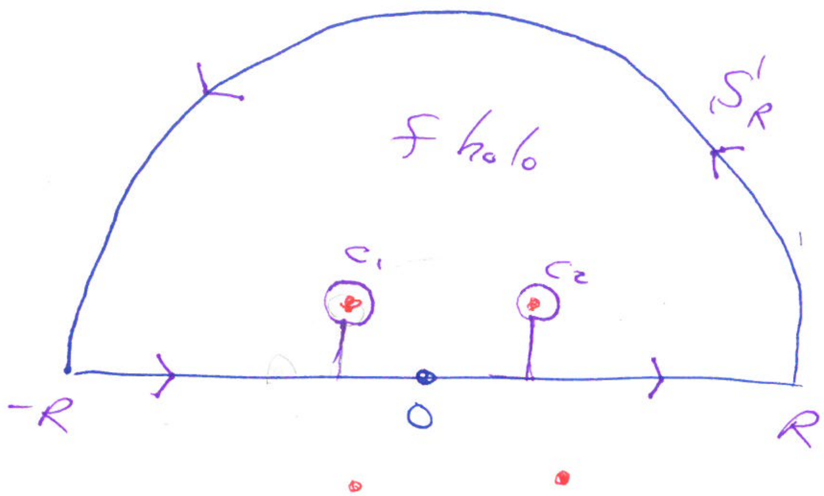
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$



$$f(z) = \frac{1}{1+z^4}$$

pole if $z^4+1=0$

or $z^4 = -1 = e^{i\pi}$

$$z \in \{e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}\}$$

$$0 = \frac{1}{2\pi i}\left[\int_{-R}^{R} \frac{dx}{1+x^4} + \int_{S'_R} \frac{dz}{1+z^4} + \int_{-c_1} \frac{dz}{1+z^4} + \int_{-c_2} \frac{dz}{1+z^4}\right]$$

On S_R: $z = Re^{i\theta}$ $\theta: 0 \to \pi$

$$\left|\int_{S_R} \frac{dz}{1+z^4}\right| \le \left|\int_{\theta=0}^{\pi} \frac{Rie^{i\theta}\,d\theta}{1+R^4e^{4i\theta}}\right|$$

$$\le \int_{\theta=0}^{\pi} \frac{R\,d\theta}{R^4-1} = \frac{R}{R^4-1}\pi \xrightarrow[R\to\infty]{} 0$$

$$|\text{Contr}| \le \pi R \cdot \frac{1}{R^4-1} \xrightarrow[R\to\infty]{} 0$$

$$\frac{1}{2\pi i}\oint_{C_1}\frac{dz}{1+z^4}$$

C_1 disk centered at $e^{i3\pi/4}$ of radius ε

$$\frac{1}{1+z^4} = \frac{1}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$

$$= \frac{1}{\underset{\displaystyle z_1}{\underset{\shortparallel}{z-e^{i3\pi/4}}}}\left[\underbrace{\frac{1}{(\quad)(\quad)(\quad)}}_{g(z_1)+g'(z_1)(z-z_1)+\frac{g''(z_1)}{2}(z-z_1)^2+\cdots}\right]$$

none vanish at $e^{i3\pi/4}$

$$= \frac{g(z)}{z-z_1} = \frac{g(z_1)}{z-z_1} + \underbrace{g'(z_1) + \frac{g''(z_1)}{2!}(z-z_1) + \cdots}_{\text{has a primitive}}$$

$$\frac{1}{2\pi i}\oint_{C_1}\frac{g(z)}{z-z_1}dz = \frac{1}{2\pi i}\oint_{C_1}\frac{g(z_1)}{z-z_1}dz + 0$$

$$= g(z_1)$$

DIGRESSION

$$\frac{1}{2\pi i}\oint_C \frac{1}{(z-z_0)^n}\,dz$$ circle radius ε about z_0

$$z = z_0 + \varepsilon e^{i\theta} \qquad dz = i\varepsilon e^{i\theta}d\theta$$

$$= \frac{1}{2\pi i}\int_{\theta=0}^{2\pi} \frac{1}{\varepsilon^n e^{in\theta}}\, i\varepsilon e^{i\theta}d\theta$$

$$= \frac{1}{2\pi}\varepsilon^{1-n}\int_{\theta=0}^{2\pi} \underbrace{e^{i(1-n)\theta}}_{\cos((1-n)\theta)+i\sin((1-n)\theta)}\,d\theta$$

$$= \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n\neq 1 \end{cases}$$

$$\frac{1}{2\pi i}\oint_{C_1}\sum_{n=-N}^{M} a_n (z-z_0)^n dz$$

$$= \frac{1}{2\pi i}\oint_{C_1} a_{-1}\frac{1}{z-z_0}\,dz + 0$$

$$= \frac{1}{2\pi i}\, 2\pi i\, a_{-1} = a_{-1}$$

Have 3rd piece $= \frac{1}{2\pi i}\int_{C_1} \frac{dz}{1+z^4}$ centred at $z_1 = e^{i3\pi/4}$

$g(z_1)$ which is

$$\frac{1}{\left(e^{i3\pi/4} - e^{i\pi/4}\right)\left(e^{i3\pi/4} - e^{i5\pi/4}\right)\left(e^{i3\pi/4} - e^{i\frac{7\pi}{4}}\right)}$$

Fourth piece $\frac{1}{2\pi i}\oint_{C_2} \frac{dz}{1+z^4}$ similar

$$\frac{1}{z - z_2} \quad \underbrace{\frac{1}{(z-z_1)(z-z_3)(z-z_4)}}_{h(z)}$$

$$\frac{1}{2\pi i}\oint_{C_2} \frac{dz}{1+z^4} = \frac{1}{2\pi i}\oint_{C_2} \frac{h(z_2) + h'(z_2)(z-z_2) + \cdots}{z - z_2}$$

$$= \frac{1}{2\pi i}\oint_{C_2} \frac{h(z_2)\,dz}{z - z_2}$$

$$= h(z_2)$$

$$= \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}$$

Put it together!

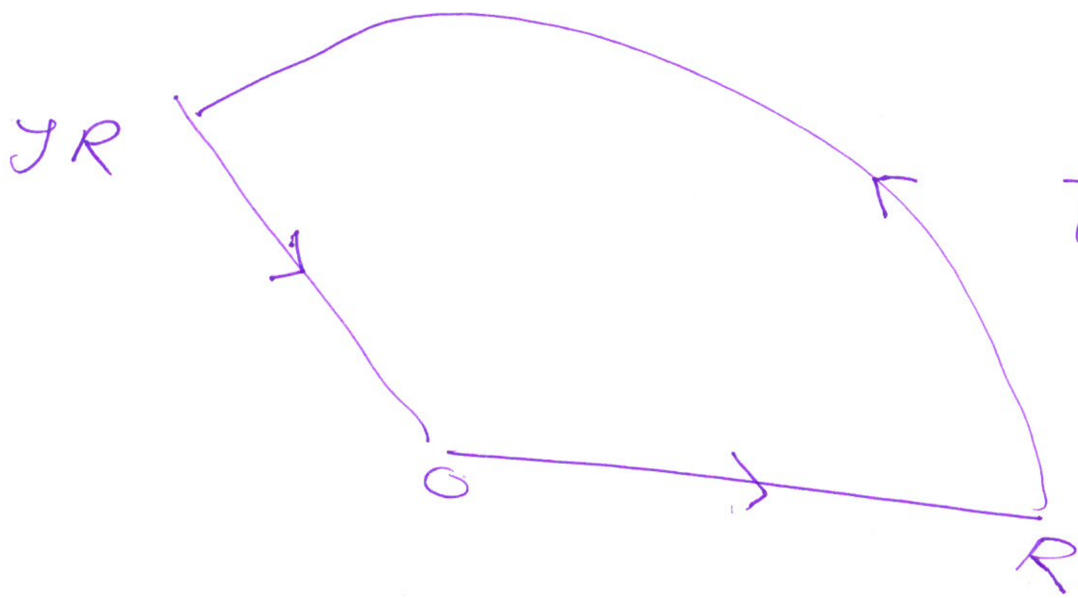
$$\frac{1}{2\pi i}\int_{-R}^{R}\frac{dx}{1+x^4} = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$+ \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}$$

$$+ \left(\text{Goes to } 0 \text{ as } R\to\infty\right)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$\zeta = e^{2\pi i/3} \qquad \zeta^3 = 1$$



$$\frac{1}{1+x^3}$$