Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 2: 11-1-21: https://youtu.be/yivEV2yhxgA

Lecture 19: 10/27/17: Riemann Mapping Theorem (Proof) https://youtu.be/x0Yy1Ivn1c4 (2015 Lecture)

Plan for the day: Lecture 2: November , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Review Montel's theorem / analysis results
- Sketch the proof of the Riemann Mapping Theorem
- Give examples of items in the proof
- Fill in most of the theoretical details

General items.

- Discuss differences b/w real and complex

Let Ω be an open subset of $\mathbb{C}$. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be **normal** if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω (the limit need not be in $\mathcal{F}$).

The family $\mathcal{F}$ is said to be **uniformly bounded on compact subsets of Ω** if for each compact set $K \subset \Omega$ there exists $B > 0$, such that

$$|f(z)| \leq B_K \quad \text{for all } z \in K \text{ and } f \in \mathcal{F}.$$

Also, the family $\mathcal{F}$ is **equicontinuous** on a compact set K if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $z, w \in K$ and $|z - w| < \delta$, then

$$|f(z) - f(w)| < \epsilon \qquad \text{for all } f \in \mathcal{F}.$$

Montel's Theorem:

Theorem 3.3 *Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then:*

(i) *$\mathcal{F}$ is equicontinuous on every compact subset of Ω.*

(ii) *$\mathcal{F}$ is a normal family.*

Proposition 3.5 *If Ω is a connected open subset of $\mathbb{C}$ and $\{f_n\}$ a sequence of injective holomorphic functions on Ω that converges uniformly on every compact subset of Ω to a holomorphic function f, then f is either injective or constant.*

$f_n : \mathbb{D} \to \mathbb{D}$

$f_n(z) = z/n \qquad \lim_{n\to\infty} f_n(z) = 0$

$f_n(z) = z + 1/n$

range issues

$f_n(z) = \dfrac{z + 1/n}{100{,}000} \qquad \lim_{n\to\infty} f_n(z) = \dfrac{z}{100{,}000}$

Assume not injective, not constant

Proof: Assume limit is not constant & [Assume] z_1, z_2 st $f(z_1) = f(z_2)$

$g_n(z) = f_n(z) - f_n(z_1)$: zero at z_1 (f_n is 1-1)

$g(z) = f(z) - f(z_1)$: zero at z_1 and z_2 (by assumption)

$\dfrac{1}{2\pi i} \oint_\gamma \dfrac{g_n'(z)}{g_n(z)}\, dz = 0$ same as $n \to \infty$ $\lim \int = \int \lim$ $\dfrac{1}{2\pi i} \oint_\gamma \dfrac{g'(z)}{g(z)}\, dz = 1$

by accumulation no other zeros if γ has a small radius

What about real case?

Seq $f_n(x)$ real analytic, injective, converges uniformly on compact sets

Must $\lim f_n$ be either constant or injective?

Try $\frac{1}{n} z + \left(1 - \frac{1}{n}\right) z^2 = f_n(z)$

Goes from z to z^2 so maybe....

$\frac{1}{n} x + \left(1 - \frac{1}{n}\right) x^2 = f_n(x)$

if works for real, works for complex

function real analytic but not complex differentiable...

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

$g^{(n)}(0) = 0$ (L'Hopital)

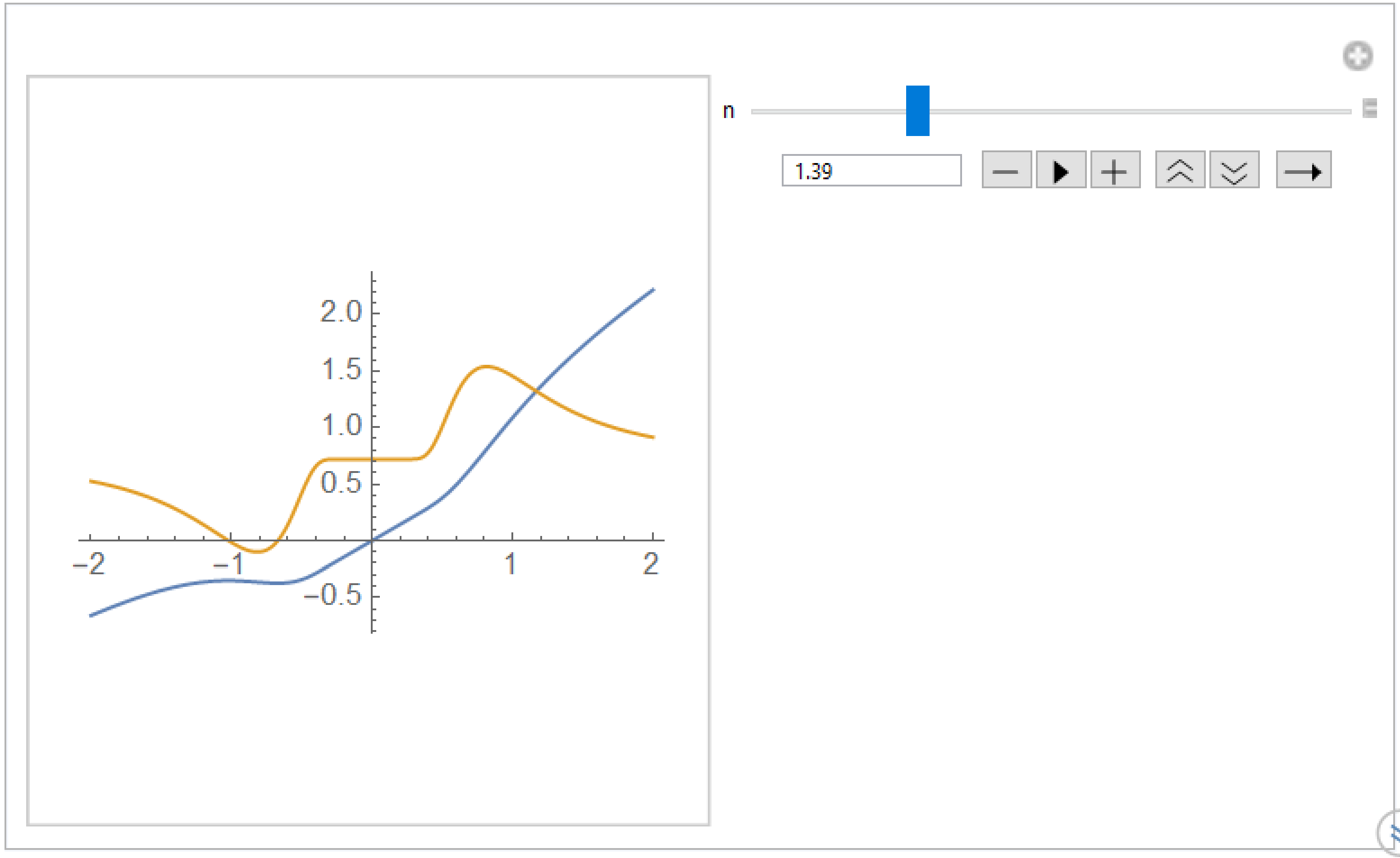
Cannot use if real analytic, can use if just say differentiable

$g(x)$ not 1-1

$g_n(x) =$  $+$  $\xrightarrow[n \to \infty]{} g(x)$

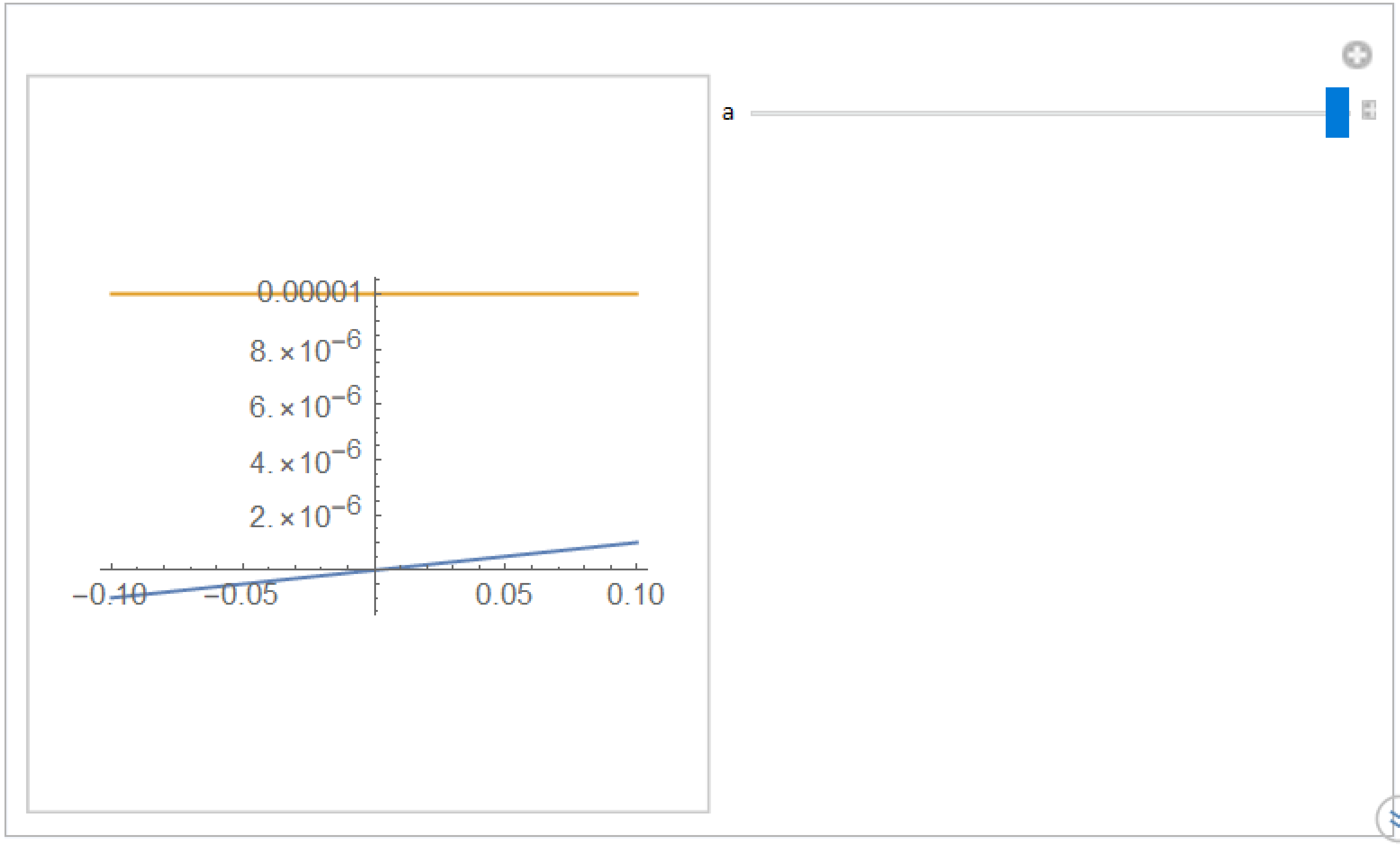
```
In[83]:= f[x_] := If[x ≠ 0, Exp[-1/x^2], 0]
         g[x_, n_] := x/n + f[x]
         dg[x_, n_] := If[x ≠ 0, 1/n + (2/x^3) Exp[-1/x^2], 0]
         Manipulate[Plot[{g[x, n], dg[x, n]} , {x, -2, 2}],
          {n, .05, 5}]
```

Out[86]=



Injective at the key value of n

```
A[x_, a_] := If[x ≠ 0, Exp[-1 / (a^2 + x^2)], Exp[-1 / a^2]]
B[x_, a_] := a x + f[x, a];
dB[x_, a_] :=
  a + If[x ≠ 8989, ((2 x) / (a^2 + x^2)^2) Exp[-1 / x^2], 0];
Manipulate[Plot[{B[x, a], dB[x, a]}, {x, -.1, .1}],
 {a, 1, .00001}]
```



$a = 1/n$

as $n \to \infty$

have $a \to 0$

$B_a(x) \to \begin{cases} e^{-1/x^2} \\ 0 \end{cases}$

Theorem 3.1 (Riemann mapping theorem) *Suppose Ω is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that*

$$F(z_0) = 0 \quad \text{and} \quad F'(z_0) > 0.$$

Corollary 3.2 *Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.*

Step 1. Use logarithm to say wlog map from disk to disk.

Step 2. Use Montel to get a map with maximal derivative at origin.

Step 3. Show it is conformal (if not contradicts maximality).

$\Omega \subset \mathbb{D}$

$f_* : \Omega \to \mathbb{D}$

$f_*(f_*(z))$

Step 1: use logarithm

Ω open, simply connected, not all of $\mathbb{C}$

wlog, $0 \notin \Omega$

$z \in \Omega$ $\quad f(z)$

$w \in \Omega$ $\quad f(w)$

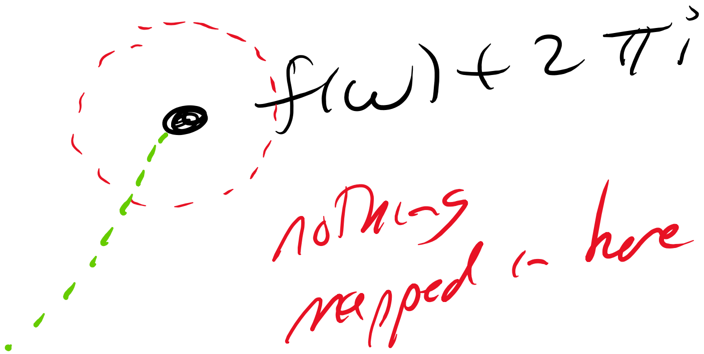
Claim $f(z) \neq f(w) + 2\pi i$

$f(z) = \log z$

$e^{f(z)} = e^{\log z} = z$

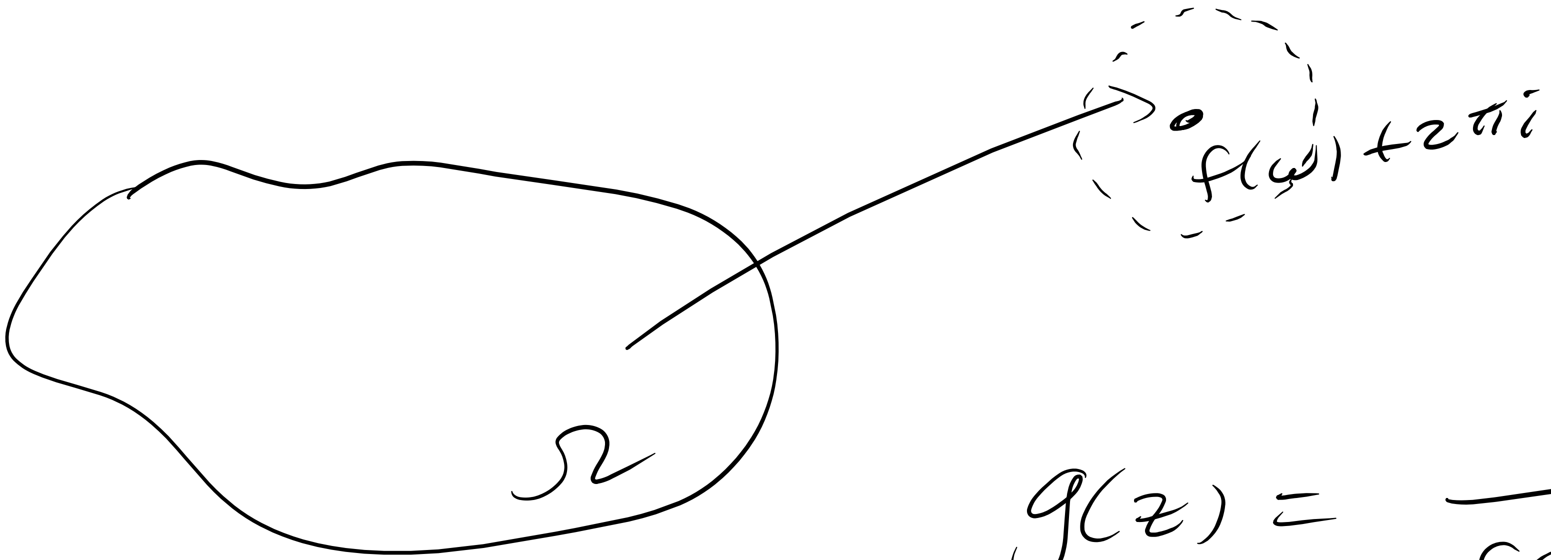
$e^{f(z)} = e^{f(z) + 2\pi i} = e^{f(w)}$

$z = w$ violates 1-1



Fact

$z_n \to z_*$



$$g(z) = \frac{1}{f(z) - [f(w) + 2\pi i]}$$

$f(z) = 1/z$