# Math 383: Complex Analysis: Fall ‘21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 31: 12-3-21:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

# Plan for the day: Lecture 31: December 2, 2021:

**https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf**

- Eigenvalues of matrices
- Random Matrix Theory and L-functions

**General items.**

- The unreasonable effectiveness of mathematics in the natural sciences: https://www.maths.ed.ac.uk/~v1ranick/papers/wigner.pdf
  Eugene Wigner -- Reprinted from Communications in Pure and Applied Mathematics, Vol. 13, No. I (February 1960). New York: John Wiley & Sons, Inc. Copyright © 1960 by John Wiley & Sons, Inc.

# Eigenvalues of Matrices

Defns: $A$ is symmetric if $A = A^T$

$A$ is complex Hermitian if $A = \bar{A}^T$

ex: $\begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}$ real symmetric

$\begin{pmatrix} 3 & 4+i \\ 4-i & 2 \end{pmatrix}$ complex Hermitian

Defn: use $H$ for complex conjugate transpose: $\bar{A}^T = A^H$

$\vec{v} \in \mathbb{C}^n$ then $\|\vec{v}\|^2 = \vec{v}^H \vec{v}$

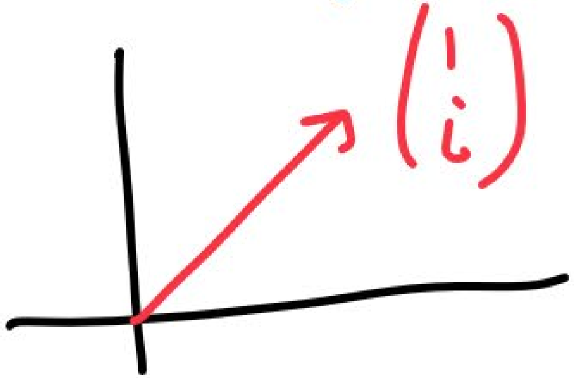
Note H important!

$\vec{v} = \begin{pmatrix}1\\ i\end{pmatrix}$ then $\vec{v}^T \vec{v} = (1 \ \ i)\begin{pmatrix}1\\ i\end{pmatrix} = 1^2 + i^2 = 0$

but $\vec{v} \neq \vec{0}$

However $\vec{v}^H \vec{v} = (1 \ \ -i)\begin{pmatrix}1\\ i\end{pmatrix} = 1^2 - i^2 = 2$

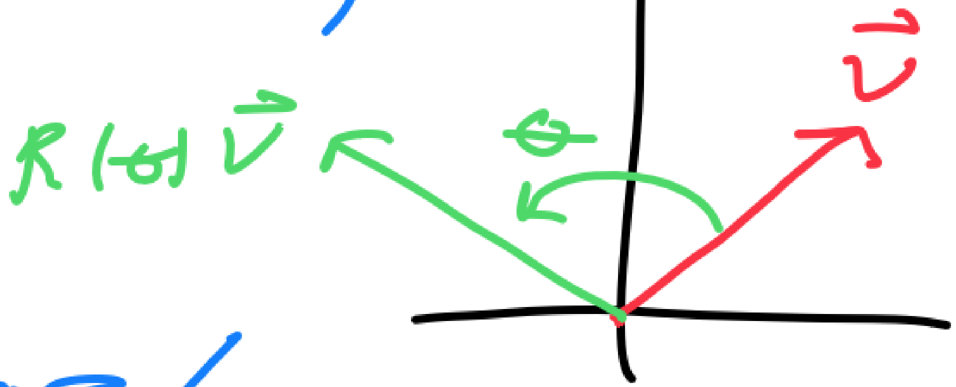
so $\|\vec{v}\| = \sqrt{2}$



Eigenvalues/Eigenvectors of real matrices not necessarily real.

$$A = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotates by $\theta$




Clearly no real eigenvectors/eigenvalues

Thm: If $A = A^H$ then eigenvalues real

Proof: Assume $A\vec{v} = \lambda \vec{v}$ with $\vec{v} \neq \vec{0}$

Consider $\vec{v}^H A \vec{v}$

It is $(\vec{v}^H A)\vec{v}$ and $\vec{v}^H(A\vec{v})$

$= (\vec{v}^H A^H)\vec{v}$ $\quad$ $= \vec{v}^H(\lambda\vec{v})$

$= (A\vec{v})^H \vec{v}$ $\quad$ $= \lambda \vec{v}^H \vec{v}$

$= (\lambda\vec{v}^H)\vec{v}$ $\quad$ as $\vec{v}^H\vec{v} \neq 0$,

$= \bar{\lambda}\vec{v}^H\vec{v}$ $\quad$ $\lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$

__Corollary__: Eigenvalues of real symm matrices are real.

Importance: can order.

Think energy levels

What other types of matrices?

Defn: $U$ is unitary if $U^H U = U U^H = I$

$Q$ is orthogonal if $Q^T Q = Q Q^T = I$

(let's take real)

Thm: Eigenvalues of unitary matrices are of the form $e^{i\theta}$

Proof: $U\vec{v} = \lambda \vec{v}$, $U^H U = I$, $\vec{v} \neq \vec{0}$

$$\|\vec{v}\|^2 = \vec{v}^H \vec{v} = \vec{v}^H U^H U \vec{v}$$

$$= (U\vec{v})^H U\vec{v} = (\lambda \vec{v})^H (\lambda \vec{v})$$

$$= \bar{\lambda} \lambda \vec{v}^H \vec{v} = |\lambda|^2 \|\vec{v}\|^2 \Rightarrow |\lambda| = 1$$

## Thm: Eigenvalue Trace Lemma:

$$Tr(A) = \sum_{k=1}^{N} \lambda_k(A)$$

$$Tr(A) = a_{11} + \cdots + a_{NN}$$

Proof Sketch: Trivial if $A$ is diagonal.

Trivial if $A$ is upper triangular.

Always have $S$ st $S^{-1}AS = T$

$$Tr(S^{-1}AS) = Tr(ASS^{-1}) = Tr(A) \quad \blacksquare$$

# Method of moments

have $\lambda_1, \ldots, \lambda_N$

$m^{th}$-moment: $\frac{1}{N} \sum_{k=1}^{N} \lambda_k^m$

If know moments, know $\lambda$'s as only finitely many.

"Pass to the limit..."

# From the Manhattan Project to Elliptic Curves

Steven J Miller
Dept of Math/Stats, Williams College

sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu
**http://www.williams.edu/Mathematics/sjmiller**

**MASON IV, March 7, 2020**

# Introduction

## Goals

- Determine correct scale and statistics to study eigenvalues and zeros of $L$-functions.
- See similar behavior in different systems.
- Discuss the tools and techniques needed to prove the results.
- Find sub-systems with interesting behavior!

## Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the $t_i$?

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Examples: Spacings between

- Energy Levels of Nuclei.
- Eigenvalues of Matrices.
- Zeros of *L*-functions.
- Summands in Zeckendorf Decompositions.
- Primes.
- $n^k \alpha \bmod 1$.

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## Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

## Classical Random Matrix Theory

With Olivia Beckwith, Leo Goldmakher, Chris Hammond, Steven Jackson, Cap Khoury, Murat Koloğlu, Gene Kopp, Victor Luo, Adam Massey, Eve Ninsuwan, Vincent Pham, Karen Shen, Jon Sinsheimer, Fred Strauch, Nicholas Triantafillou, Wentao Xiong

## Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

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Classical Mechanics: 3 Body Problem intractable.

Heavy nuclei (Uranium: $200+$ protons / neutrons) worse!

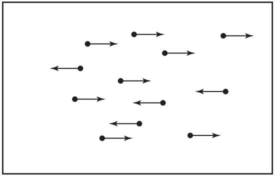
Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n = E_n\psi_n$$

$H$ : matrix, entries depend on system
$E_n$ : energy levels
$\psi_n$ : energy eigenfunctions

## Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

## Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix $p$, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of $A$.

## Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at $x_0$:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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To each $A$, attach a probability measure:

$$\mu_{A,N}(x) \quad = \quad \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$\begin{aligned} \mu_{A,N}(x) &= \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) \\ \int_a^b \mu_{A,N}(x)dx &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N} \end{aligned}$$

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\text{k}^{\text{th}}\text{ moment} &= \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.
\end{aligned}$$

# Wigner's Semi-Circle Law

## Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all $A$, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

## SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of $A$, but choose the matrix elements randomly and independently.

### Eigenvalue Trace Lemma

Let $A$ be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

## SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\mathrm{Ave}(\lambda_i(A)^2) \sim N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

# SKETCH OF PROOF: Averaging Formula

Recall $k$-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average $k$-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of $k$-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

## SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ji}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \quad \cdot \prod_{\substack{(k,l)\neq(i,j) \\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} \;=\; 1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

## SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

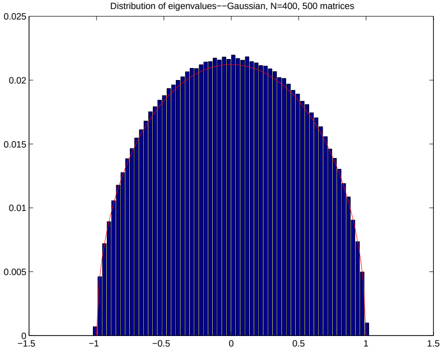
$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Main contribution when the $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).

*Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices* (with Adam Massey and John Sinsheimer), Journal of Theoretical Probability **20** (2007), no. 3, 637–662.

http://arxiv.org/abs/math/0512146

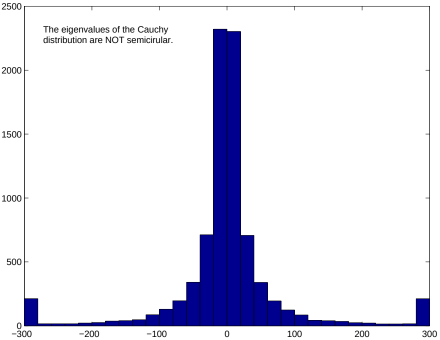
# Numerical examples




500 Matrices: Gaussian $400 \times 400$

$$p(x) \;=\; \frac{1}{\sqrt{2\pi}}\; e^{-x^2/2}$$

## Numerical examples




Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

I. Zakharevich, *A generalization of Wigner's law*, Comm. Math. Phys. **268** (2006), no. 2, 403–414.

http://web.williams.edu/Mathematics/sjmiller/public_html/book/papers/innaz.pdf

## Block Circulant Ensemble

With Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong.

## The Ensemble of $m$-Block Circulant Matrices

Symmetric matrices periodic with period $m$ on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & {\color{red}c_2} & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & {\color{blue}d_2} & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & {\color{red}c_2} & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & {\color{blue}d_2} & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & {\color{red}c_2} & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & {\color{blue}d_2} \\
\hline
{\color{red}c_2} & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & {\color{blue}d_2} & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right).$$

Choose distinct entries i.i.d.r.v.

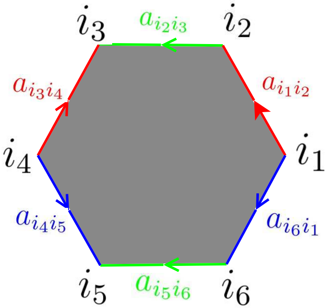
## Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as $m$ stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$\begin{aligned} M_n(N) &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \dots, i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}. \end{aligned}$$

where the sum is over oriented matchings on the edges $\{(1,2),(2,3),\dots,(n,1)\}$ of a regular $n$-gon.

## Oriented Matchings and Dualization

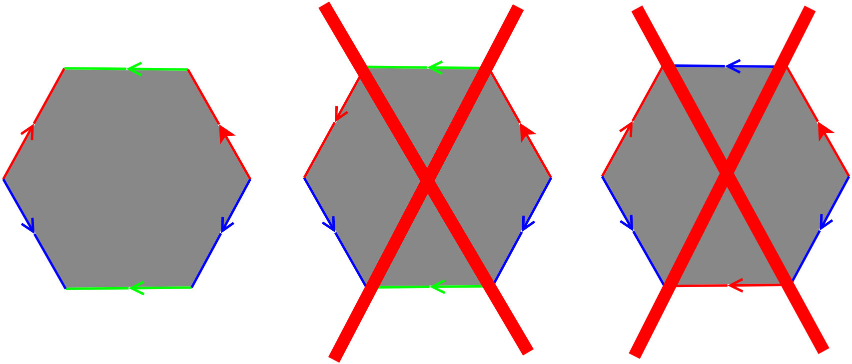



$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right)$$

**Figure:** An oriented matching in the expansion for $M_n(N) = M_6(8)$.

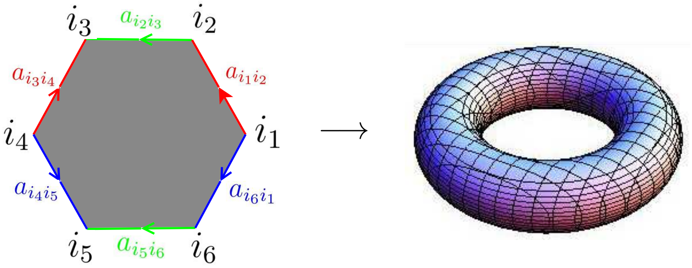
## Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



## Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.




Contribution from such a pairing is $m^{-2g}$, where $g$ is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

## Computing the Even Moments

### Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus $g$ surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

## Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} \ = \ (2k-1)!!\, c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} \ = \ \left(\frac{1+x}{1-x}\right)^r.$$

Thus, we write

$$M_{2k} \ = \ m^{-(k+1)} (2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\begin{aligned}
\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k,m) \\
&= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left( \left( \frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z} \\
&= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{1}{(\ell-1)!} \left( \frac{-t^2}{m} \right)^{\ell-1}.
\end{aligned}$$

## Results

Fourier transform and algebra yields

**Theorem: Kologlu, Kopp and Miller**

The limiting spectral density function $f_m(x)$ of the real symmetric $m$-block circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

# Results (continued)

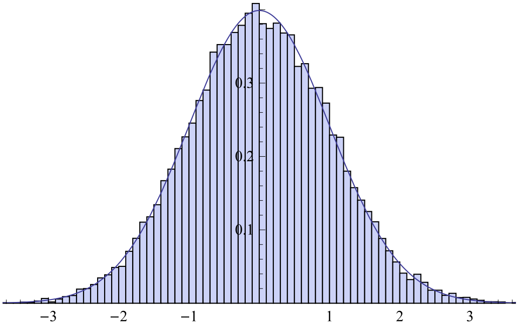



**Figure:** Plot for $f_1$ and histogram of eigenvalues of 100 circulant matrices of size $400 \times 400$.

## Results (continued)

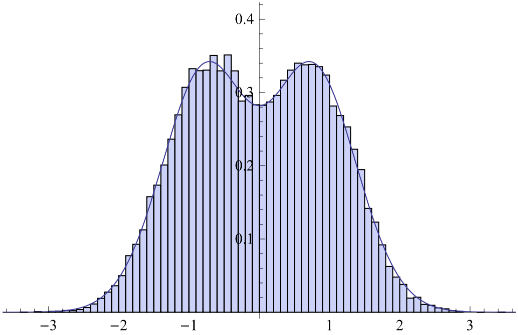



**Figure:** Plot for $f_2$ and histogram of eigenvalues of 100 2-block circulant matrices of size $400 \times 400$.

## Results (continued)

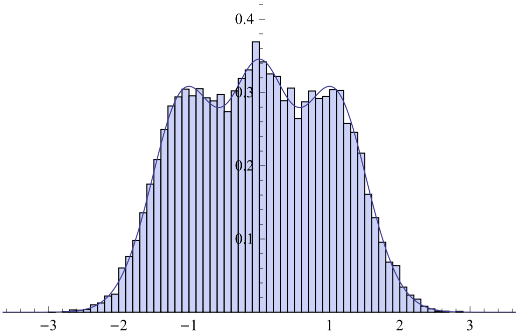



**Figure:** Plot for $f_3$ and histogram of eigenvalues of 100 3-block circulant matrices of size $402 \times 402$.

## Results (continued)

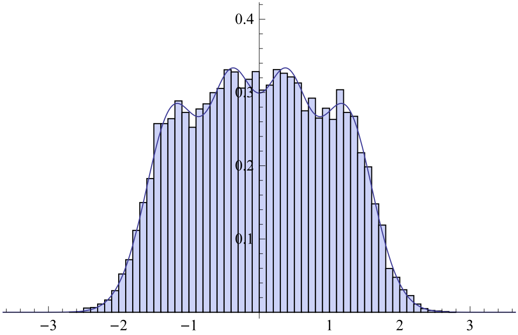



**Figure:** Plot for $f_4$ and histogram of eigenvalues of 100 4-block circulant matrices of size $400 \times 400$.

## Results (continued)

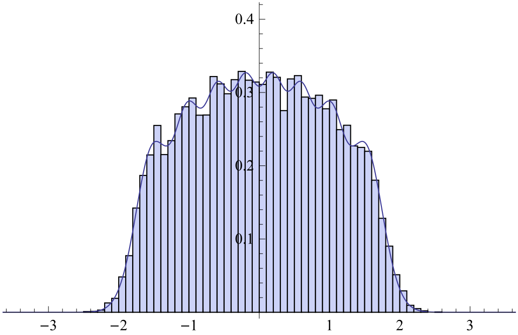



**Figure:** Plot for $f_8$ and histogram of eigenvalues of 100 8-block circulant matrices of size $400 \times 400$.

# Results (continued)

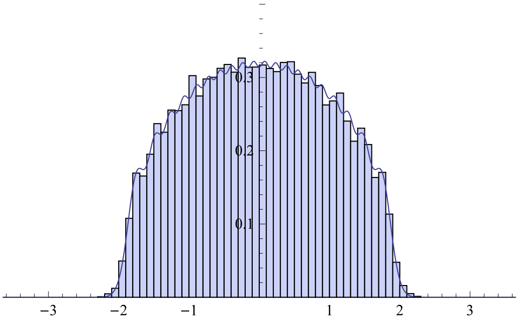



**Figure:** Plot for $f_{20}$ and histogram of eigenvalues of 100 20-block circulant matrices of size $400 \times 400$.

## Results (continued)

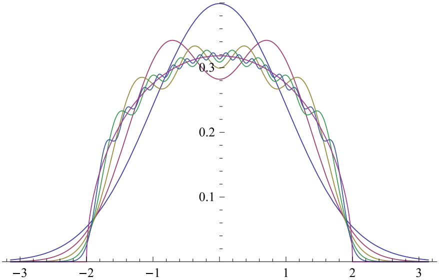



**Figure:** Plot of convergence to the semi-circle.

*The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices* (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

## Other Ensembles: $N \times N$ $(k, w)$-checkerboard ensemble

Matrices $M = (m_{ij}) = M^T$ with $a_{ij}$ iidrv, mean 0, variance 1, finite higher moments, $w$ fixed and

$$m_{ij} = \begin{cases} a_{ij} & \text{if } i \not\equiv j \bmod k \\ w & \text{if } i \equiv j \bmod k. \end{cases}$$

Example: $(3, w)$-checkerboard matrix:

$$\begin{pmatrix} w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\ a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\ a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w \end{pmatrix}$$

# Split Eigenvalue Distribution: Checkerboard and Generalizations

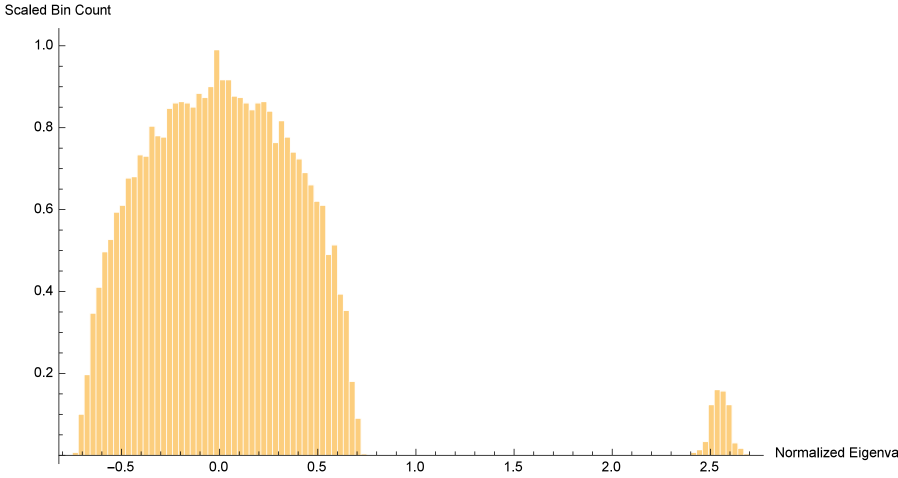



**Figure:** Histogram of normalized eigenvalues for 100 100 × 100

## Split Eigenvalue Distribution: Checkerboard and Generalizations

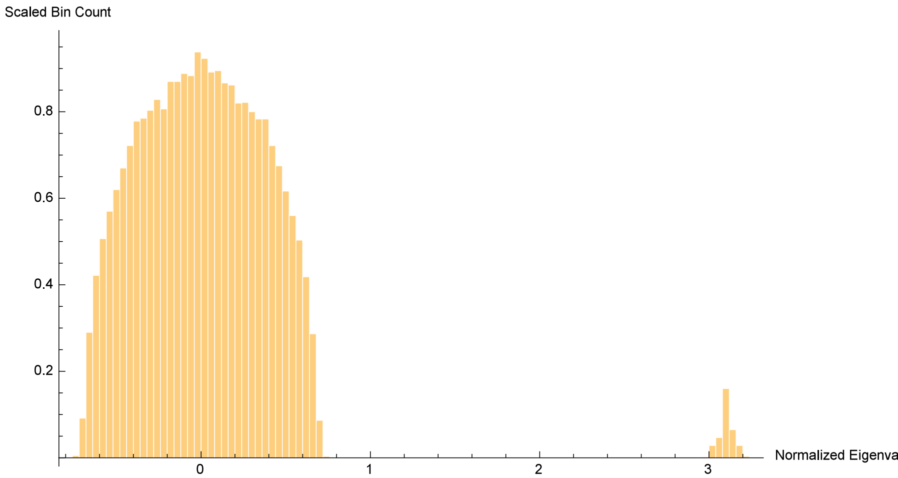



**Figure:** Histogram of normalized eigenvalues for 100 150 × 150

# Split Eigenvalue Distribution: Checkerboard and Generalizations

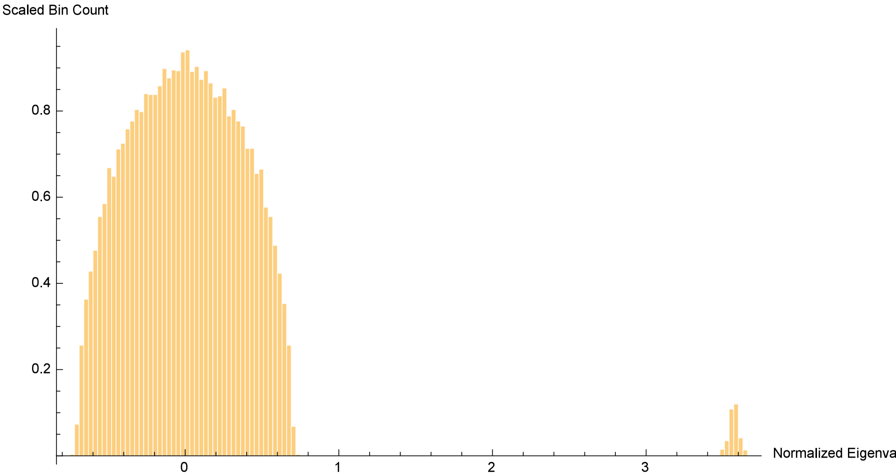



**Figure:** Histogram of normalized eigenvalues for 100 200 × 200

# Split Eigenvalue Distribution: Checkerboard and Generalizations


**Figure:** Histogram of normalized eigenvalues for 100 250 × 250

# Split Eigenvalue Distribution: Checkerboard and Generalizations

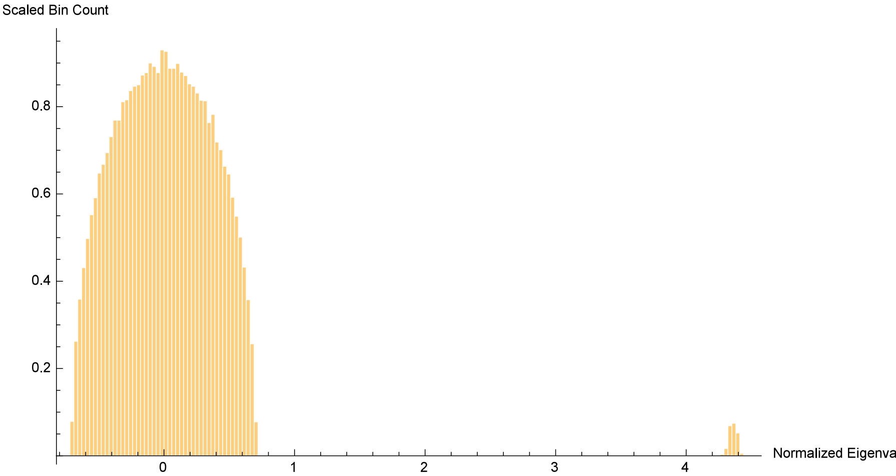



**Figure:** Histogram of normalized eigenvalues for 100 300 $\times$ 300

# Split Eigenvalue Distribution: Checkerboard and Generalizations

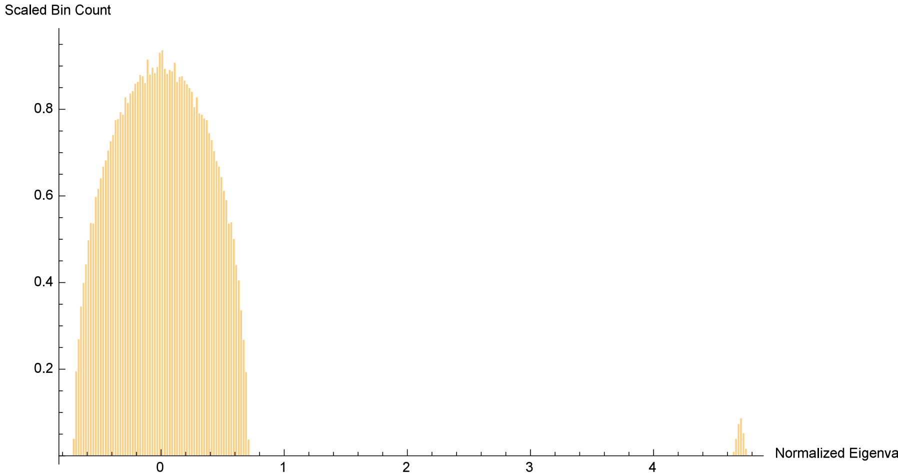



**Figure:** Histogram of normalized eigenvalues for 100 $350 \times 350$

## Split Eigenvalue Distribution: Checkerboard and Generalizations

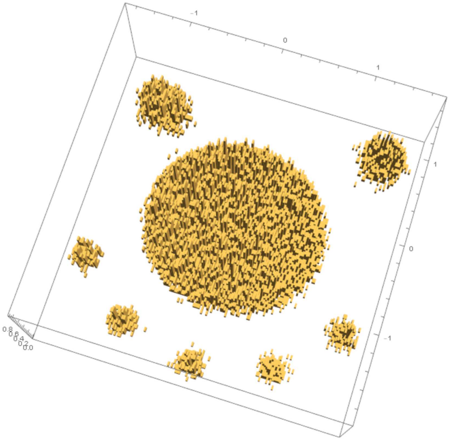



**Figure:** Complex Generalization I.

# Split Eigenvalue Distribution: Checkerboard and Generalizations

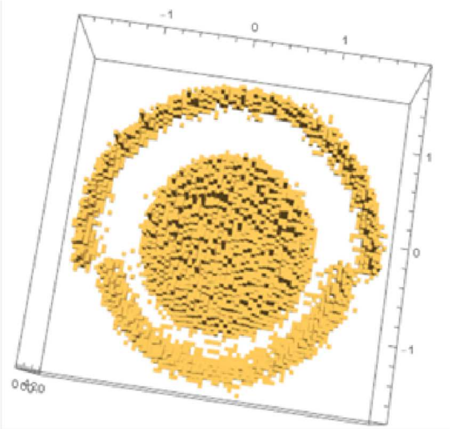



**Figure:** Complex Generalization II.

# Introduction to $L$-Functions

# Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right]\cdots$$

$$= \sum_n \frac{1}{n^s}.$$

## Riemann Zeta Function (cont)

$$\begin{aligned} \zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \leq x\} \end{aligned}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

## Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

**Functional Equation:**

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

**Riemann Hypothesis (RH):**

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

**Observation:** Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

## General $L$-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

**Functional Equation:**

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f) L(s,f) \;=\; \Lambda(1-s,f).$$

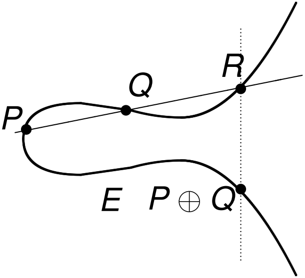
**Generalized Riemann Hypothesis (RH):**

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

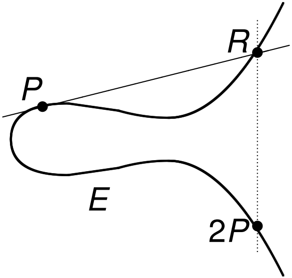
**Observation:** Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

## Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.




Addition of distinct points $P$ and $Q$




Adding a point $P$ to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

## Elliptic curve $L$-function

$E: y^2 = x^3 + ax + b$, associate $L$-function

$$L(s, E) \ = \ \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \ = \ \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

# Elliptic curve *L*-function

$E : y^2 = x^3 + ax + b$, associate $L$-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

**Birch and Swinnerton-Dyer Conjecture**

Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$.

## Properties of zeros of $L$-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from $L$-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

## Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

# Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \quad = \quad -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \;=\; -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

## Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p\log\left(1-p^{-s}\right) \\
&= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

# Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
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\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.
$$

## Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p\log\left(1-p^{-s}\right)$$

$$= \sum_p\frac{\log p\cdot p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p\log p\int\phi(s)p^{-s}ds.$$

# Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
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&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.
$$

**Knowledge of zeros gives info on coefficients.**

## Explicit Formula: Example

Dirichlet *L*-functions: Let $\phi$ be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet *L*-function from a non-trivial character $\chi$ with conductor $m$ and zeros $\rho = \frac{1}{2} + i\gamma_\chi$. Then

$$\begin{aligned}\sum_\rho \phi\left(\gamma_\chi \frac{\log(m/\pi)}{2\pi}\right) &= \int_{-\infty}^{\infty} \phi(y)dy \\ &-2\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)\frac{\chi(p)}{p^{1/2}} \\ &-2\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)}{p} + O\left(\frac{1}{\log m}\right).\end{aligned}$$

# Katz-Sarnak
# Density Conjectures

## Measures of Spacings: $n$-Level Density and Families

Let $g_i$ be even Schwartz functions whose Fourier Transform is compactly supported, $L(s,f)$ an $L$-function with zeros $\frac{1}{2} + i\gamma_f$ and conductor $Q_f$:

$$D_{n,f}(g) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

- Properties of $n$-level density:
  - ◇ Individual zeros contribute in limit
  - ◇ Most of contribution is from low zeros
  - ◇ Average over similar $L$-functions (family)

## $n$-Level Density

$n$-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of $L$-functions ordered by conductors, $g_k$ an even Schwartz function: $D_{n,\mathcal{F}}(g) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right) \cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, $n$-level density converges to

$$\int g(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} \;=\; \int \widehat{g}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

**Conjecture (Katz-Sarnak)**

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

## 1-Level Densities

Let $\mathcal{G}$ be one of the classical compact groups: Unitary, Symplectic, Orthogonal (or SO(even), SO(odd)).
If $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$, 1-level density of $\mathcal{G}$ is

$$\widehat{g}(0) \quad - \quad c_{\mathcal{G}} \, \frac{g(0)}{2},$$

where

$$c_{\mathcal{G}} \;=\; \begin{cases} 0 & \mathcal{G} \text{ is Unitary} \\ 1 & \mathcal{G} \text{ is Symplectic} \\ -1 & \mathcal{G} \text{ is Orthogonal.} \end{cases}$$

## Identifying the Symmetry Groups

- Often suggested by monodromy group in the function field.
- Tools: Explicit Formula, Summation Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation: **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

*The low lying zeros of a* $GL(4)$ *and a* $GL(6)$ *family of L-functions* (with Eduardo Dueñez), Compositio Mathematica **142** (2006), no. 6, 1403–1425.

http://arxiv.org/abs/math/0506462

## Explicit Formula

- $\pi$: cuspidal automorphic representation on $\mathrm{GL}_n$.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake params: $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n \left(1 - \alpha_{\pi,i}(p)p^{-s}\right)^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu} \widehat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

## Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are $\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} \ = \ \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}$.

**Theorem (Dueñez-Miller)**

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of $L$-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Breaks analysis of compound families into simple ones.

*The effect of convolving families of L-functions on the underlying group symmetries* (with Eduardo Dueñez), Proceedings of the London Mathematical Society, 2009; doi: 10.1112/plms/pdp018.

http://arxiv.org/pdf/math/0607688.pdf

## 1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\nu^{\text{th}} \text{ moment} : \lambda_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu$$

$$S_1(\mathcal{F}) = -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right]$$

$$S_2(\mathcal{F}) = -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]$$

The corresponding classical compact group determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

## Takeaways

**Very similar to Central Limit Theorem.**

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.
- First moment zero save for families of elliptic curves.
- Higher moments control convergence and can depend on arithmetic of family.

## Correspondences

Similarities between $L$-Functions and Nuclei:

$$\begin{array}{rcl} \text{Zeros} & \longleftrightarrow & \text{Energy Levels} \\ \text{Schwartz test function} & \longrightarrow & \text{Neutron} \\ \text{Support of test function} & \longleftrightarrow & \text{Neutron Energy.} \end{array}$$

# Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.
2. Explicit Formula: Relates sums over zeros to sums over primes.
3. Averaging Formulas: Orthogonality of characters, Legendre Sums, Petersson Formula, Kuznestov Formula

## Applications of $n$-level density

Bounding the order of vanishing at the central point.
Average rank $\cdot\, \phi(0) \ \leq\ \int \phi(x) W_{G(\mathcal{F})}(x)dx$ if $\phi$ non-negative.

**Theorem (Miller, Hughes-Miller)**

*Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any $r$ there is a $c_r$ such that probability of at least $r$ zeros at the central point is at most $c_n r^{-n}$.*

Better results from 2-level than Iwaniec-Luo-Sarnak for $r \geq 5$.

*Low lying zeros of L–functions with orthogonal symmetry* (with Christopher Hughes), Duke Mathematical Journal **136** (2007), no. 1, 115–172. http://arxiv.org/abs/math/0507450

## Elliptic Curves: First Zero Above Central Point

Eduardo Dueñez, Duc Khim Huynh, Jon P. Keating and Nina Snaith

## Theoretical results: $y^2 = x^3 + A(T)x + B(T)$

### Theorem: M– '04

For small support, one-param family of rank $r$ over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

$$\text{where} \quad \mathcal{G} = \begin{cases} \mathrm{SO(odd)} & \text{if half odd} \\ \mathrm{SO(even)} & \text{if all even} \\ \text{weighted average} & \text{otherwise.} \end{cases}$$

**Supports Katz-Sarnak, B-SD, and Independent model in limit.**

Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

## Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank $r$ over $\mathbb{Q}(T)$.
Natural sub-families:

- Curves of rank $r$.
- Curves of rank $r + 2$.

## Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank $r$ over $\mathbb{Q}(T)$.
Natural sub-families:

- Curves of rank $r$.
- Curves of rank $r + 2$.

**Question:** Does the sub-family of rank $r + 2$ curves in a rank $r$ family behave like the sub-family of rank $r + 2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r + 2$?

## Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank $r$ one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r+2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

## Excess Rank

One-parameter family, rank $r$ over $\mathbb{Q}(T)$.
Density Conjecture (Generic Family) $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

| | | |
|---|---|---|
| Percent with rank r | $\approx$ | 32% |
| Percent with rank r+1 | $\approx$ | 48% |
| Percent with rank r+2 | $\approx$ | 18% |
| Percent with rank r+3 | $\approx$ | 2% |

Problem: small data sets, sub-families, convergence rate $\log$(conductor).

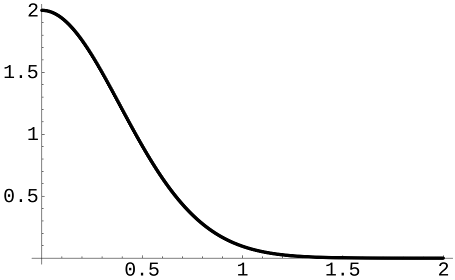
## Data on Excess Rank

$$y^2 + y = x^3 + Tx$$

Each set is 2000 curves, last has conductors of size $10^{17}$, (small on logarithmic scale).

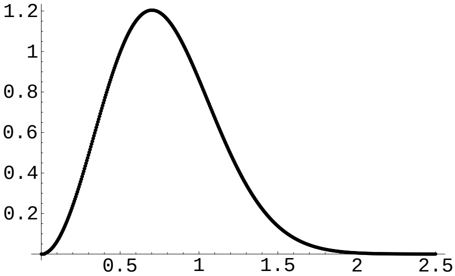
| $t$-Start | Rk 0 | Rk 1 | Rk 2 | Rk 3 | Time (hrs) |
|---|---|---|---|---|---|
| -1000 | 39.4 | 47.8 | 12.3 | 0.6 | <1 |
| 1000 | 38.4 | 47.3 | 13.6 | 0.6 | <1 |
| 4000 | 37.4 | 47.8 | 13.7 | 1.1 | 1 |
| 8000 | 37.3 | 48.8 | 12.9 | 1.0 | 2.5 |
| 24000 | 35.1 | 50.1 | 13.9 | 0.8 | 6.8 |
| 50000 | 36.7 | 48.3 | 13.8 | 1.2 | 51.8 |

## RMT: Theoretical Results ($N \to \infty$)



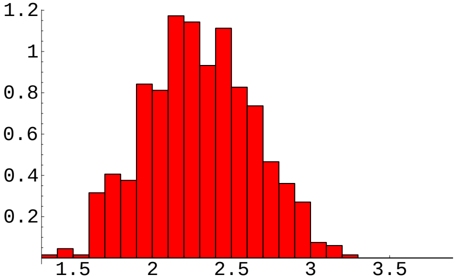

1st normalized evalue above 1: SO(even)

## RMT: Theoretical Results ($N \to \infty$)




1st normalized evalue above 1: SO(odd)

# Rank 2 Curves: 1st Norm. Zero above the Central Point

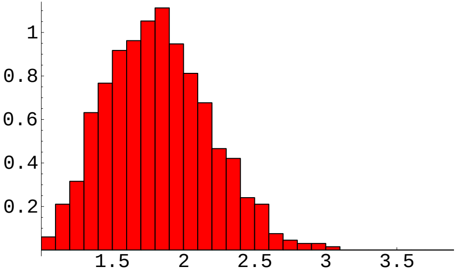



665 rank 2 curves from

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

## Rank 2 Curves: 1st Norm. Zero above the Central Point

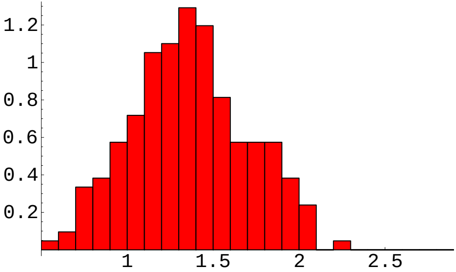



665 rank 2 curves from

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

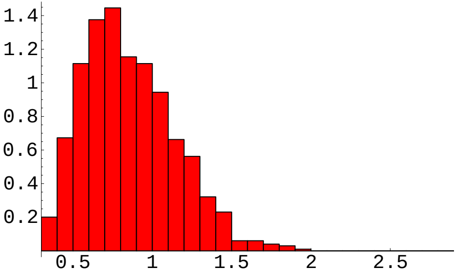
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

## Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0



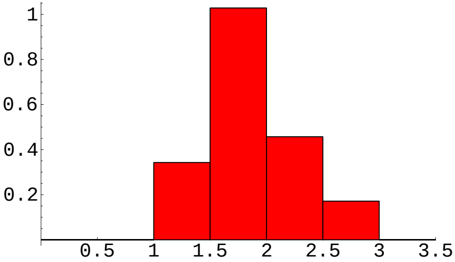

209 rank 0 curves from 14 rank 0 families,
$\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

## Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0



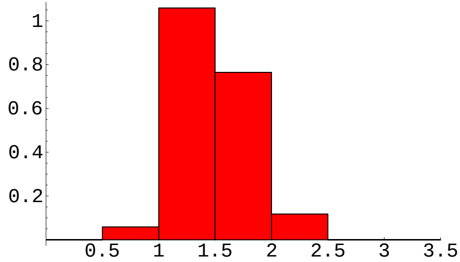

996 rank 0 curves from 14 rank 0 families,
$\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

## Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point




35 curves, $\log(\mathrm{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

## Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
## 1st Normalized Zero above Central Point




34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$,
$\sigma_\mu = .34$

## Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i.e., shifted by the same amount).

## Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of $j^{\text{th}}$ normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

| | 863 Rank 0 Curves | 701 Rank 2 Curves | t-Statistic |
|---|---|---|---|
| **Median** $z_2 - z_1$ | 1.28 | 1.30 | |
| **Mean** $z_2 - z_1$ | 1.30 | 1.34 | -1.60 |
| **StDev** $z_2 - z_1$ | 0.49 | 0.51 | |
| **Median** $z_3 - z_2$ | 1.22 | 1.19 | |
| **Mean** $z_3 - z_2$ | 1.24 | 1.22 | 0.80 |
| **StDev** $z_3 - z_2$ | 0.52 | 0.47 | |
| **Median** $z_3 - z_1$ | 2.54 | 2.56 | |
| **Mean** $z_3 - z_1$ | 2.55 | 2.56 | -0.38 |
| **StDev** $z_3 - z_1$ | 0.52 | 0.52 | |

# Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j$ = imaginary part of the $j^{\text{th}}$ norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

| | 64 Rank 2 Curves | 23 Rank 4 Curves | t-Statistic |
|---|---|---|---|
| **Median** $z_2 - z_1$ | 1.26 | 1.27 | |
| **Mean** $z_2 - z_1$ | 1.36 | 1.29 | 0.59 |
| **StDev** $z_2 - z_1$ | 0.50 | 0.42 | |
| **Median** $z_3 - z_2$ | 1.22 | 1.08 | |
| **Mean** $z_3 - z_2$ | 1.29 | 1.14 | 1.35 |
| **StDev** $z_3 - z_2$ | 0.49 | 0.35 | |
| **Median** $z_3 - z_1$ | 2.66 | 2.46 | |
| **Mean** $z_3 - z_1$ | 2.65 | 2.43 | 2.05 |
| **StDev** $z_3 - z_1$ | 0.44 | 0.42 | |

# Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j$ = imaginary part of the $j^{\text{th}}$ norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

| | **701 Rank 2 Curves** | **64 Rank 2 Curves** | **t-Statistic** |
|---|---|---|---|
| **Median** $z_2 - z_1$ | 1.30 | 1.26 | |
| **Mean** $z_2 - z_1$ | 1.34 | 1.36 | 0.69 |
| **StDev** $z_2 - z_1$ | 0.51 | 0.50 | |
| **Median** $z_3 - z_2$ | 1.19 | 1.22 | |
| **Mean** $z_3 - z_2$ | 1.22 | 1.29 | 1.39 |
| **StDev** $z_3 - z_2$ | 0.47 | 0.49 | |
| **Median** $z_3 - z_1$ | 2.56 | 2.66 | |
| **Mean** $z_3 - z_1$ | 2.56 | 2.65 | 1.93 |
| **StDev** $z_3 - z_1$ | 0.52 | 0.44 | |

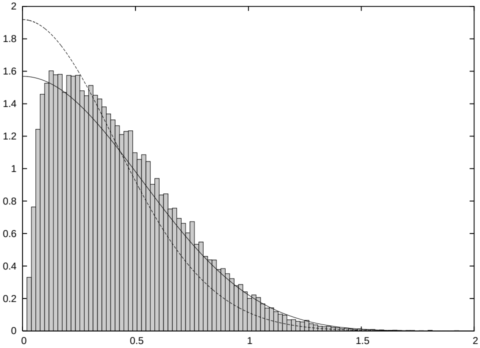
## New Model for Finite Conductors

- Replace conductor $N$ with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with $L$-function Ratios Conj.
  - ◇ Do the number theory computation.

- Excised Orthogonal Ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Char. polys $\Lambda_A(\theta) = \det(I - e^{i\theta}A^{-1})$ model $L(1/2 + it, E)$.
  - ◇ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(0)| \geq ce^N$.

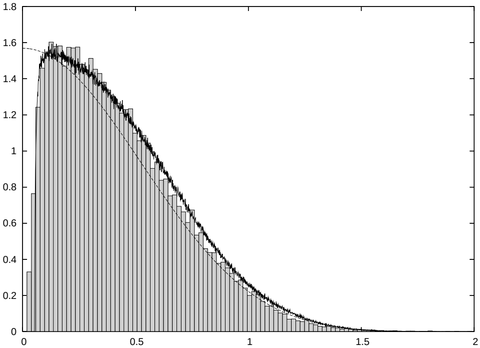
- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

## Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$




Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_{\mathrm{eff}}$ (solid), standard $N_0$ (dashed).

# Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400{,}000$

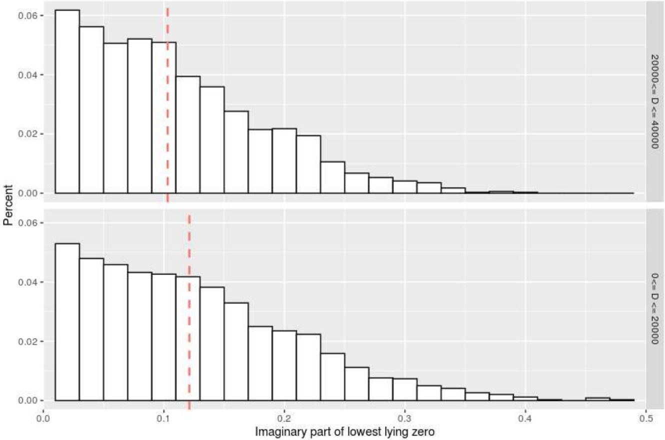



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): $N_{\text{eff}}$ = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

*The lowest eigenvalue of Jacobi Random Matrix Ensembles and Painlevé VI*, (with E. Dueñez, D. K. Huynh, J. Keating and N. Snaith), Journal of Physics A: Mathematical and Theoretical **43** (2010) 405204 (27pp). http://arxiv.org/pdf/1005.1298

*Models for zeros at the central point in families of elliptic curves* (with E. Dueñez, D. K. Huynh, J. Keating and N. Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp). http://arxiv.org/pdf/1107.4426

## Generalizations: With Owen Barrett and Nathan Ryan




Quadratic twists of a weight 3 level 8 form.

# Open Questions and References

## Open Questions: Low-lying zeros of *L*-functions

- Generalize excised ensembles for higher weight $\mathrm{GL}_2$ families where expect different discretizations.

- Obtain better estimates on vanishing at the central point by finding optimal test functions for the second and higher moment expansions.

- Further explore *L*-function Ratios Conjecture to predict lower order terms in families, compute these terms on number theory side.

See Dueñez-Huynh-Keating-Miller-Snaith, Miller, and the Ratios papers.

# Publications: Random Matrix Theory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   http://arxiv.org/abs/math/0312215
2. *Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices* (with Adam Massey and John Sinsheimer), Journal of Theoretical Probability **20** (2007), no. 3, 637–662.
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3. *The distribution of the second largest eigenvalue in families of random regular graphs* (with Tim Novikoff and Anthony Sabelli), Experimental Mathematics **17** (2008), no. 2, 231–244.
   http://arxiv.org/abs/math/0611649
4. *Nuclei, Primes and the Random Matrix Connection* (with Frank W. K. Firk), Symmetry **1** (2009), 64–105; doi:10.3390/sym1010064. http://arxiv.org/abs/0909.4914
5. *Distribution of eigenvalues for highly palindromic real symmetric Toeplitz matrices* (with Steven Jackson and Thuy Pham), Journal of Theoretical Probability **25** (2012), 464–495.
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6. *The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices* (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812
7. *Distribution of eigenvalues of weighted, structured matrix ensembles* (with Olivia Beckwith, Karen Shen), submitted December 2011 to the Journal of Theoretical Probability, revised September 2012.
   http://arxiv.org/abs/1112.3719 .
8. *The expected eigenvalue distribution of large, weighted $d$-regular graphs* (with Leo Goldmahker, Cap Khoury and Kesinee Ninsuwan), preprint.

# Publications: *L*-Functions


1. *The low lying zeros of a* $GL(4)$ *and a* $GL(6)$ *family of L-functions* (with Eduardo Dueñez), Compositio Mathematica **142** (2006), no. 6, 1403–1425. http://arxiv.org/abs/math/0506462
2. *Low lying zeros of L–functions with orthogonal symmetry* (with Christopher Hughes), Duke Mathematical Journal **136** (2007), no. 1, 115–172. http://arxiv.org/abs/math/0507450
3. *Lower order terms in the 1-level density for families of holomorphic cuspidal newforms*, Acta Arithmetica **137** (2009), 51–98. http://arxiv.org/abs/0704.0924
4. *The effect of convolving families of L-functions on the underlying group symmetries* (with Eduardo Dueñez), Proceedings of the London Mathematical Society, 2009; doi: 10.1112/plms/pdp018. http://arxiv.org/pdf/math/0607688.pdf
5. *Low-lying zeros of number field L-functions* (with Ryan Peckner), Journal of Number Theory **132** (2012), 2866–2891. http://arxiv.org/abs/1003.5336
6. *The low-lying zeros of level 1 Maass forms* (with Levent Alpoge), preprint 2013. http://arxiv.org/abs/1301.5702
7. *The n-level density of zeros of quadratic Dirichlet L-functions* (with Jake Levinson), submitted September 2012 to Acta Arithmetica. http://arxiv.org/abs/1208.0930
8. *Moment Formulas for Ensembles of Classical Compact Groups* (with Geoffrey Iyer and Nicholas Triantafillou), preprint 2013.

# Publications: Elliptic Curves


1. 1- *and* 2*-level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **140** (2004), 952–992. http://arxiv.org/pdf/math/0310159
2. *Variation in the number of points on elliptic curves and applications to excess rank*, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120. http://arxiv.org/abs/math/0506461
3. *Investigations of zeros near the central point of elliptic curve L-functions*, Experimental Mathematics **15** (2006), no. 3, 257–279. http://arxiv.org/pdf/math/0508150
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5. *Towards an 'average' version of the Birch and Swinnerton-Dyer Conjecture* (with John Goes), Journal of Number Theory **130** (2010), no. 10, 2341–2358. http://arxiv.org/abs/0911.2871
6. *The lowest eigenvalue of Jacobi Random Matrix Ensembles and Painlevé VI*, (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), Journal of Physics A: Mathematical and Theoretical **43** (2010) 405204 (27pp). http://arxiv.org/pdf/1005.1298
7. *Models for zeros at the central point in families of elliptic curves* (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp). http://arxiv.org/pdf/1107.4426
8. *Effective equidistribution and the Sato-Tate law for families of elliptic curves* (with M. Ram Murty), Journal of Number Theory **131** (2011), no. 1, 25–44. http://arxiv.org/abs/1004.2753
9. *Moments of the rank of elliptic curves* (with Siman Wong), Canad. J. of Math. **64** (2012), no. 1, 151–182. http://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/mwMomentsRanksEC812final.pdf

# Publications: $L$-Function Ratio Conjecture


1. *A symplectic test of the L-Functions Ratios Conjecture*, Int Math Res Notices (2008) Vol. 2008, article ID rnm146, 36 pages, doi:10.1093/imrn/rnm146. http://arxiv.org/abs/0704.0927

2. *An orthogonal test of the L-Functions Ratios Conjecture*, Proceedings of the London Mathematical Society 2009, doi:10.1112/plms/pdp009. http://arxiv.org/abs/0805.4208

3. *A unitary test of the L-functions Ratios Conjecture* (with John Goes, Steven Jackson, David Montague, Kesinee Ninsuwan, Ryan Peckner and Thuy Pham), Journal of Number Theory **130** (2010), no. 10, 2238–2258. http://arxiv.org/abs/0909.4916

4. *An Orthogonal Test of the L-functions Ratios Conjecture, II* (with David Montague), Acta Arith. **146** (2011), 53–90. http://arxiv.org/abs/0911.1830

5. *An elliptic curve family test of the Ratios Conjecture* (with Duc Khiem Huynh and Ralph Morrison), Journal of Number Theory **131** (2011), 1117–1147. http://arxiv.org/abs/1011.3298

6. *Surpassing the Ratios Conjecture in the 1-level density of Dirichlet L-functions* (with Daniel Fiorilli). submitted September 2012 to Proceedings of the London Mathematical Society. http://arxiv.org/abs/1111.3896