whenever } n \geq N \text{ and } x \in A \).

(6.2.5) **Cauchy Criterion for Uniform Convergence:** A sequence of functions \( (f_n) \) defined on a set \( A \subseteq \mathbb{R} \) converges uniformly on \( A \) if and only if for every \( \epsilon > 0 \), there exists an \( N \in N \) such that \( |f_n(x) - f_m(x)| < \epsilon \quad \text{whenever } n, m \geq N \text{ and } x \in A \).

(6.2.6) **Continuous Limit Theorem:** Let \( (f_n) \) be a sequence of functions defined on \( A \subseteq \mathbb{R} \) that converges uniformly on \( A \) to a function \( f \). If each \( f_n \) is continuous at \( c \in A \), then \( f \) is continuous at \( c \).
(6.3.1) **Differentiable Limit Theorem:** Let \( f_n \to f \) pointwise on the closed interval \([a, b]\), and assume that each \( f_n \) is differentiable. If \( (f'_n) \) converges uniformly on \([a, b]\) to a function \( f \), then the function \( f \) is differentiable and \( f' = g \).

(6.3.2) **Weaker Differentiability Limit Theorem:** Let \((f_n)\) be a sequence of differentiable functions defined on the closed interval \([a, b]\), and assume \((f'_n)\) converges uniformly on \([a, b]\). If there exists a point \( x_0 \in [a, b] \) where \( f_n(x_0) \) is convergent, then \((f_n)\) converges uniformly on \([a, b]\).

(6.3.3) **Stronger Differentiable Limit Theorem:** Let \((f'_n)\) be a sequence of differentiable functions defined on the closed interval \([a, b]\), and \((f'_n)\) converges uniformly to a function \( g \) on \([a, b]\). If there exists a point \( x_0 \in [a, b] \) where \( f_n(x_0) \) is convergent, then \((f_n)\) converges uniformly. Moreover, the limit function \( f = \lim f_n \) is differentiable and satisfies \( f' = g \).

**Series of Functions**

(6.4.1) **Convergence of Series of Functions:** For each \( n \in \mathbb{N} \), let \( f_n \) and \( f \) be functions defined on a set \( A \subseteq \mathbb{R} \). The infinite series \( \sum f_n(x) \) **converges pointwise** on \( A \) to \( f(x) \) if the sequence \( s_n(x) \) of partial sums defined by \( s_n(x) = f_1(x) + f_2(x) + \ldots + f_n(x) \) converges pointwise to \( f(x) \). The series **converges uniformly** on \( A \) to \( f \) if the sequences \( s_n(x) \) converges uniformly on \( A \) to \( f(x) \).

(*) If have series in which functions \( f_n \) are continuous, then by the Algebraic Continuity Theorem the partial sums will be continuous as well.

(6.4.2) **Term by Term Continuity Theorem.** Let \( f_n \) be continuous functions defined on a set \( A \subseteq \mathbb{R} \), and assume that \( \sum f_n \) converges uniformly to a function \( f \). Then, \( f \) is continuous on \( A \). **Proof idea:** Apply Continuous Limit Theorem (6.2.6) to partial sums \( s_k = f_1 + f_2 + \ldots + f_k \).

(6.4.3) **Term by Term Differentiability Theorem.** Let \( f_n \) be differentiable functions defined on an interval \( A \), and assume that \( \sum f'_n(x) \) converges uniformly to a limit \( g(x) \) in \( A \). If there exists a point \( x_0 \in [a, b] \) where \( \sum f_n(x_0) \) converges, then the series \( \sum f_n(x) \) converges uniformly to a differentiable function \( f(x) \) satisfying \( f'(x) = g(x) \) on \( A \). In other words, \( f(x) = \sum f_n(x) \) and \( f'(x) = \sum f'_n(x) \). **Proof idea:** Apply the Stronger Differentiable Limit Theorem to the partial sums \( s_k = f_1 + f_2 + \ldots + f_k' \), and observe that the Algebraic Differentiability Theorem (5.2.4) implies that \( s_k' = f_1' + f_2' + \ldots + f_k' \).

(6.4.4) **Cauchy Criterion for Uniform Convergence of a Series.** A series \( \sum f_n \) converges uniformly on \( A \subseteq \mathbb{R} \) if and only if for every \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that \( |f_{m+1}(x) + f_{m+2}(x) + \ldots + f_n(x)| < \epsilon \) whenever \( n > m \geq N \) and \( x \in A \).

(6.4.5) **Weierstrass M-Test.** For each \( n \in \mathbb{N} \), let \( f_n \) be a function defined on a set \( A \subseteq \mathbb{R} \) and let \( M_n > 0 \) be a real number satisfying \( |f_n(x)| \leq M_n \) for all \( x \in A \). If \( \sum M_n \) converges, then \( \sum f_n \) converges uniformly on \( A \). **Proof idea:** Cauchy Criterion and the triangle inequality.

**Power Series:** functions of the form \( f(x) = \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \)
(6.5.1) If a power series \( \sum a_n x^n \) converges at some point \( x_0 \in \mathbb{R} \), then it converges absolutely for any \( x \) satisfying \( |x| < |x_0| \). **Proof Idea:** Since the series converges, then the sequence of terms is bounded (converges to 0). Using the hypothesis (if \( x \in \mathbb{R} : |x| < |x_0| \)), find series of \( M|x/x_0|^n \) to be geometric with ratio \( |x/x_0| < 1 \), so converges and thus by Comparison Test, converges absolutely.

(*) Implies that the set of points for which a given power series converges must necessarily be \{0\}, \( \mathbb{R} \), or a bounded interval centered around \( x = 0 \). \( R \) is referred to as the radius of convergence of a power series.

(6.5.2) If a power series \( \sum a_n x^n \) converges absolutely at a point \( x_0 \), then it converges uniformly on the closed interval [\(-c, c\)] where \( c = |x_0| \). **Proof Idea:** Application of the Weierstrass M-Test.

(*) if the power series \( g(x) = \sum a_n x^n \) converges conditionally at \( x = R \), then it is possible for it to diverge when \( x = -R \). Sample with \( R = 1: \sum (-1)^n x^n/n \).

(6.5.3) **Abel’s Lemma.** Let \( b_n \) satisfy \( b_1 \geq b_2 \geq b_3 \geq ... \geq 0 \), and let \( \sum a_n \) be a series for which the partial sums are bounded. In other words, assume that there exists \( A > 0 \) such that \( |a_1 + a_2 + ... + a_n| \leq A \) for all \( n \in \mathbb{N} \). Then for all \( n \in \mathbb{N} \), \( |a_1 b_1 + a_2 b_2 + ... + a_n b_n| \leq Ab_1 \).

(6.5.4) **Abel’s Theorem.** Let \( g(x) = \sum a_n x^n \) be a power series that converges at the point \( x = R > 0 \). Then the series converges uniformly on the interval \([0, R]\). (Similar result for \( x = -R \).)

(6.5.5) If a power series converges pointwise on the set \( A \subseteq \mathbb{R} \), then it converges uniformly on any compact set \( K \subseteq A \). **Proof idea:** Apply Abel’s Theorem (6.5.4) to the max and min of the compact set \( K \).

(*) Power series is continuous at every point at which it converges.

(6.5.6) If \( \sum a_n x^n \) converges for all \( x \in (-R, R) \), then the differentiated series \( \sum n a_n x^{n-1} \) converges at each \( x \in (-R, R) \) as well. Consequently, the convergence is uniform on compact sets contained in (-\( R \), \( R \)).

(*) Series can converge at endpoint, but differentiated series can diverge. Ex: \( \sum x^n/n \) at \( x = -1 \).

(6.5.7) Assume \( f(x) = \sum a_n x^n \) converges on an interval \( A \subseteq \mathbb{R} \). Then, the function \( f \) is continuous on \( A \) and differentiable on any open interval (-\( R \), \( R \)) \( \subseteq A \). Moreover, the derivative is given by \( f'(x) = \sum n a_n x^{n-1} \) and \( f \) is infinitely differentiable on (-\( R \), \( R \)), and the successive derivatives can be obtained via term by term differentiation of the appropriate series.
Results from psets:

4W:
- The limit of a sequence, if it exists, must be unique. First, assume \( \lim a_n = a \) and \( \lim a_n = b \), and proceed to show that \( a = b \).
- (Reverse Triangle Inequality): \( |a + b| \leq |a| + |b| \) \( \Rightarrow \) Inverse Triangle Inequality: \( |a - b| \geq |a| - |b| \).
- For sequences \((x_n), (y_n)\):
  - \((x_n)\) and \((y_n)\) divergent but \((x_n + y_n)\) convergent; \(x_n = n, y_n = -n\).
  - \((x_n)\) convergent and \((y_n)\) convergent, and \((x_n + y_n)\) converges; impossible by the ALT
  - \((b_n)\) convergent with \(b_n \neq 0 \ \forall \ n : (1/b_n)\) convergent; \(b_n = 1/n\)
  - unbounded \((a_n)\) and convergent \((b_n)\) and \((a_n - b_n)\) bounded; impossible
  - \((a_n), (b_n)\) such that \((a_n b_n)\) converges but \((b_n)\) does not; \((a_n) = 0, (b_n) = n\).

4F:
- (Squeeze Theorem): If \(x_n \leq y_n \leq z_n \ \forall \ n \in \mathbb{N}\) and \(\lim x_n = \lim z_n = L\), then \(\lim y_n = L\).
- (Cesaro Means): If \((x_n)\) is a convergent sequence, then the sequence given by the averages \(y_n = \frac{1}{n} \sum_{k=1}^{n} x_k\) also converges to the same limit. Note: it is possible for \((y_n)\) of averages to converge even if \((x_n)\) does not. Example: \(x_n = (-1)^n\)
- (Limit Superior): \(\limsup a_n = \lim_{n \to \infty} y_n\) where \(y_n = \sup \{a_k : k \geq n\}\)
  - \(y_k\) converges
  - \(\liminf a_n = \lim_{n \to \infty} x_n\) where \(x_n = \inf \{a_k : k \geq n\}\)
  - \(\liminf a_n \leq \limsup a_n\) for every bounded sequence.
    - Strict inequality when \(\liminf a_n = -1, \limsup a_n = 1\).
  - \(\liminf a_n = \limsup a_n\) if and only if \(\lim an\) exists, and all three values are equal.

5W:
- For \((a_n), (b_n)\) Cauchy, we have that:
  - \(c_n = |a_n - b_n|\) is Cauchy while \(c_n = (-1)^n a_n\) is not Cauchy.

5F:
- (Infinite product) \(\prod b_n = b_1 b_2 b_3 \ldots\)
  - Understood in terms of sequence of partial products \(p_m = \prod_{n=1}^{m} b_n = b_1 b_2 \ldots b_m\)
  - The sequence of partial products converges if and only if \(\sum a_n\) converges.
- If \(a_n > 0\) \(\land\) \(\lim (n a_n) = L \neq 0\), then \(\sum a_n\) diverges.
- Assume that \(a_n > 0\) \(\land\) \(\lim n^2 a_n\) exists. Then \(\sum a_n\) converges.
- For sequence \((a_n)\):
  - If \(\sum a_n\) converges absolutely, then \(\sum a_n^2\) converges absolutely
FALSE: If $\sum a_n$ converges and $(b_n)$ converges, then $\sum a_nb_n$ converges. Counterexample: $a_n = b_n = (-1)^n(\sqrt{n})^{-1}$

- If $\sum a_n$ converges conditionally, then $\sum n^2a_n$ diverges.

- (Ratio Test): Given series $\sum a_n$ with $a_n \neq 0$, the Ratio Test states that if $(a_n)$ satisfies $\lim |a_{n+1}/a_n| = r < 1$, then the series converges absolutely.