Contour Integration

Trigonometric Integrals

Idea is $\int_{\theta=0}^{2\pi}$ becomes $\int_{|z|=1}$ with

$z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ so $d\theta = -\frac{i}{z}dz$

and $\cos\theta = \dfrac{z + 1/z}{2}$ and $\sin\theta = \dfrac{z - 1/z}{2i}$

Example: $\displaystyle\int_0^{2\pi} (\sin\theta)^{2n} d\theta$

Soln 1: $\displaystyle I_n = \int_0^{2\pi} (\sin\theta)^{2n} d\theta$

$$\begin{aligned} I_{n+1} &= \int_0^{2\pi} \sin^2\theta \cdot (\sin\theta)^{2n} d\theta \\ &= \int_0^{2\pi} (1-\cos^2\theta)(\sin\theta)^{2n} d\theta \\ &= I_n - \int_0^{2\pi} \cos\theta \cdot (\sin\theta)^{2n} \cos\theta \, d\theta \end{aligned}$$

$\int$ by parts, get recurrence relation: $u = \cos\theta$, $dv = (\sin\theta)^{2n}\cos\theta$; $du = -\sin\theta \, d\theta$, $v = \dfrac{(\sin\theta)^{2n+1}}{2n+1}$

$$I_{n+1} = I_n - \cos\theta \cdot \frac{(\sin\theta)^{2n+1}}{2n+1}\Bigg|_0^{2\pi} - \int_0^{2\pi} \frac{(\sin\theta)^{2n+2}}{2n+1} d\theta$$

$$I_{n+1} = I_n - \frac{1}{2n+1} I_{n+1} \rightarrow I_{n+1} = \frac{2n+1}{2n+2} I_n$$

Now solve!

CONTOUR INTEGRATION & TRIG (CONTINUED)

We have $I_n = \int_0^{2\pi} (\sin\theta)^{2n}\, d\theta$

and $I_{n+1} = \frac{2n+1}{2n+2} I_n$, $I_0 = 2\pi$.

Thus $I_n = \frac{2n-1}{2n} I_{n-1} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} I_{n-2} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} I_{n-3}$

Continuing we find

$$I_n = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2} I_0$$

Let $m!! = m(m-2)(m-4)\cdots \begin{cases} 1 & \text{if } m \text{ odd} \\ 2 & \text{if } m \text{ even} \end{cases}$

$$I_n = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{(2n)!!}{(2n)!!} I_0$$

$$= \frac{(2n)!}{2^n n! \cdot 2^n n!} I_0 \quad \text{as } (2n)!! = (2n)(2n-2)(2n-4)\cdots 2 = 2^n \cdot n!$$

$$= \frac{1}{4^n} \frac{(2n)!}{n!\,n!} I_0 = \frac{2\pi}{4^n}\binom{2n}{n} = \frac{2\pi}{2^{2n}}\binom{2n}{n}$$

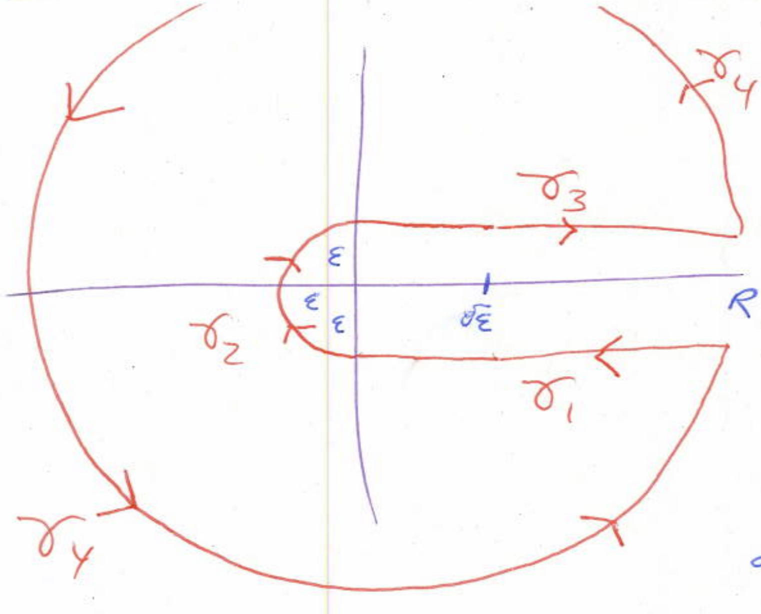
Alternate Method

$$I_n = \int_0^{2\pi} \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^{2n} d\theta$$

as $\int_0^{2\pi} e^{ik\theta}\, d\theta = \begin{cases} 2\pi & k=0 \\ 0 & k \neq 0 \end{cases}$

answer is just coeff of $e^{i0\theta}$, which is $\binom{2n}{n}\frac{(-1)^n}{2^{2n} i^{2n}} = \frac{1}{2^{2n}}\binom{2n}{n}$

BRANCH CUT: CONTOUR INTEGRATION



Want to find

$$\int_0^\infty \frac{dx}{1+x^3}$$

note denom is well behaved

idea is to exploit multi-valuedness of complex logarithm. Sadly

lots of technical details.

Key facts: $\lim_{u \to 0} u \log u = 0$

Proof: $u = 1/v$, $v \to \infty$ and apply L'Hopital

$$\lim_{u \to 0} u \log u = \lim_{v \to \infty} \frac{-\log v}{v} = \lim_{v \to \infty} \frac{-1/v}{1} = 0$$

Claim: Study integral of $\dfrac{\log z}{z^3+1}$

on γ_3, $z = x + i\varepsilon$ on γ_1, $z = x - i\varepsilon$

$r_x = (x^2+\varepsilon^2)^{1/2}$ so $\varepsilon \le r_x \le 2R$ for R big

For γ_1 and γ_3: let's integrate from $\sqrt{\varepsilon}$ to R, as contribution from 0 to $\sqrt{\varepsilon}$ is negligible

↳ To see this, note $0 \le x \le \sqrt{\varepsilon} \Rightarrow r_x \geq \varepsilon$ and $r_x \le 2\sqrt{\varepsilon}$

$$\int_0^{\sqrt{\varepsilon}} \frac{\log r_x + i\theta_x}{(x+i\varepsilon)^3+1}\, dx \le \frac{\sqrt{\varepsilon}\,(\log 2\sqrt{\varepsilon} + 2\pi)}{1/2} \to 0$$

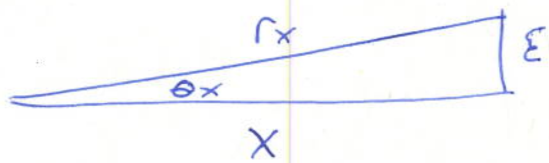
and similarly for $x - i\varepsilon$

Branch Cut: Continued

So there's no harm in having γ_1, γ_3 from $\sqrt{\varepsilon}$. Why do we do this? The problem is the "angle" in the logarithm:

$$\text{on } \gamma_1 : \quad \log(x+i\varepsilon) = \log r_x + i\theta_x$$

$$\text{on } \gamma_3 : \quad \log(x-i\varepsilon) = \log r_x + (2\pi - \theta_x)\, i$$



If $x \geq \sqrt{\varepsilon}$ then $\sin\theta_x \leq \frac{\varepsilon}{x} \leq \frac{\varepsilon}{\sqrt{\varepsilon}}$

so $\sin\theta_x \leq \sqrt{\varepsilon}$ so $\theta_x \leq 2\sqrt{\varepsilon}$

By "removing" small x we get an upper bound of θx.

We find

$$\int_{\gamma_1} \frac{\log z}{z^3+1}\, dz + \int_{\gamma_3} \frac{\log z}{z^3+1}\, dz$$

$$= \int_R^{\sqrt{\varepsilon}} \frac{\log r_x + (2\pi - \theta_x)\, i}{(x-i\varepsilon)^3+1}\, dx + \int_{\sqrt{\varepsilon}}^R \frac{\log r_x + i\theta x}{(x+i\varepsilon)^3+1}\, dx$$

The θx integrals are bounded by $\max_{\sqrt{\varepsilon} \leq x \leq R} \theta x \cdot \int_{\sqrt{\varepsilon}}^R \frac{dx}{(x-\varepsilon)^3+1}$;

as the x-integral is bounded indep of R and the max θx is at most $2\sqrt{\varepsilon}$, this tends to zero.

Next we see the $\log r_x$ pieces are negligible.

BRANCH CUT: CONTINUED

Studying integral of $\log r_x$:

↳ Remember $\varepsilon \le r_x \le 2R$

so $|\log r_x| \le |\log \varepsilon| + \log 2R$

have $\displaystyle\int_{\sqrt{\varepsilon}}^{R} \log r_x \left[\frac{1}{(x+i\varepsilon)^3+1} - \frac{1}{(x-i\varepsilon)^3+1}\right] dx$

idea is diff is small - a multiple of ε

$$\le \int_{\sqrt{\varepsilon}}^{R} \left[|\log \varepsilon| + \log 2R\right] \left| \frac{x^3 - 3i\varepsilon x^2 - 3\varepsilon^2 x + i\varepsilon^3 + 1 - x^3 - 3i\varepsilon x^2 + 3\varepsilon^2 x + i\varepsilon^3 + 1}{\left[(x+i\varepsilon)^3+1\right]\left[(x-i\varepsilon)^3+1\right]} \right| dx$$

$$\le \varepsilon\left[|\log \varepsilon| + \log 2R\right] \int_{\sqrt{\varepsilon}}^{R} \frac{2010(x^2+x+1)}{\left((x-\varepsilon)^3+1\right)^2} dx$$

(as worst case, ie denom biggest, when have $x - i\varepsilon = x - \varepsilon$; of course this can't happen, but gives upper bound).

$\le \varepsilon\left[|\log \varepsilon| + \log 2R\right] 2010!$ (as integral is bounded indep of ε and R)

So long as $\varepsilon \log 2R \to 0$, this goes to 0 (already showed $\varepsilon \log \varepsilon \to 0$).

See need $\varepsilon \log 2R \to 0$, say $\varepsilon = \frac{1}{R}$

BRANCH CUT: CONTINUED

Almost done with analysis of $\int_{\gamma_1+\gamma_3}$

Showed Θ_x terms don't contribute as $\varepsilon \to 0$

Showed $\log r_x$ terms " " " $\varepsilon \to 0, R \to \infty$

provided $\varepsilon \log 2R \to 0$

left with

$$\int_{R}^{\sqrt{\varepsilon}} \frac{2\pi i}{(x-i\varepsilon)^3+1}\,dx \xrightarrow[R\to\infty]{\varepsilon\to 0} -\int_0^\infty \frac{2\pi i\,dx}{x^3+1}$$

We are just left with $\int_{\gamma_4}$, but this isn't too bad as $\left|\frac{\log z}{z^3+1}\right|$, on γ_4, is at most $\frac{\log R + 2\pi}{(R^3-1)}$

The arc has length $< 2\pi R$, so contribution $\leq 2\pi R \cdot \frac{\log R + 2\pi}{R^3-1}$,

which tends to 0 as $R \to \infty$.

Now just use

$$\frac{1}{2\pi i} \oint_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} \frac{\log z\,dz}{z^3+1} = \sum \operatorname{Res}_{z_0}\left(\frac{\log z}{z^3+1}\right)$$

BRANCH CUT: RECAP

Let's recap:

There was a lot of book-keeping to do.

(1) Choose params ε and R

(2) For γ_1, γ_3 make $x \geq \sqrt{\varepsilon}$ to force $|\theta_x| \leq 2\sqrt{\varepsilon}$, which led to those integrals being negligible.

(3) The $\log r_x$ on γ_1 and γ_3 is negligible provided $\varepsilon \log 2R \to 0$. This is due to $\frac{1}{(x-i\varepsilon)^3+1}$ being close to $\frac{1}{(x+i\varepsilon)^3+1}$ as $\varepsilon \to 0$

(4) For γ_2, there is a negligible contribution as $\varepsilon \log \varepsilon \to 0$ as $\varepsilon \to 0$.

(5) For γ_4, there is negligible contribution as the integrand is essentially bounded by $\frac{\log R}{R^3-1}$; as the arc has length at most $2\pi R$, we see there is essentially no contribution for large R.

Contour Integration

Trigonometric Integrals

$$I_n = \int_0^{2\pi} (\sin\theta)^{2n}\, d\theta$$

$$= \oint_{|z|=1} \left(\frac{z - \frac{1}{z}}{2i}\right)^{2n} \left(\frac{-i}{z}\right) dz$$

$$= 2\pi i \, \frac{-i}{(2i)^{2n}} \, \frac{1}{2\pi i} \oint_{|z|=1} \left(z - \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

need residue, which comes from constant term of $\left(z - \frac{1}{z}\right)^{2n}$, which is $\binom{2n}{n}(-1)^n z^n \left(\frac{1}{z}\right)^n$

$$= \frac{2\pi}{2^{2n}(-1)^n} (-1)^n \binom{2n}{n} = \frac{2\pi}{2^{2n}} \binom{2n}{n}$$

Checks

$\underline{n=1}$: $\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{1}{2}\int_0 (\sin^2\theta + \cos^2\theta)\, d\theta = \frac{2\pi}{2}$ ✓

$\underline{n=2}$: $\int_0^{2\pi} \sin^4\theta\, d\theta = \int_0 (1 - \cos^2\theta)\sin^2\theta\, d\theta$

$$= \int_0^{2\pi} \sin^2\theta\, d\theta - \int_0^{2\pi} (\sin\theta\cos\theta)^2\, d\theta$$

$$= \frac{2\pi}{2} - \frac{1}{4}\int_0^{2\pi} (\sin(2\theta))^2\, d\theta$$

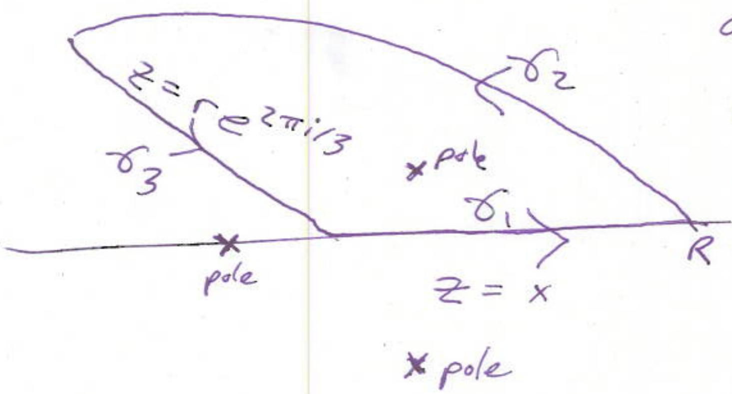
$$= \frac{2\pi}{2} - \frac{1}{4}\,\frac{1}{2}\int_0^{2\pi} \left(\sin(2\theta)^2 + \cos(2\theta)^2\right) d\theta$$

$$= \frac{2\pi}{2} - \frac{1}{8}2\pi = 2\pi\cdot\frac{3}{8}; \text{ as } \frac{1}{2^4}\binom{4}{2} = \frac{6}{16} = \frac{3}{8} \checkmark$$

CONTOUR INTEGRATION

We did $\int_0^\infty \frac{dx}{x^3+1}$ by using branch cuts. Could we do this another way? Note $(e^{2\pi i/3})^3 = 1$ and the poles of $\frac{1}{x^3+1}$ are at $e^{2\pi i/6}, e^{2\pi i 3/6}, e^{2\pi i 5/6}$

Consider contour



only pole inside is at $e^{2\pi i/6}$

How do we find residue?

Well, this is a simple zero of z^3+1, so

$$g(z) = z^3+1$$
$$= 0 + g'(e^{2\pi i/6})(z-e^{2\pi i/6}) + \frac{1}{2} g''(e^{2\pi i/6})(z-e^{2\pi i/6})^2 + \cdots$$

Thus $$\frac{1}{g(z)} = \frac{1}{g'(e^{2\pi i/6})(z-e^{2\pi i/6})} \cdot \frac{1}{1+\frac{1}{2}\frac{g''(e^{2\pi i/6})}{g'(e^{2\pi i/6})}(z-e^{2\pi i/6})+\cdots}$$

Thus residue is just $1/g'(e^{2\pi i/6}) = 1/3e^{4\pi i/6}$

(More generally, if $h(z)$ is holo at z_0 and $g(z)$ has a simple zero at z_0, then $Res_{z_0}(h/g)$ is $h(z_0)/g'(z_0)$.)

Thus $$\frac{1}{2\pi i}\oint_\gamma \frac{dz}{z^3+1} = \frac{1}{3e^{4\pi i/6}} = \frac{1}{3}e^{4\pi i/3}$$

CONTOUR INTEGRATION

Have $\frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z^3+1} = \frac{1}{3} e^{4\pi i/3}$

Clear $\oint_{\gamma_2} \frac{dz}{z^3+1} \xrightarrow[R\to\infty]{} 0$ (bounded by $\frac{\frac{1}{3} \cdot 2\pi R}{R^3-1}$)

Now $\frac{1}{2\pi i}\oint_{\gamma_1} \frac{dz}{z^3+1} = \frac{1}{2\pi i} \int_0^R \frac{dx}{x^3+1}$, goes to what we want

while $\frac{1}{2\pi i} \int_{\gamma_3} \frac{dz}{z^3+1} = \frac{1}{2\pi i} \int_R^0 \frac{e^{2\pi i/3}\, dr}{r^3+1}$

$$= -\frac{1}{2\pi i} \int_0^R \frac{e^{2\pi i/3}\, dx}{x^3+1} \qquad (r=x)$$

$$= \frac{1}{2\pi i} \int_0^R \frac{e^{2\pi i 5/6}\, dx}{x^3+1}$$

Thus $\frac{1}{2\pi i} \int_0^R \frac{1+e^{2\pi i 5/6}}{1+x^3}\, dx \longrightarrow \frac{1}{3} e^{4\pi i/3}$

so $\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi i}{3} \frac{e^{4\pi i/3}}{1+e^{2\pi i 5/6}}$ $\qquad i = e^{\pi i/2} = e^{2\pi i/4}$

$\frac{1}{4} + \frac{2}{3} = \frac{11}{12}$

$$= \frac{2\pi}{3} \frac{e^{2\pi i\, 11/12}}{1+e^{2\pi i 5/6}}$$

This is $\frac{2\pi}{\sqrt{3}}$: "clear" denominator with $1+e^{-2\pi i\, 10/12}$

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3} \frac{e^{2\pi i\, 11/12} + e^{2\pi i\, 1/12}}{1+2\mathrm{Re}(e^{2\pi i 5/6})+1}$$

Good! have a real number!

numerator is $2\mathrm{Re}(e^{2\pi i/12})$

$$= \frac{2\pi}{3} \frac{2\sqrt{3}/2}{3}$$

$$= \frac{2\pi}{3\sqrt{3}}$$