Math 383: Complex Analysis: Fall ‘23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 01: 9-08-23: https://youtu.be/sljeScdiiR0

Definition of differentiability, differences between real and complex differentiability.

Definitions of the Derivative

Definition of the derivative:

$f'(x)$ or df/dx is $\lim_{h \to 0} (f(x+h) - f(x)) / h$

Exercise: what is the definition that works well for multivariable calculus (NOT THIS!!!)

For complex functions we use

$f'(z) = \lim_{h \to 0} (f(z+h) - f(z)) / h$

When does this limit exist?

For what functions $f(z)$ will $f'(z)$ exist?

Possibilities: $f(z)$ is a complex polynomial: $a_n z^n + \ldots + a_1 z + a_0$

We are using $z = x + i y$ where x, y are real numbers and $i = \sqrt{-1}$.

Consider f(z) = complex conjugate of z
So if z = x + iy then complex conjugate is x - iy.

f'(z) = lim_{h -> 0} (f(z+h) - f(z)) / h

Want the same limit no matter how h goes to zero.

ComplexConjugate(z+h) = ComplexConjugate(z) + ComplexConjugate(h).

So f'(z) = lim_{h -> 0} ComplexConjugate(h)/h.

One path: approach along the real axis: h = h_x + i 0. So ComplexConjuage(h) = h, limit equals 1.
Another path: approach along the imaginary axis: h = 0 + i h_y, ComplexConjugate(h) = -i h_y, so limit is
lim_{h_y -> 0} -i h_y / (i h_y) = lim_{h_y -> 0} (-1) = -1.
Limits differ, so the complex conjugation function is not complex differentiable!

Complex deriv:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

1) $f(z) = C \longrightarrow \lim_{h \to 0} \frac{C - C}{h} = \lim_{h \to 0} 0 = 0$

2) $f(z) = z \Longrightarrow \lim_{h \to 0} \frac{(z_0+h) - z_0}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

Complex conj: $z = x + iy$ then $\overline{z} = x - iy$, $i = \sqrt{-1}$

3) $f(z) = \overline{z} \longrightarrow \lim_{h \to 0} \frac{\overline{z_0+h} - \overline{z_0}}{h} = \lim_{h \to 0} \overline{h}/h = \begin{cases} 1 & h \in \mathbb{R} \\ -1 & h \in i\mathbb{R} \end{cases}$

Not diff!

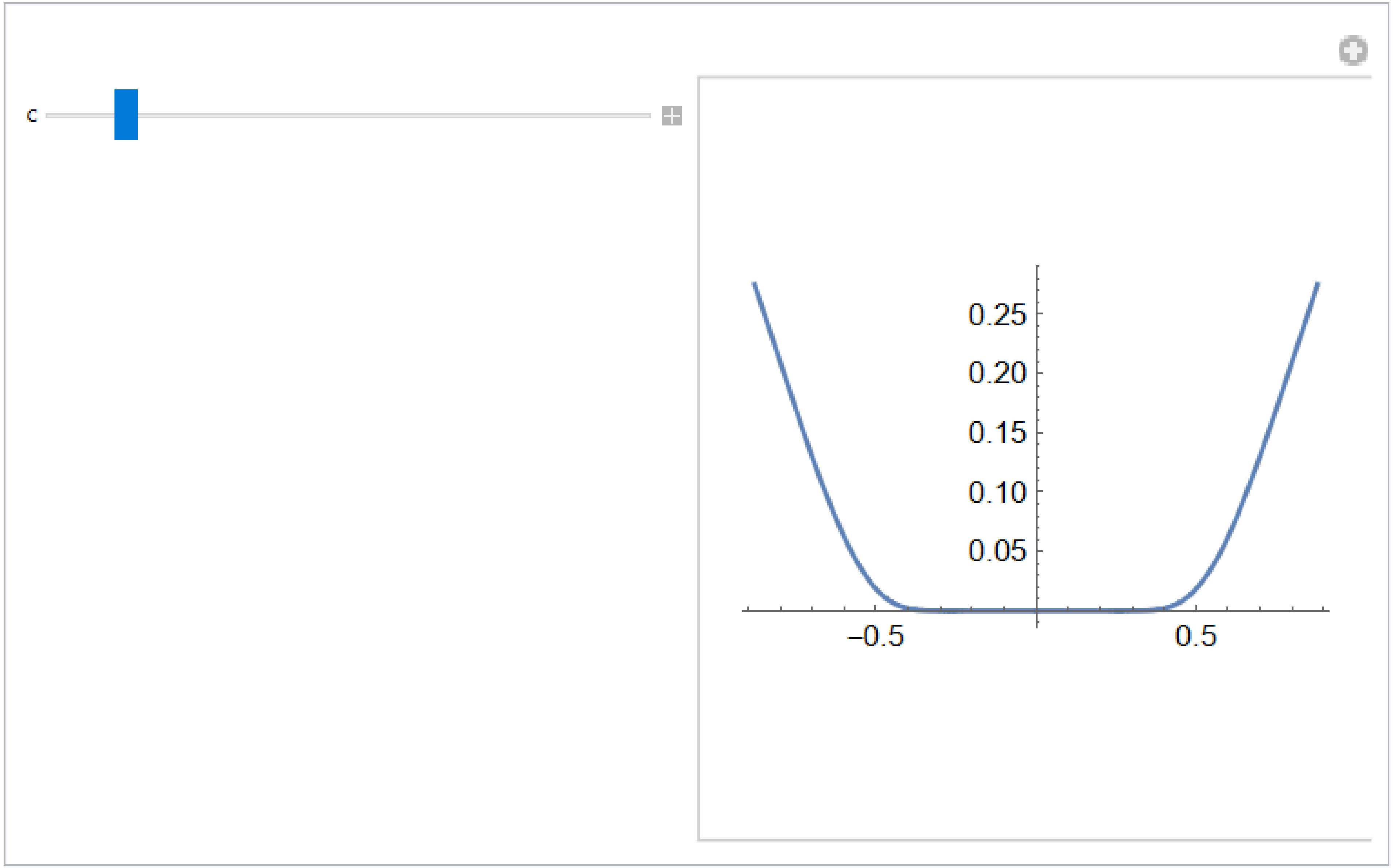
Real Analysis versus Complex Analysis

1. Assume f is differentiable, must f' be differentiable? False for Reals, true for Complex.
   Consider: $f(x) = x^3 \sin(1/x)$ if x is not zero and 0 if x is zero.
   Go back to the definition: look at $f(h)/h$
   Exercise: show f''' or f''' does not exist
   So if complex differentiable once, lather rinse repeat, infinitely differentiable!

2. Assume f has a Taylor series that converges for all x; must f equal its Taylor Series in a neighborhood of zero?   False for Reals, true for Complex.
   Consider $f(x) = \exp(-1/x^2)$ if x is not zero and 0 if x is zero.
   Exercise: show $f^{(n)}(0) = 0$, so Taylor series is identically zero, but f is not zero function.
   Not even complex continuous -- take $z = 1/iy$ with y going to zero, see function explodes

In[1]:=
```
f[x_] := If[x == 0, 0, Exp[-1 / x^2]];
Manipulate[Plot[f[x], {x, -c, c}], {c, 1, .001} ]
```

Out[2]=

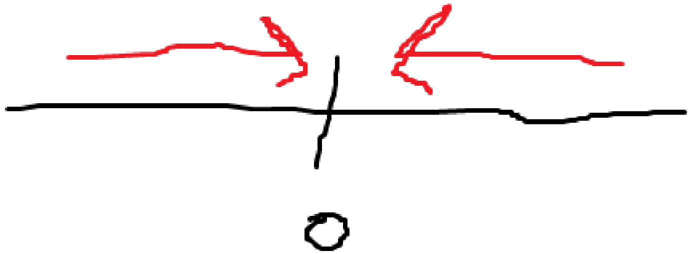


Real vs Complex

3. Consider f bounded and differentiable, is it constant? Real no, Complex yes.
   Real: consider sin(x), cos(x), arctan'(x) = 1/(1+x^2)

4. Consider differentiable f and sequence of points x_n such that f(x_) = 0 and lim_{n -> oo} x_n = x* and f(x*) = 0; must f be identically zero?
   Consider f(x) = x^3 sin(1/x). This is differentiable, has infinitely many zeros that get closer and closer to 0, f(0) = 0, but f is not identically 0.

https://youtu.be/sljeScdiiR0



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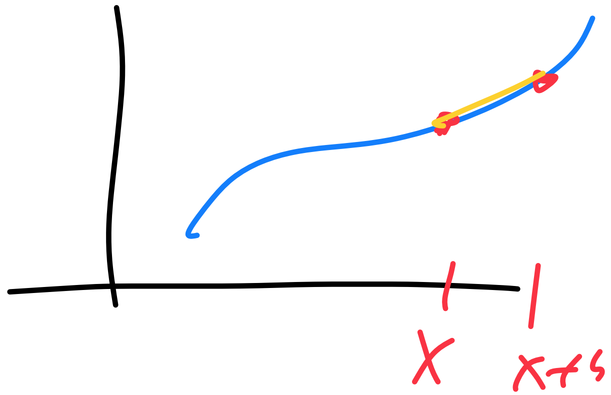
Lecture 02: 9-10-23: https://youtu.be/xYFwqGrs9mQ

Experimental Mathematics:
Formulas for derivatives, product rule, Cauchy-Riemann Equations, Green’s Theorem

Definitions of the Derivative: From 1-dimension to several.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

Try $x = a+h$ so $h \to 0$ same as $x \to a$

$$f'(a) \Rightarrow \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = 0$$

Means the deriv of f at a is $f'(a)$

Definitions of the Derivative: Several Variables.

$$\frac{\partial f}{\partial x_k}(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{e}_k) - f(\vec{a})}{h}$$

e_k is unit vector $(0, \ldots, 0, 1, 0, \ldots 0)$

f is differentiable if

$$\lim_{\vec{x} \to \vec{a}} \frac{\| f(\vec{x}) - f(\vec{a}) - (Df)(\vec{a}) \cdot (\vec{x} - \vec{a}) \|}{\| \vec{x} - \vec{a} \|} = 0$$

where $(Df)(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \ldots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$

Find $f(x,y)$ st $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ but f not diff

Important Derivatives:

What are the derivatives of $f(x) = x^3$, $g(x) = x^{3/2}$ and $h(x) = x^{\sqrt{2}}$?

Rule: $(x^n)' = nx^{n-1}$

$f'(x) = 3x^2$

$(x+h)^3$ 5

$x^3 + 3x^2h + 3xh^2 + h^3$

$g'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$g'(x) = \lim_{h \to 0} \frac{(x+h)^{\frac{3}{2}} - x^{\frac{3}{2}}}{h}$

Multiply by $\frac{(x+h)^{\frac{3}{2}} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}}$

$g'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{((x+h)^{3/2} + x^{3/2})h}$

Goes to $2x^{3/2}$

$A(x) = g(x)^2 = x^3$

Thus $A'(x) = 2g(x)g'(x) = 3x^2$

$h'(x) = \sqrt{2}\, x^{\sqrt{2}-1}$

$h(x) = x^{\sqrt{2}}$

$= e^{\sqrt{2}\ln x}$

Chain Rule

need to know exp, ln

Product Rule: Experimental Discovery!

$f(x) = x^n$ $g(x) = x^m$

$f'(x) = n x^{n-1}$ $g'(x) = m x^{m-1}$

$h(x) = f(x)g(x) = x^{n+m}$

$h'(x) = (n+m) x^{n+m-1}$

$= n\, x^{n+m-1}$

$+ m\, x^{n+m-1}$

Conj: $(f(x)g(x))' =$ function of f, g, f', g'

Note: $f'(x)g(x) + f(x)g'(x) = (f(x)g(x))'$ here

Ex: $f(x) = \sin x$ $g(x) = \cos x$ $h(x) = f(x)g(x) =$

$= \frac{1}{2}\sin(2x)$

Definitions of the Complex Derivative and Properties.

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

differentiable

Proposition 2.2 *If f and g are holomorphic in Ω, then:*

(i) *$f + g$ is holomorphic in Ω and $(f+g)' = f' + g'$.*

(ii) *fg is holomorphic in Ω and $(fg)' = f'g + fg'$.*

(iii) *If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and*

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z) \quad \text{for all } z \in \Omega.$$

Cauchy-Riemann Equations – Experimental Discovery: f(z) = u(x,y) + i v(x,y)

Notation: $\frac{\partial u}{\partial x} = u_x \qquad \frac{\partial u}{\partial y} = u_y$

$f(z) = z$

$= x + iy$

so $u(x,y) = x \qquad v(x,y) = y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = 1$

Maybe $u_x = v_y$

$u_y = \pm v_x$?

$g(z) = z^2$

$g(z) = x^2 - y^2 + 2ixy$

$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

Maybe $u_x = v_y$

$u_y = -v_x$

C-R Eqs

Two limits: along x-axis, along y-axis

$$f'(z) = \lim_{h\to 0} \frac{[u(x+h,y)+iv(x+h,y)] - [u(x,y)+iv(x,y)]}{h}$$

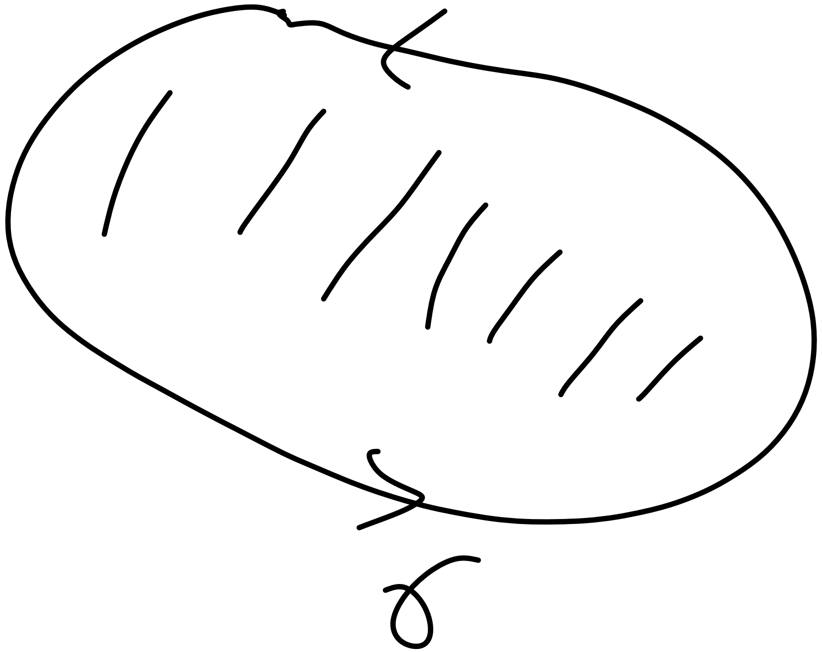
$$= \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y)-v(x,y)}{h}$$

$$= u_x(x,y) + i\, v_x(x,y)$$

Compute $f'(z)$ going down y-axis $\Rightarrow f'(z) = v_y(x,y) - i\, u_y(x,y)$

CHECK THIS

Green's Thm



$$\oint_\gamma P\,dx + Q\,dy = \iint_{\text{Region}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$F(x,y) = \big(P(x,y), Q(x,y)\big)$$

$$f(z) = \text{“}\big(u(x,y), v(x,y)\big)\text{''} \text{ or } u + iv$$

Writing $f = u + iv$, we find after separating real and imaginary parts and using $1/i = -i$, that the partials of u and v exist, and they satisfy the following non-trivial relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the **Cauchy-Riemann** equations, which link real and complex analysis.

We can clarify the situation further by defining two differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Proposition 2.3 *If f is holomorphic at z_0, then*

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0 \quad \textit{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Also, if we write $F(x, y) = f(z)$, then F is differentiable in the sense of real variables, and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2.$$

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Lecture 03: 9-13-23: No class, do exam: Real Analysis Review (limsup/liminf, strange functions): https://youtu.be/DLyzZhJN58w (slides); watched at home: Differentiating Term By Term, Analytic Functions, Path Integrals: https://youtu.be/e60Dh8cAIhQ (2017). Green's Theorem in a day: https://youtu.be/Iq-Og1GAtOQ

Lecture 04: 9-15-23: https://youtu.be/-hqBpme4Q2A

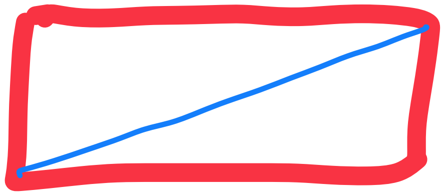
Primitives and Goursat’s Theorem, Green Review (if time)

Suppose f is a function on the open set Ω. A **primitive** for f on Ω is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.

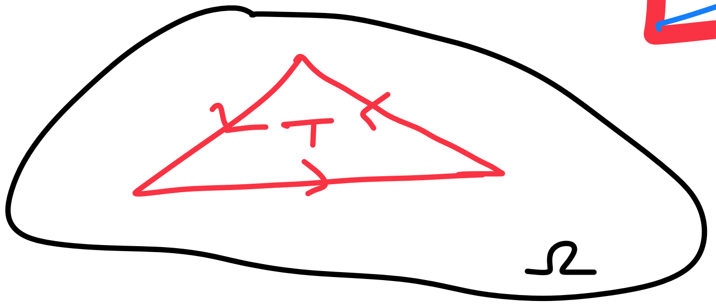
Goursat's theorem

Theorem 1.1 *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

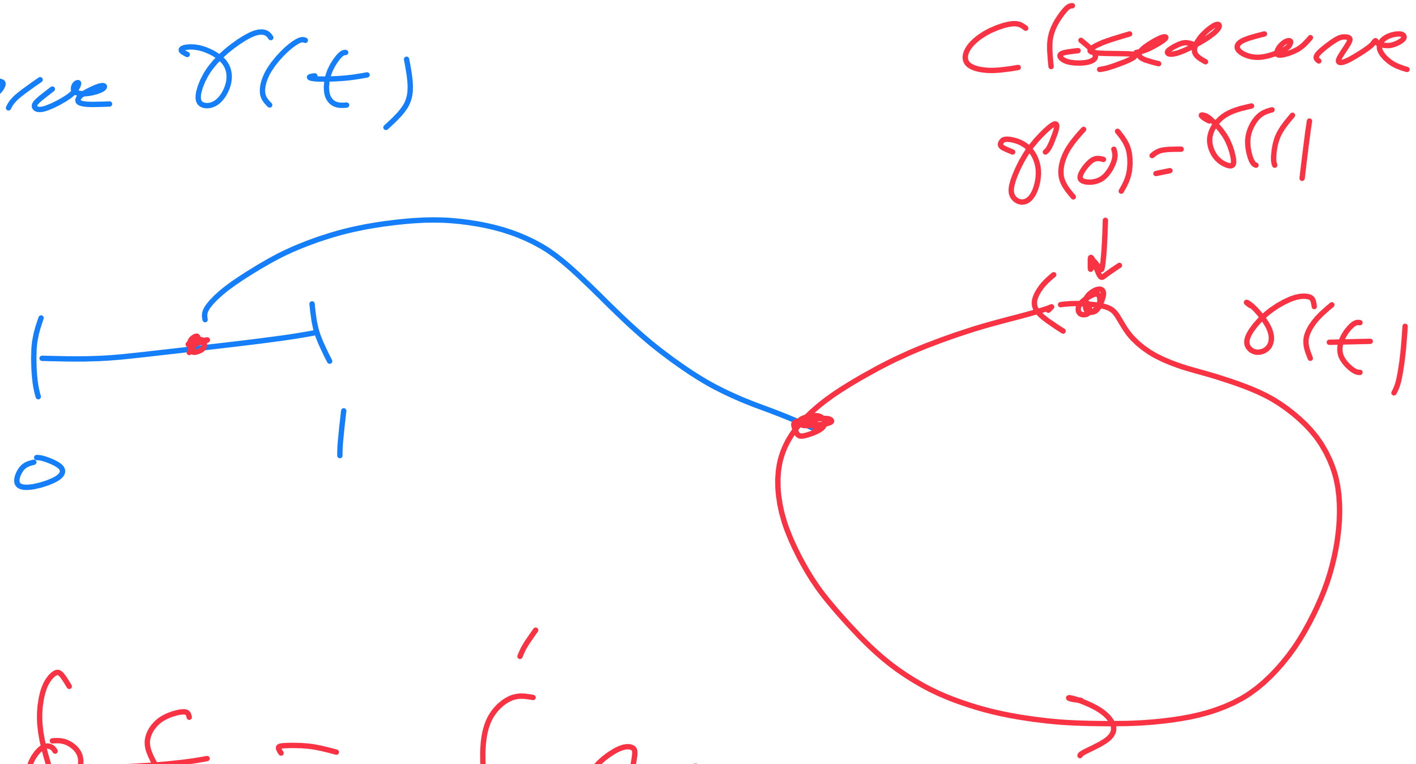
$$\int_T f(z)\, dz = 0$$



whenever f is holomorphic in Ω.



Curve $\gamma(t)$



$$\oint_\gamma f = \int_{t=0}^{1} \underbrace{f(\gamma(t))\,\gamma'(t)}_{\frac{d}{dt}\left[F(\gamma(t))\right]}\,dt$$

Imagine $F' = f$

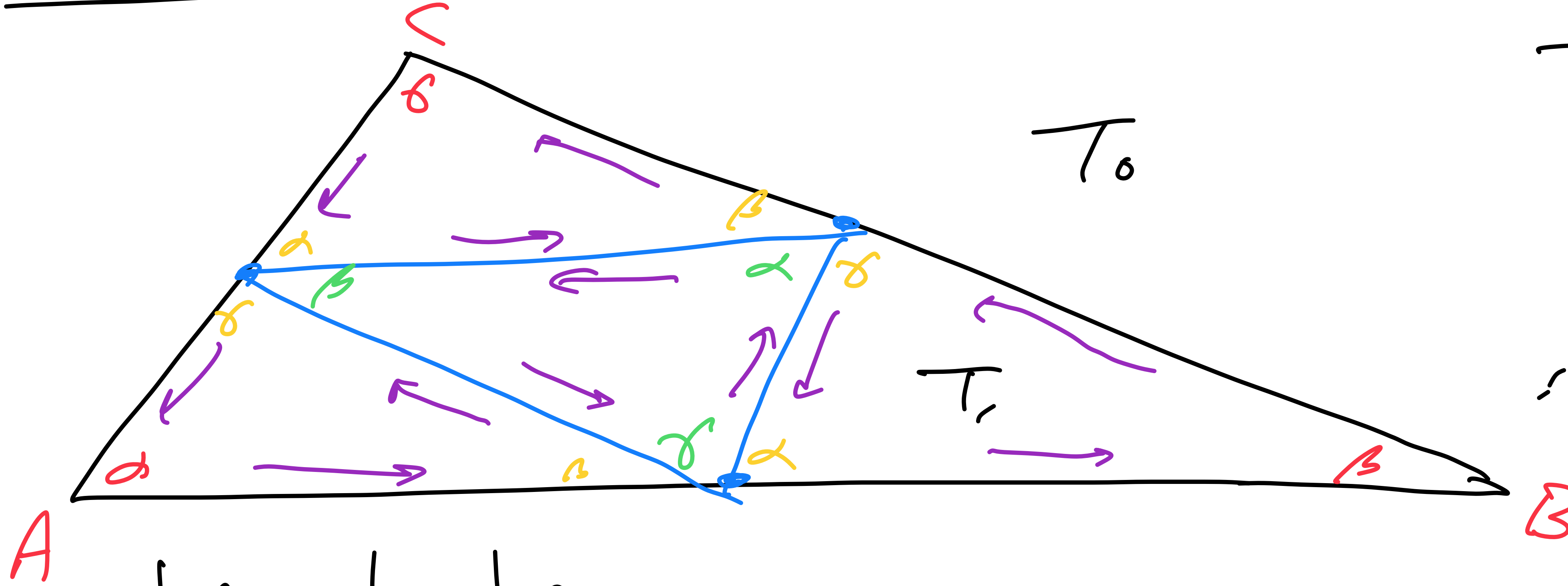
$$= F(\gamma(t))\Big|_0^1 = F(\gamma(1)) - F(\gamma(0)) = 0$$

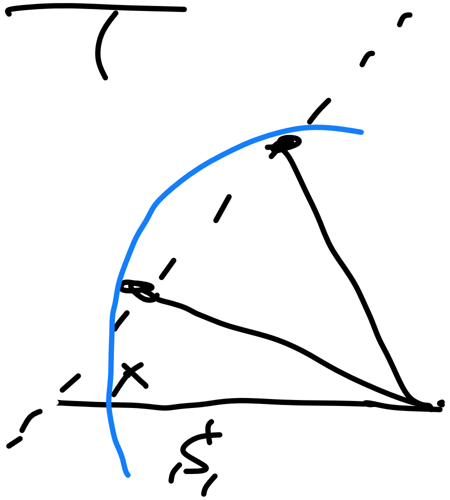
as $\gamma(1) = \gamma(0)$

Search for Primitives

f	F
$z^n \longrightarrow$	$\dfrac{z^{n+1}}{n+1}$ if $n \neq -1$
$\dfrac{1}{z} \longrightarrow$	??
$\sum_{n=0}^{N} a_n z^n \longrightarrow$	$\sum_{n=0}^{N} \dfrac{a_n z^{n+1}}{n+1}$
$\sum_{n=0}^{\infty} a_n z^n \longrightarrow$	$\sum_{n=0}^{\infty} \dfrac{a_n z^n}{n+1}$ if $\lvert z\rvert < R$ where original f conv abs for $\lvert z\rvert < R$

Goursat's Theorem





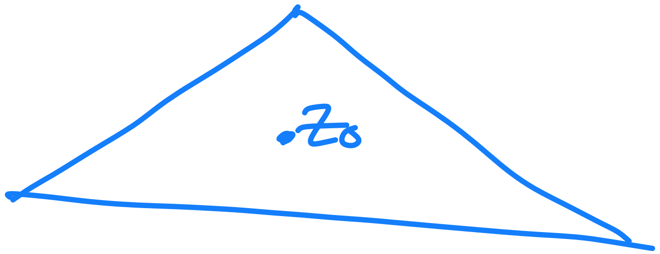
$$\left|\int_{T_0} f\right| = \left|\int_{T_{11}} f + \int_{T_{12}} f + \int_{T_{13}} f + \int_{T_{14}} f\right|$$

$$\left|\int_{T_0} f\right| \le 4 \max_{1 \le j \le 4} \left|\int_{T_{1j}} f\right|$$

Call the associated triangle to the max j to be Triangle T_1

$$\left|\int_{T_0} f\right| \le 4^n \left|\int_{T_n} f\right|$$

very small



As f is holomorphic:

$$\underbrace{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)}$$

let $\varepsilon_{z_0}(z) = \left(\dfrac{f(z)-f(z_0)}{z-z_0}\right) - f'(z_0)$.

$$\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)+\varepsilon_{z_0}(z)$$

as $z\to z_0$, $\varepsilon_{z_0}(z)\to 0$

$$f(z) = f(z_0)+f'(z_0)(z-z_0)+\varepsilon_{z_0}(z)(z-z_0)$$

$$f(z) = \underbrace{f(z_0)}_{\downarrow \text{primitive}} + \underbrace{f'(z_0)(z-z_0)}_{\downarrow \text{primitive}} + \underbrace{\varepsilon_{z_0}(z)(z-z_0)}_{\downarrow}$$

$$f(z_0)z + \qquad f'(z_0)\underbrace{\frac{(z-z_0)^2}{2}} + \quad ??$$

$$\int_{T_n} f = \underbrace{\int_{T_n} f(z_0)z}_{0} + \underbrace{\int_{T_n} f'(z_0)\frac{(z-z_0)^2}{2}}_{0} + \int_{T_n} \varepsilon_{z_0}(z)(z-z_0)$$

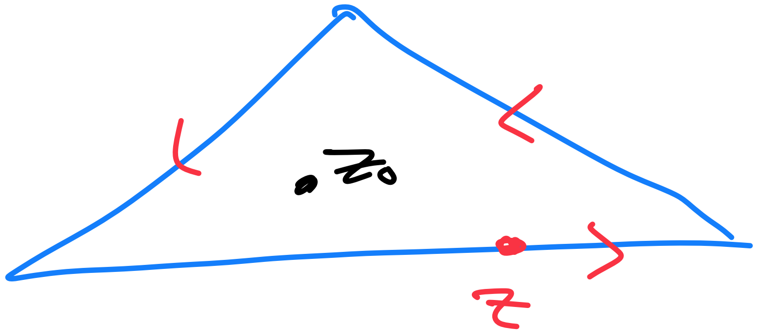
$$\left|\int_{T_n} f\right| = \left|\int_{T_n} \varepsilon_{z_0}(z)(z-z_0)\,dz\right| \leq \operatorname{Max}|\varepsilon_{z_0}(z)| * \operatorname{Max}|z-z_0|$$

$$* \operatorname{Perim}(T_n)$$

$$\left|\int_{T_n} f\right| = \left|\int_{T_n} \varepsilon_{z_0}(z)(z-z_0)\,dz\right| \le \max|\varepsilon_{z_0}(z)| * \max|z-z_0|$$

$$* \operatorname{Perim}(T_n)$$

use $|z-z_0| \le \operatorname{Perim}(T_n)$



$$\le \max|\varepsilon_{z_0}(z)| * \operatorname{Perim}(T_n)^2$$

$$\le \max|\varepsilon_{z_0}(z)| * \frac{1}{4^n} \operatorname{Perim}(T)^2$$

$$\left|\int_T f\right| \le 4^n \max_{z\in T_n}|\varepsilon_{z_0}(z)| * \frac{1}{4^n} \operatorname{Perim}(T)^2$$

$$\le \text{Const} \cdot \max_{z\in T_n}|\varepsilon_{z_0}(z)| \longrightarrow 0$$

$\blacksquare$

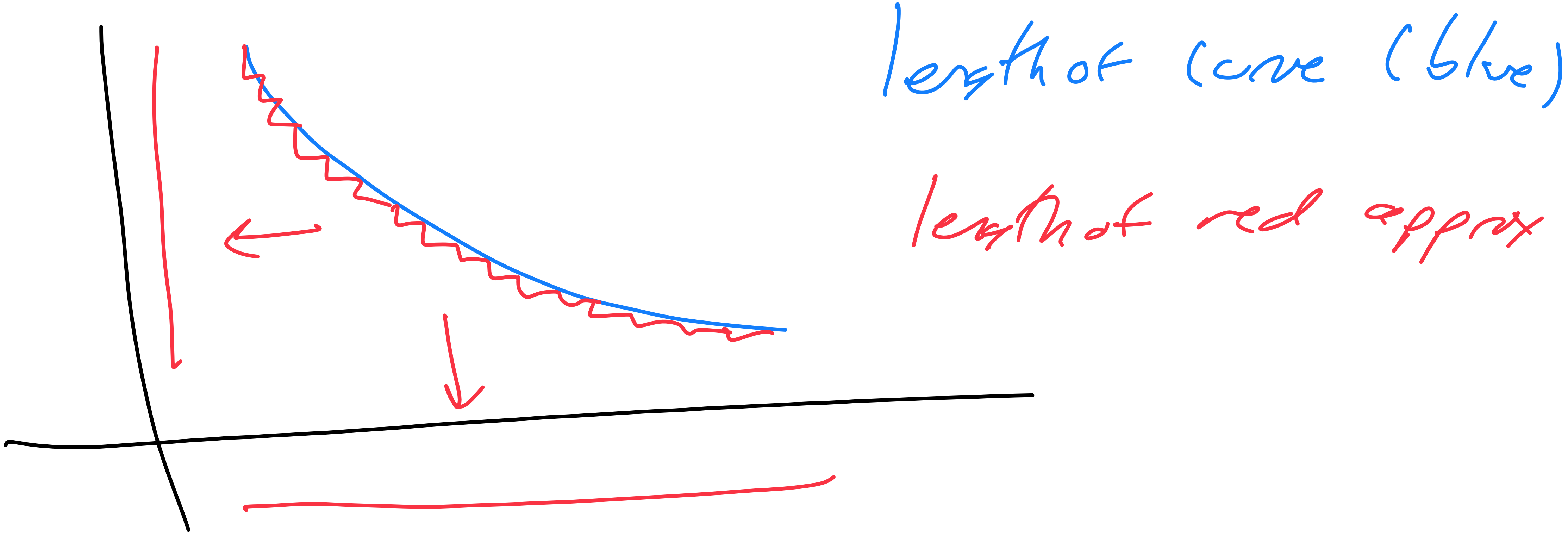
Aside: 1, 2, ..., 9

$$\frac{\square}{\square\square} + \frac{\square}{\square\square} + \frac{\square}{\square\square}$$

Use each once and only once

Can make the sum equal 1!

$9! \approx 330{,}000$



$$ds \neq dx + dy$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

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Lecture 05: 9-18-23: https://youtu.be/pTyXgBAGN7A

Primitive Theorem, Cauchy's Formula

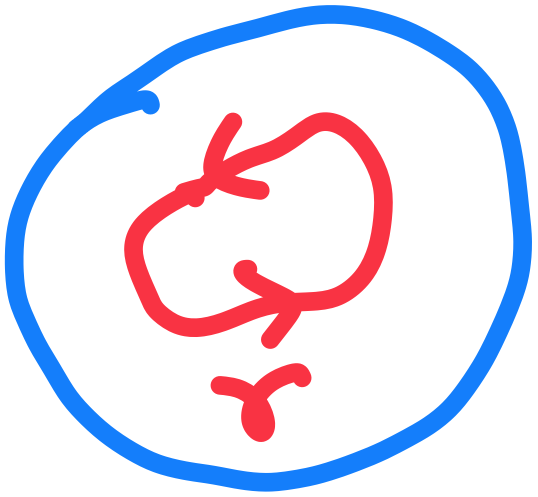
General items: • Choices have in complexification • Path independence • Level of rigor

Theorem 2.1 *A holomorphic function in an open disc has a primitive*

Theorem 2.2 (Cauchy's theorem for a disc) *If f is holomorphic in a disc, then*

$$\int_{\gamma} f(z)\,dz = 0$$

for any closed curve γ in that disc.



Proof. Since f has a primitive, we can apply Corollary 3.3 of Chapter 1.

$$\int_{\gamma} f$$

Corollary 2.3 *Suppose f is holomorphic in an open set containing the circle C and its interior. Then*

$$\int_{C} f(z)\,dz = 0.$$

Proof. Let D be the disc with boundary circle C. Then there exists a slightly larger disc D' which contains D and so that f is holomorphic on D'. We may now apply Cauchy's theorem in D' to conclude that $\int_C f(z)\,dz = 0$.

Integrating z^n about a circle centered at the origin....

$z = x + iy$ or $z = re^{i\theta}$ $\quad 0 \le r$ and $0 \le \theta < 2\pi$

$\int_{C_r} z^n dz$ where C_r is a circle of radius r centered at the origin

$\gamma(t) = re^{it} = $ "$(r\cos t, r\sin t)$" use θ instead of t

$\gamma(\theta) = re^{i\theta}$ $\quad \gamma'(\theta) = ire^{i\theta}$ $\quad$ using $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$z = re^{i\theta}$ on circle of radius r then $dz = ir\,e^{i\theta}d\theta$

$$\int_{C_r} z^n dz = \int_{\theta=0}^{2\pi} (re^{i\theta})^n\, ire^{i\theta} d\theta = i\int_{\theta=0}^{2\pi} r^{n+1} e^{i(n+1)\theta} d\theta$$

$$= ir^{n+1}\int_{\theta=0}^{2\pi} [\cos((n+1)\theta) + i\sin((n+1)\theta)]d\theta = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise} \end{cases}$$

if $n=-1$ have $\int_0^{2\pi} 1\, d\theta = 2\pi$

if $n \ne -1$ have 0

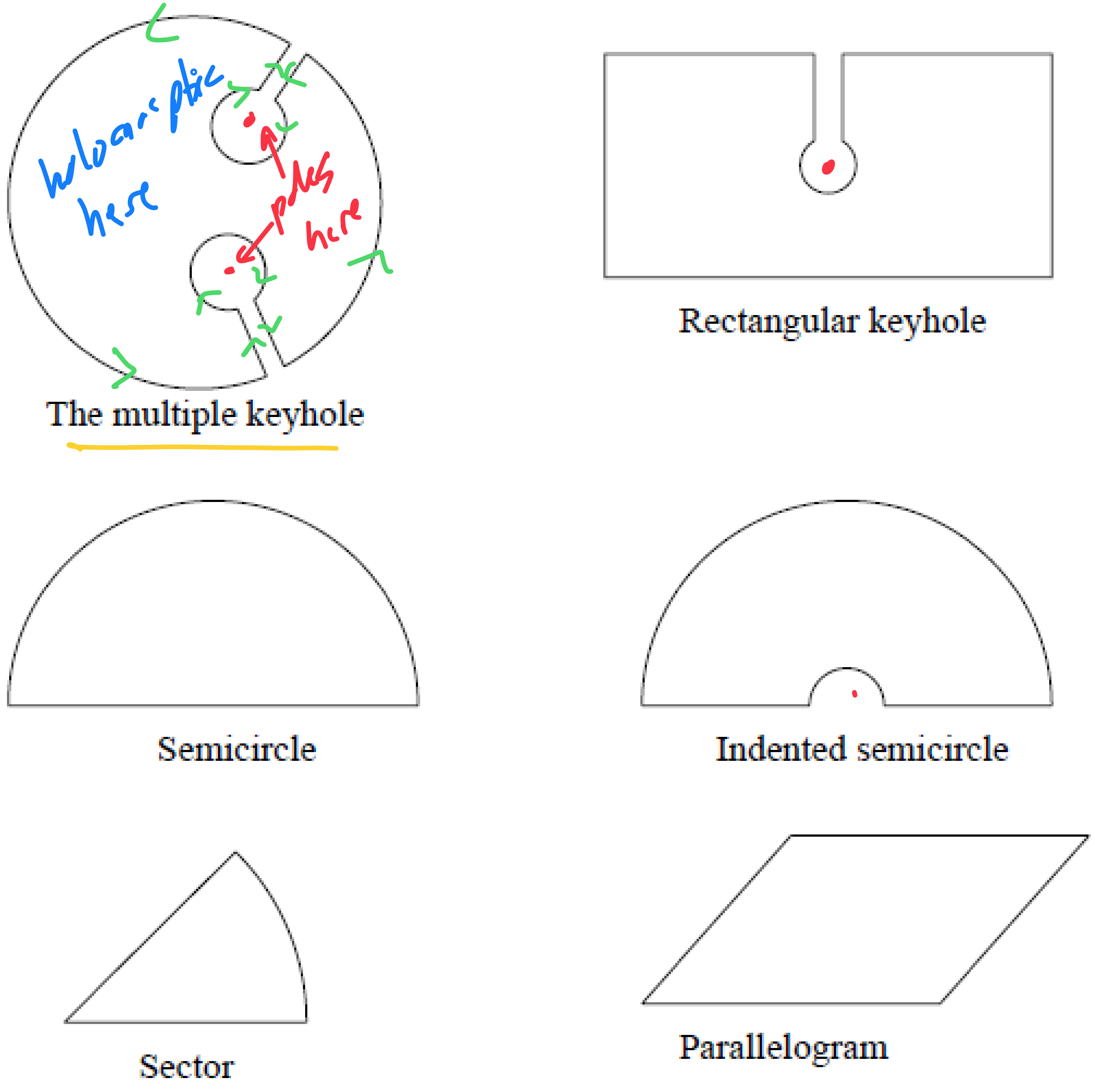


Figure 7. Examples of toy contours

$$f(z) = \sum_{n=-N}^{\infty} a_n z^n$$

only place f explodes is at z=0 if N>0

$$\int_{C_R} f(z)\,dz = \int_{C_R} \sum_{n=-N}^{\infty} a_n z^n \,dz$$

$$\overset{\text{hope}}{=} \sum_{n=-N}^{\infty} a_n \left(\int_{C_R} z^n \,dz \right)$$

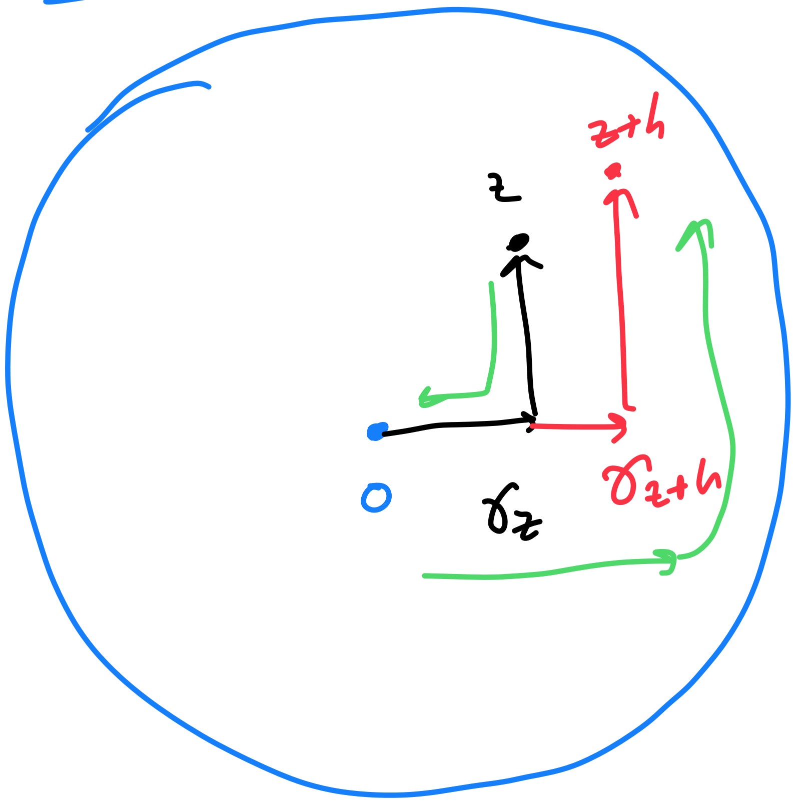
$$= 2\pi i\, a_{-1}$$

Theorem 2.1 *A holomorphic function in an open disc has a primitive in that disc.*

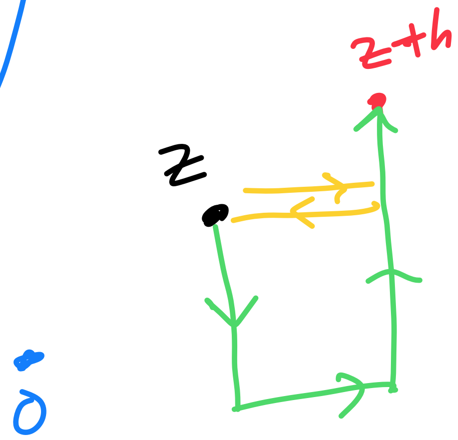
Proof

Define: $F(z) = \int_{\gamma_z} f(w)\,dw$

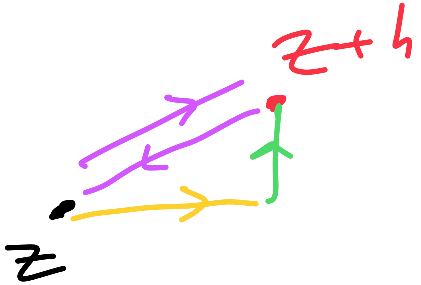
$$F(z+h) = \int_{\gamma_{z+h}} f(w)\,dw$$



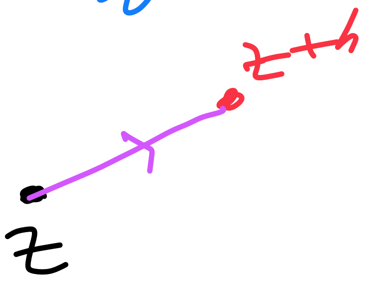
Consider: $F(z+h) - F(z) \overset{??}{=} f(z)\,h + \text{small}$

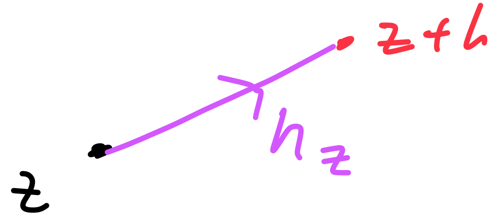


Goursat reduces to



then Goursat again





$$F(z+h) - F(z) = \int_{\eta_z} f(w)\,dw$$

along η_z have

$$f(w) = f(z) + f'(z)(w-z) + (\text{something small}) * (w-z)$$

because f is holo in a nbhd of z

$$f(w) = f(z) + \psi(w)$$

where $\psi(w) \to 0$ as $h \to 0$ for all w on η_z

because f is continuous

$$\int_{\eta_z} f(w)\,dw = \int_{\eta_z} f(z)\,dw + \int_{\eta_z} \psi(w)\,dw$$

$$\int_{\eta_z} f(z)\,dw = f(z)\,w\Big|_{z}^{z+h} = f(z)\,h$$

$\int_{\eta_z} \psi(w)\,dw$: in abs value, $\leq \max_{w \in \eta_z} |\psi(w)| * \text{Length}(\eta_z)$

$$\leq \max_{w \in \eta_z} |\psi(w)| * |h|$$

So $$\frac{F(z+h) - F(z)}{h} = f(z) + \text{Error of size at most } \max_{w \in \eta_z} |\psi(w)|$$

Goes to 0 as $h \to 0$ ∎

$z = x + iy$ or $z = re^{i\theta}$ $\bar{z} = x - iy$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\cosh(\theta) = \dfrac{e^{\theta} + e^{-\theta}}{2} = \cos(i\theta)$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx$$ Two concerns:

1) well behaved as $x \to 0$?

2) decay rapidly as $x \to \infty$?

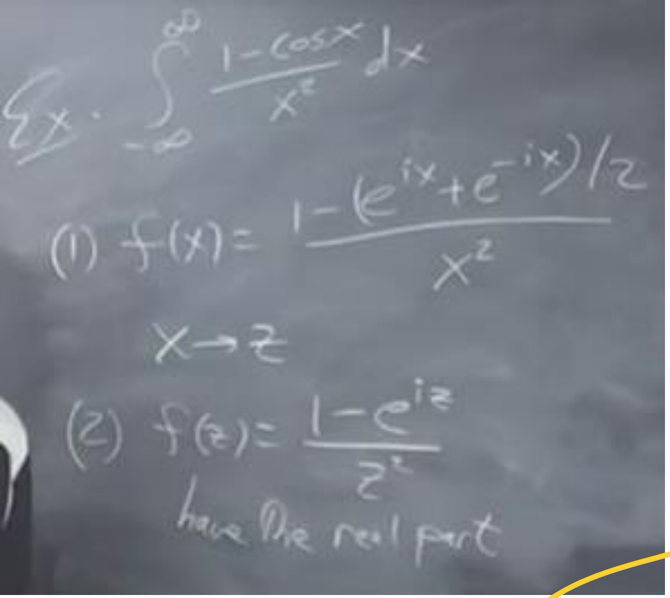
near $x \approx 0$ have $\frac{1-\cos x}{x^2} \simeq \frac{1-(1-x^2/2! + x^4/4! + \dots)}{x^2} = \frac{x^2/2!}{x^2} - \frac{x^4/4!}{x^2} + \dots$

$= \frac{1}{2} - \frac{1}{4!}x^2 + \dots$ ok

as $|x| \to \infty$ have $\left|\frac{1-\cos x}{x^2}\right| \leq \frac{1}{x^2}$ is integrable

as $x \to \infty$ $\int_R^{\infty} x^{-p}\,dx$ is finite if $p > 1$

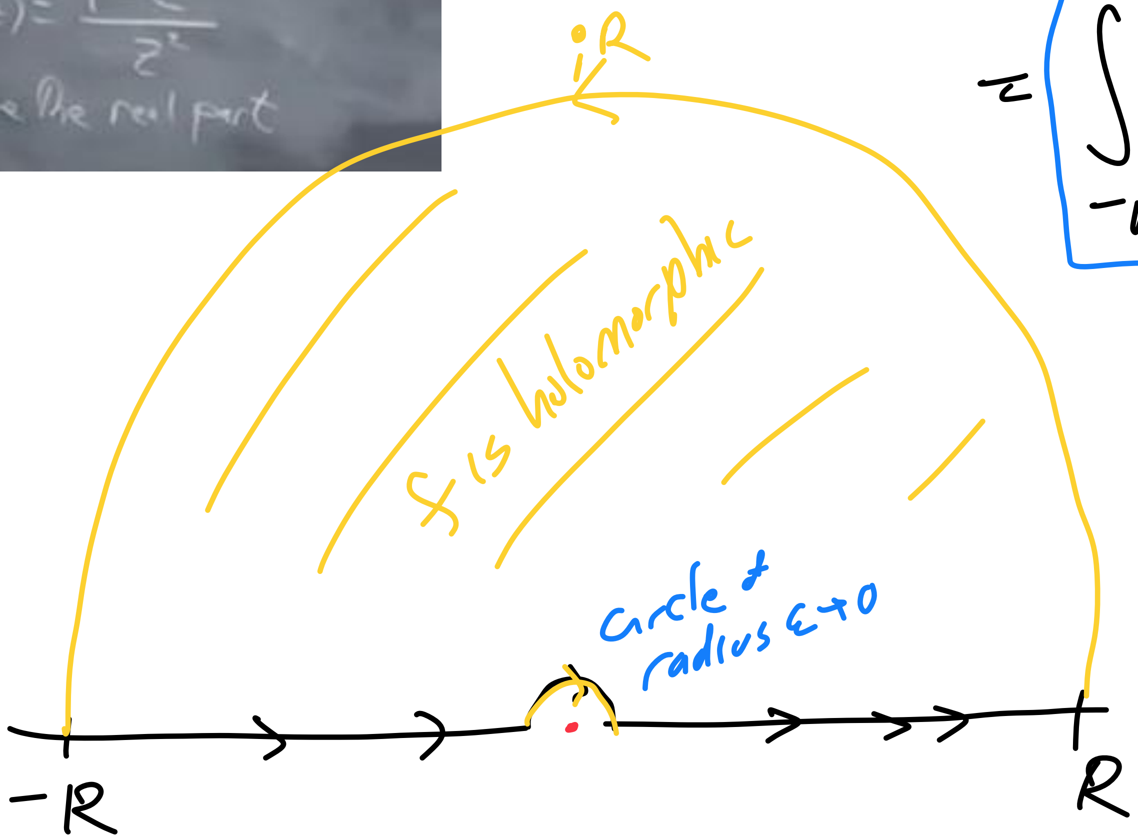
as $x \sim 0$ $\int_0^{\varepsilon} x^{-p}\,dx$ is finite if $p < 1$



$$\int_{-\infty}^{\infty} \frac{1-(e^{ix}+e^{-ix})/2}{x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

as $\varepsilon \to 0$, $R \to \infty$ goes to $\int_{-\infty}^{\infty} f(x)dx$

$$= \int_{-R}^{-\varepsilon} f(x)dx + \int_{\varepsilon}^{R} f(x)\,dx + \int_{S_\varepsilon} f(z)dz + \int_{S \text{ from } -R \text{ to } R} f(z)dz = 0$$

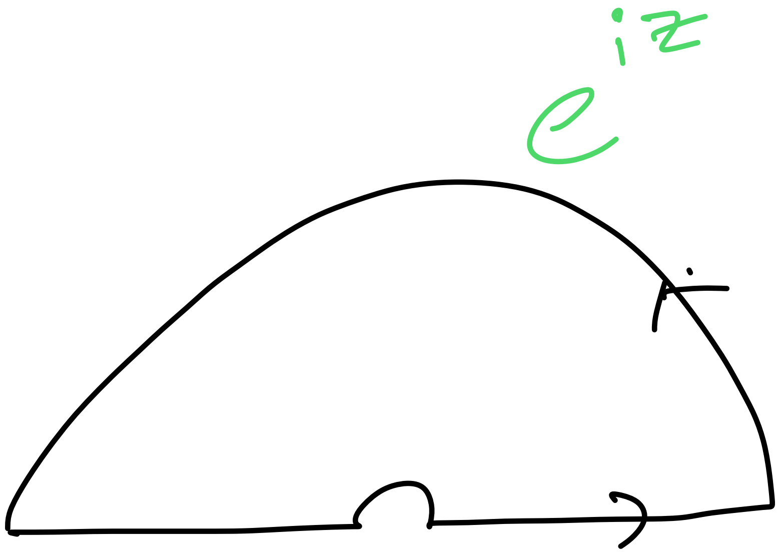


for R large

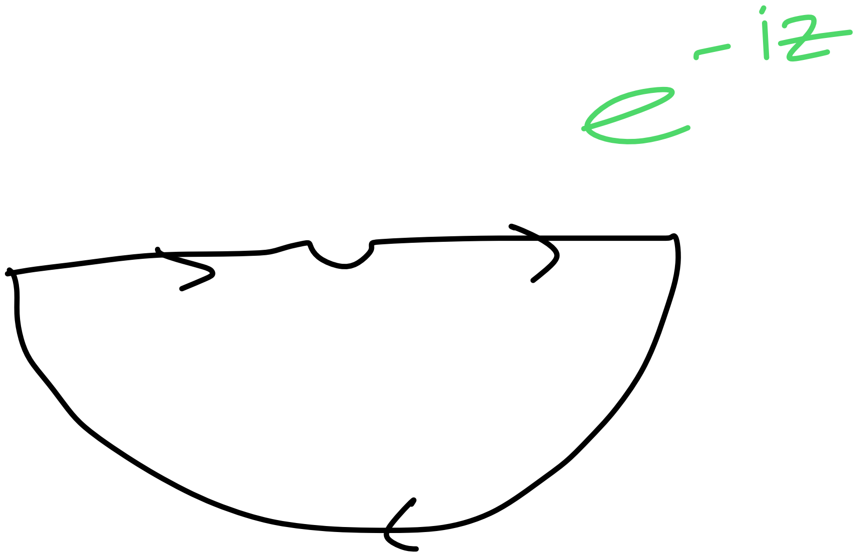
$$f(z) = \frac{1-(e^{iz}+e^{-iz})/2}{z^2}$$

$z = iR$ have $e^{-iz} = e^{R} \to \infty$

$$|e^{iz}| \le |e^{ix}e^{-y}| = 1$$



and



For more information on this problem, see the video from 2017: Lecture 05: 9/18/17: Primitive Theorem, Cauchy's Formula, Example: https://youtu.be/RkZHw4fKHfE . There we go through the details of the integration. Today we instead concentrated on why we are integrating over the region we are, and why we have the integrand we do. Explicitly, why we used exp(iz) instead of (exp(iz) + exp(-iz))/2 and why we have the detour around the origin.

For continuous functions [edit]

https://en.wikipedia.org/wiki/Contour_integration

To define the contour integral in this way one must first consider the integral, over a real variable, of a complex-valued function. Let $f : \mathbf{R} \to \mathbf{C}$ be a complex-valued function of a real variable, t. The real and imaginary parts of f are often denoted as $u(t)$ and $v(t)$, respectively, so that

$$f(t) = u(t) + iv(t).$$

Then the integral of the complex-valued function f over the interval $[a, b]$ is given by

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^b \big(u(t) + iv(t)\big)\,dt \\ &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.\end{aligned}$$

Let $f : \mathbf{C} \to \mathbf{C}$ be a continuous function on the directed smooth curve γ. Let $z : \mathbf{R} \to \mathbf{C}$ be any parametrization of γ that is consistent with its order (direction). Then the integral along γ is denoted

$$\int_\gamma f(z)\,dz$$

and is given by[6]

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

This definition is well defined. That is, the result is independent of the parametrization chosen.[6] In the case where the real integral on the right side does not exist the integral along γ is said not to exist.

Math 383: Complex Analysis: Fall '23 (Williams)

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Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 06: 9-20-23: https://youtu.be/srLW0uLwVpk

Cauchy's formula and consequences

Plan for the day: Lecture 06: September 20, 2023:

- Prove Cauchy's formulas
- See holomorphic and analytic are the same
- Apply Cauchy's formula to integrate

analytic: power series converges to the function

Remember: e^{-1/x^2}: Taylor series converges $\forall x$ but not to our fn!

General items.

- Have choices in contours and integrands
- See why we have the conditions we do

1 Goursat's theorem

Corollary 3.3 in the previous chapter says that if f has a primitive in an open set Ω, then

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in Ω. Conversely, if we can show that the above relation holds for some types of curves γ, then a primitive will exist. Our starting point is Goursat's theorem, from which in effect we shall deduce most of the other results in this chapter.

Theorem 1.1 *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)\,dz = 0$$

whenever f is holomorphic in Ω.

2 Local existence of primitives and Cauchy's theorem in a disc

Theorem 2.1 *A holomorphic function in an open disc has a primitive in that disc.*

Theorem 2.2 (Cauchy's theorem for a disc) *If f is holomorphic in a disc, then*

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in that disc.

Corollary 2.3 *Suppose f is holomorphic in an open set containing the circle C and its interior. Then*

$$\int_C f(z)\,dz = 0.$$

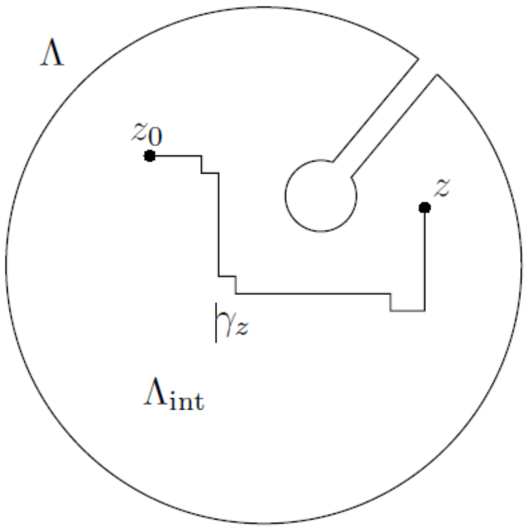
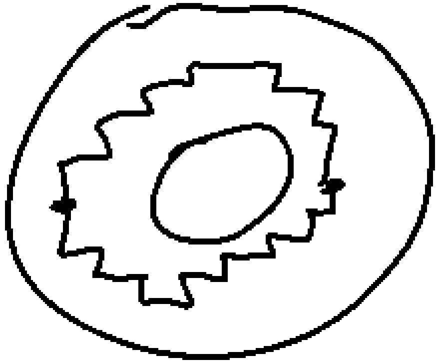
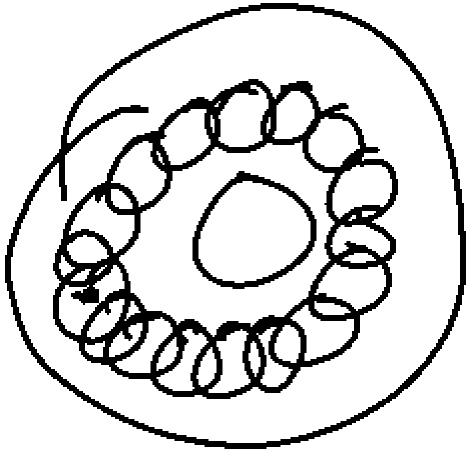
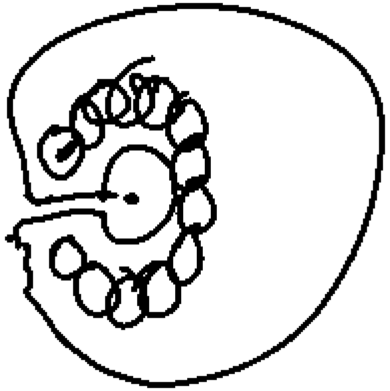


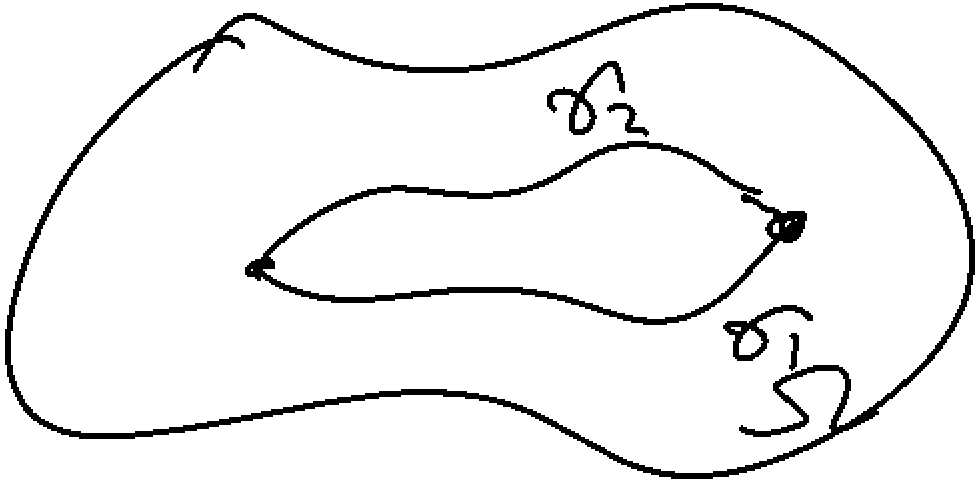
Figure 6. A curve γ_z

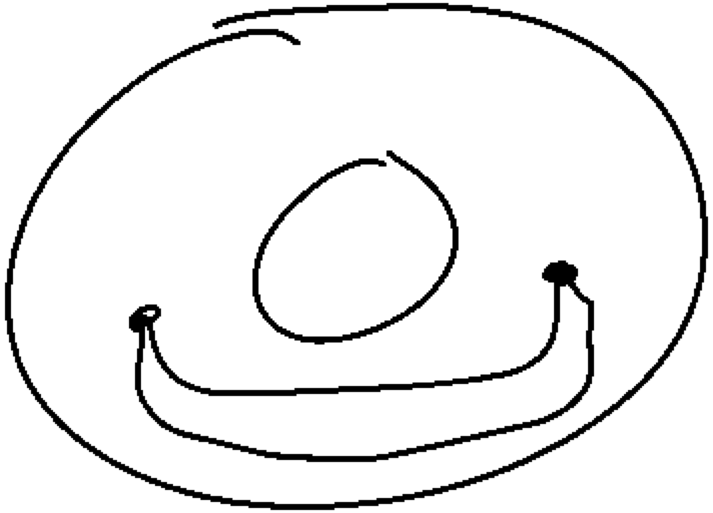
Donut/Annular Ring



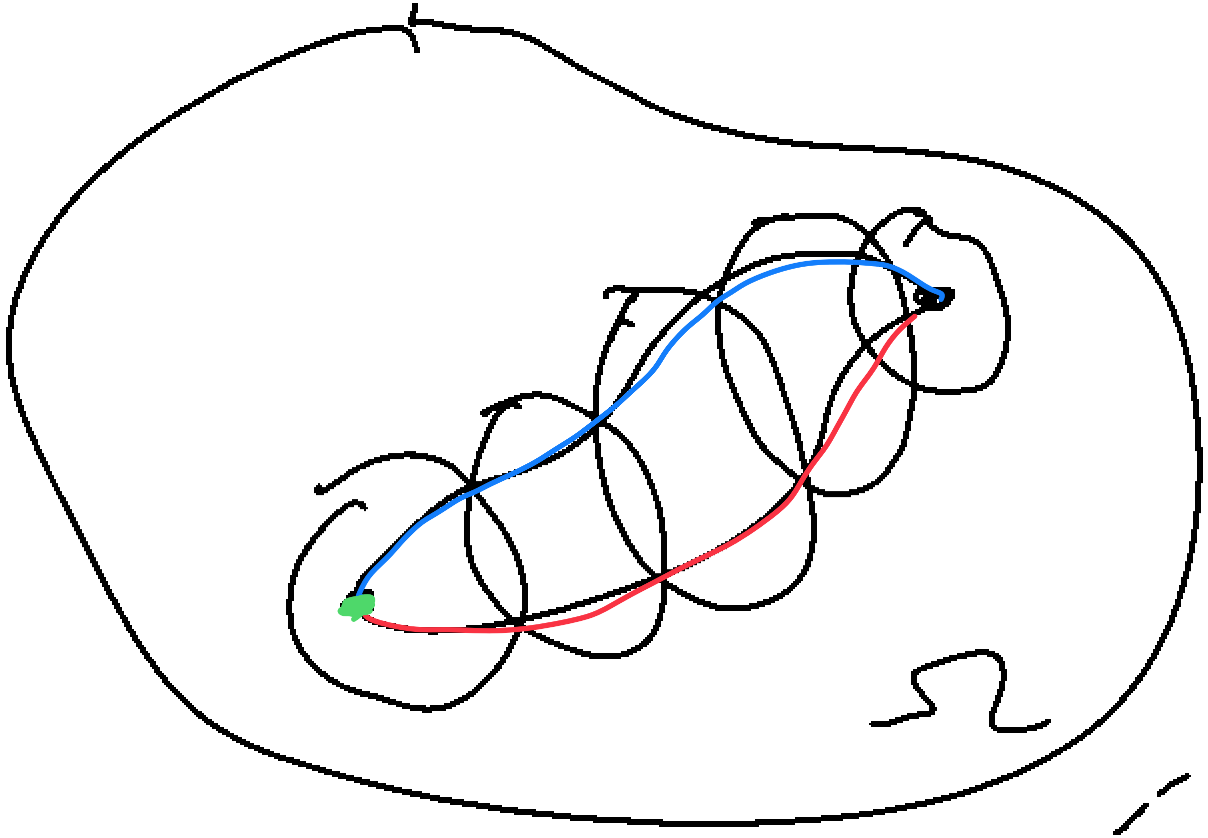


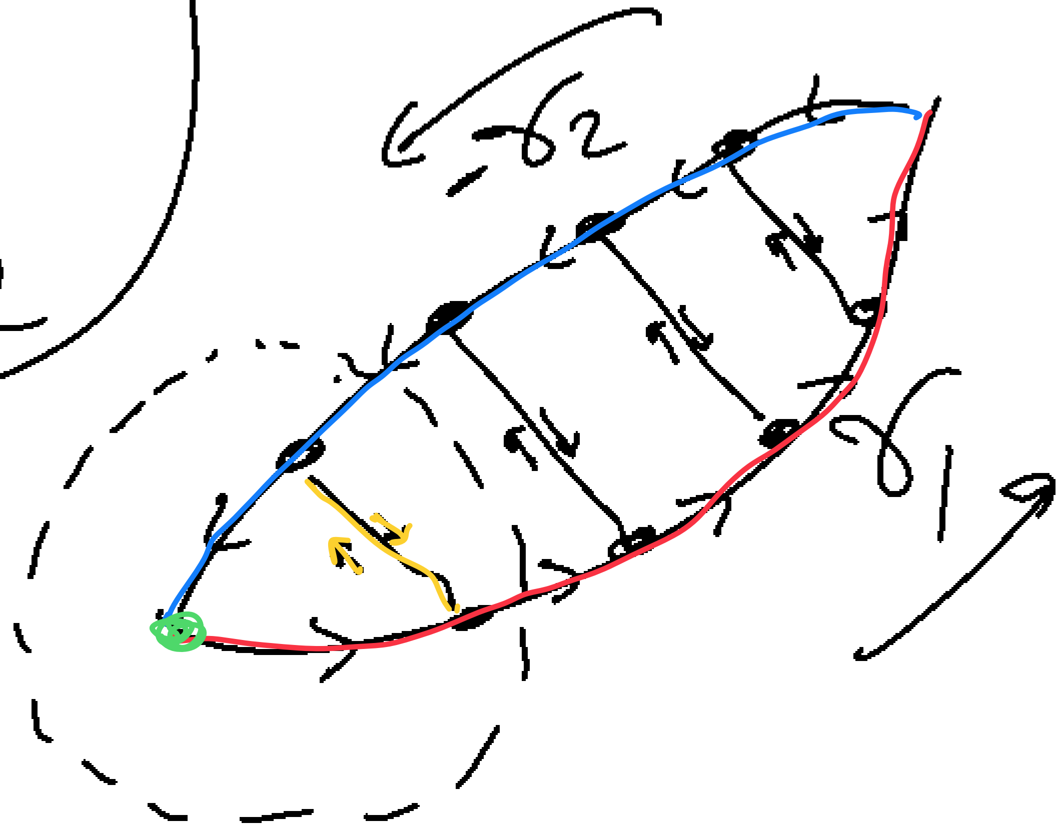




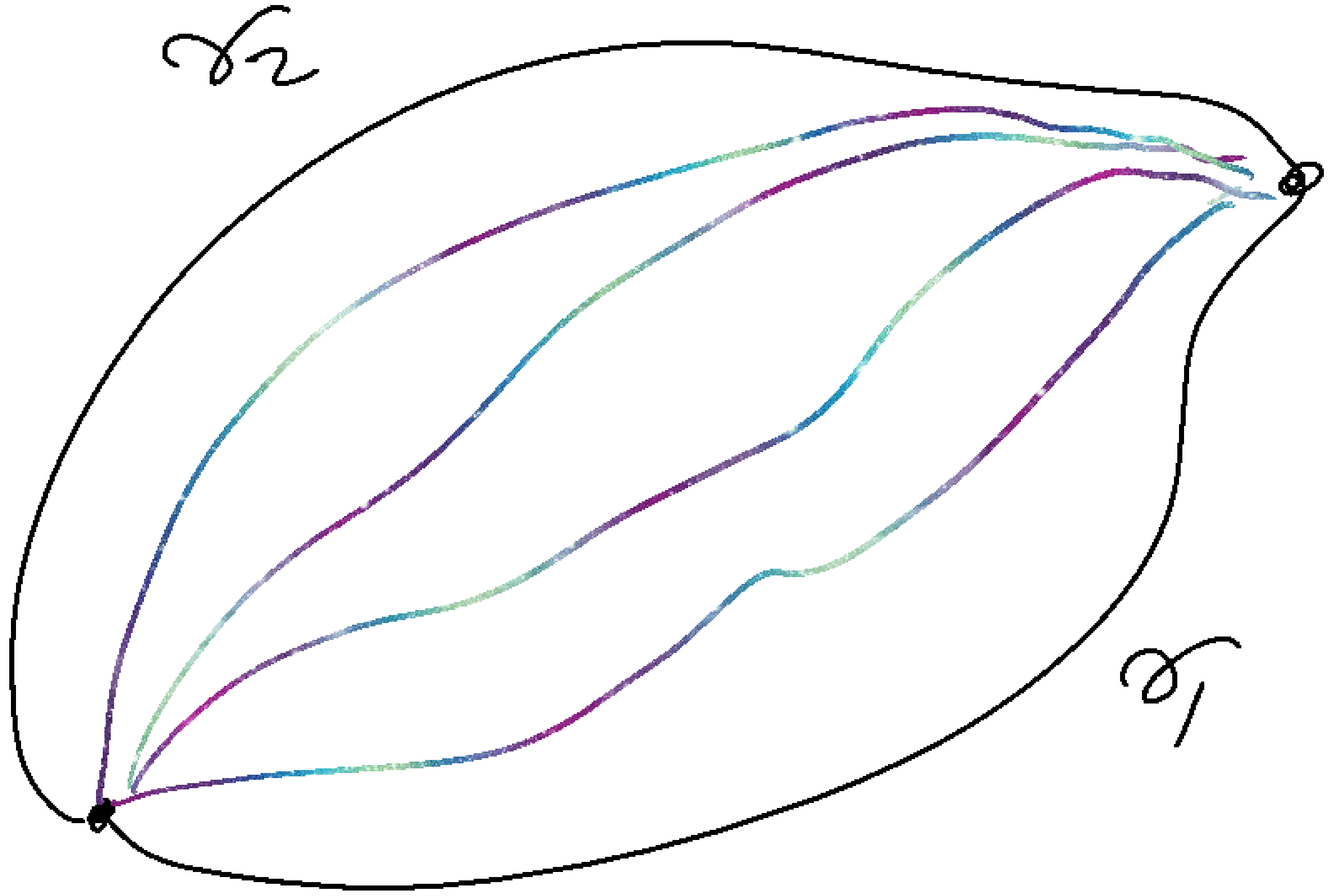


Can we go from γ_1 to γ_2 entirely in Ω





each closed curve gives zero, ok if primitives differ by a constant as would cancel


γ_2
γ_1

Theorem 4.1 *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

note ζ lies on circle C

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta \quad \text{for any point } z\in D.$$

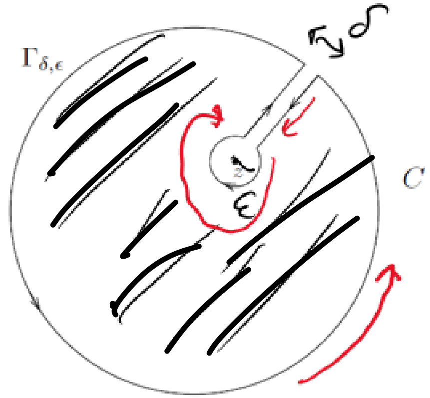


Figure 10. The keyhole $\Gamma_{\delta,\epsilon}$

Know $\displaystyle\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta=\int_{C_\varepsilon}\frac{f(\zeta)}{\zeta-z}\,d\zeta$

C_ε is a circle of radius ε centred at z.

So here, $\zeta=z+\varepsilon e^{i\theta}$

$$\underset{\text{on } C_\varepsilon}{f(\zeta)} = f(z)+f'(z)\cdot(\zeta-z)+\text{something small}$$

$$= f(z)+\text{something small by continuity}$$

$$\int_{C_\varepsilon}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\int_{\theta=0}^{2\pi}\frac{f(z)}{\cancel{\varepsilon}\cancel{e^{i\theta}}}\cdot i\cancel{\varepsilon}\cancel{e^{i\theta}}\,d\theta+\int_{\theta=0}^{2\pi}\frac{\text{something small}}{\cancel{\varepsilon}\cancel{e^{i\theta}}}\,i\cancel{\varepsilon}\cancel{e^{i\theta}}\,d\theta$$

$$=2\pi i f(z)+\left(\begin{array}{c}\text{something small in}\\ \text{abs value, goes to 0 as } \varepsilon\to 0\end{array}\right)=\int_C\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

Need for Later

$$A^n - B^n = (A-B)\left(A^{n-1} + A^{n-2}B + \dots + AB^{n-2} + B^{n-1}\right)$$

If $A=B$ is zero, so $A-B$ is a factor

Ex: $$(x+h)^n - x^n = h\left(x^{n-1} + x^{n-2}h + \dots + xh^{n-2} + h^{n-2}\right)$$

$$= hx^{n-1} + (\text{nice})\,h^2$$

$x = (y-z)-h \quad B = y-z$

$$\frac{d}{dz}(y-z)^{-1} = -1(y-z)^{-2}(-1) = 1\cdot(y-z)^{-2}$$

$$\frac{d^2}{dz^2}(y-z)^{-1} = \frac{d}{dz}\,1\cdot(y-z)^{-2} = 1\cdot(-2)(y-z)^{-3}(-1)$$

$$= 1\cdot 2\,(y-z)^{-3}$$

Corollary 4.2 *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for all z in the interior of C.

Physics Proof: $\frac{d}{dz} \int = \int \frac{d}{dz}$ Using $f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)} d\zeta$

Base Case done, Assume true for n, show true for $n+1$

$$\frac{f^{(n)}(z+h) - f^{(n)}(z)}{h} = \frac{1}{h} \frac{n!}{2\pi i} \int_C \left[\frac{f(\zeta)}{(\zeta - (z+h))^{n+1}} - \frac{f(\zeta)}{(\zeta - z)^{n+1}} \right] d\zeta$$

$$= \frac{n!}{2\pi i} \frac{1}{h} \int_C f(\zeta) \left[\frac{1}{((\zeta - z) - h)^{n+1}} - \frac{1}{(\zeta - z)^{n+1}} \right] d\zeta$$

Corollary 4.3 (Cauchy inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

$$|f^{(n)}(z_0)| \le \frac{n! \|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C}|f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

Proof: $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \zeta = z + Re^{i\theta}$

so $|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\|f\|_C \int_{\theta=0}^{2\pi} \left|\frac{Rie^{i\theta}}{(Re^{i\theta})^{n+1}}\right| d\theta$

$= \frac{n! \|f\|_C\, \cancel{2\pi} \cancel{R}}{\cancel{2\pi}\, R^{n+1}} = \frac{n! \|f\|_C}{R^n}$

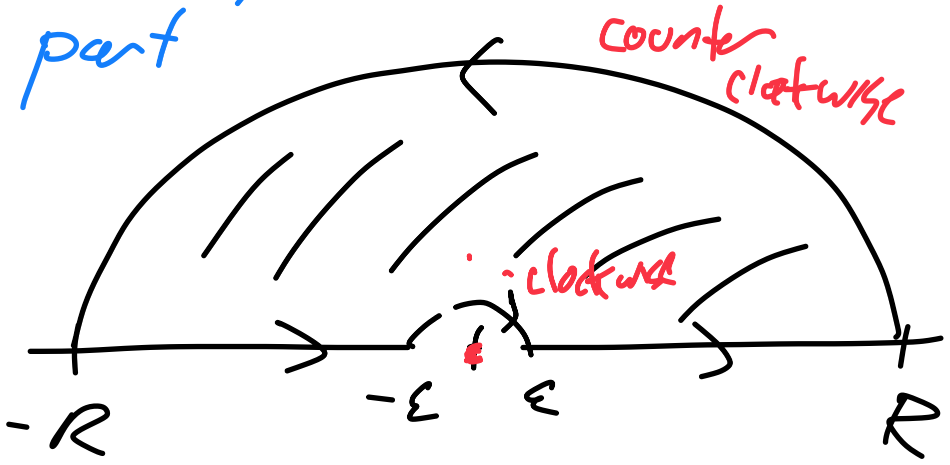
Integrate $\sin^2(x)/x^2$ and $1 / (1 + x^n)$

$$\int_{x=-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx \quad (\textstyle\int \text{ by parts})$$

$$\frac{e^{iz}}{z} \quad \text{or} \quad \frac{(e^{iz} - e^{-iz})/2i}{z}$$

Imaginary part

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} + \int_{\substack{\text{semicircle}\\ \text{radius } R}} - \int_{\substack{\text{semicircle}\\ \text{radius } \varepsilon}}$$



Cauchy Distribution

$$\frac{e^{iz}}{z} = \frac{e^{ix}e^{-y}}{x+iy}$$

if $z = x+iy$ is in H

the upper half plane ($y \geq 0$)

then $\left|\frac{e^{iz}}{z}\right| \leq \frac{1}{|z|}$

$$\left|\int_{\text{Semicircle radius } R} \frac{e^{iz}}{z}\,dz\right| \leq \int_{\theta=0}^{\pi} \frac{1}{R} R\,d\theta = \pi \neq 0$$

$z = Re^{i\theta}$

Appendix added after the lecture:

https://en.wikipedia.org/wiki/Simply_connected_space

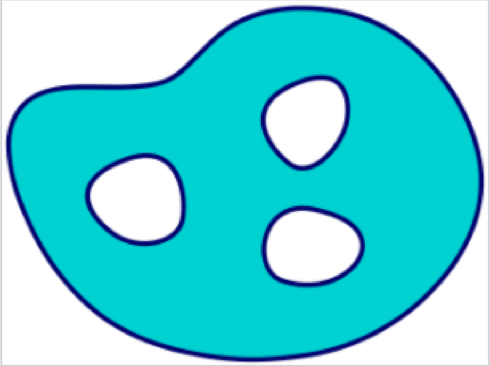
Key idea is simply connected.

Simply connected space

From Wikipedia, the free encyclopedia

In topology, a topological space is called **simply connected** (or **1-connected**, or **1-simply connected**[1]) if it is path-connected and every path between two points can be continuously

Definition and equivalent formulations [edit]



This shape represents a set that is not simply connected, because any loop that encloses one or more of the holes cannot be contracted to a point without exiting the region.

A topological space X is called *simply connected* if it is path-connected and any loop in X defined by $f : S^1 \to X$ can be contracted to a point: there exists a continuous map $F : D^2 \to X$ such that F restricted to S^1 is f. Here, S^1 and D^2 denotes the unit circle and closed unit disk in the Euclidean plane respectively.

An equivalent formulation is this: X is simply connected if and only if it is path-connected, and whenever $p : [0,1] \to X$ and $q : [0,1] \to X$ are two paths (that is, continuous maps) with the same start and endpoint ($p(0) = q(0)$ and $p(1) = q(1)$), then p can be continuously deformed into q while keeping both endpoints fixed. Explicitly, there exists a homotopy $F : [0,1] \times [0,1] \to X$ such that $F(x,0) = p(x)$ and $F(x,1) = q(x)$.

A topological space X is simply connected if and only if X is path-connected and the fundamental group of X at each point is trivial, i.e. consists only of the identity element. Similarly, X is simply connected if and only if for all points $x, y \in X$, the set of morphisms $\mathrm{Hom}_{\Pi(X)}(x,y)$ in the fundamental groupoid of X has only one element.[2]

In complex analysis: an open subset $X \subseteq \mathbb{C}$ is simply connected if and only if both X and its complement in the Riemann sphere are connected. The set of complex numbers with imaginary part strictly greater than zero and less than one, furnishes a nice example of an unbounded, connected, open subset of the plane whose complement is not connected. It is nevertheless simply connected. It might also be worth pointing out that a relaxation of the requirement that X be connected leads to an interesting exploration of open subsets of the plane with connected extended complement. For example, a (not necessarily connected) open set has connected extended complement exactly when each of its connected components are simply connected.

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Lecture 07: 9-22-23: https://youtu.be/UVvYwP7_5Ww

Evaluating Integrals

Cauchy Distribution: $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ or $\frac{1}{1+x^2}$

$$\tan(\arctan x) = x$$

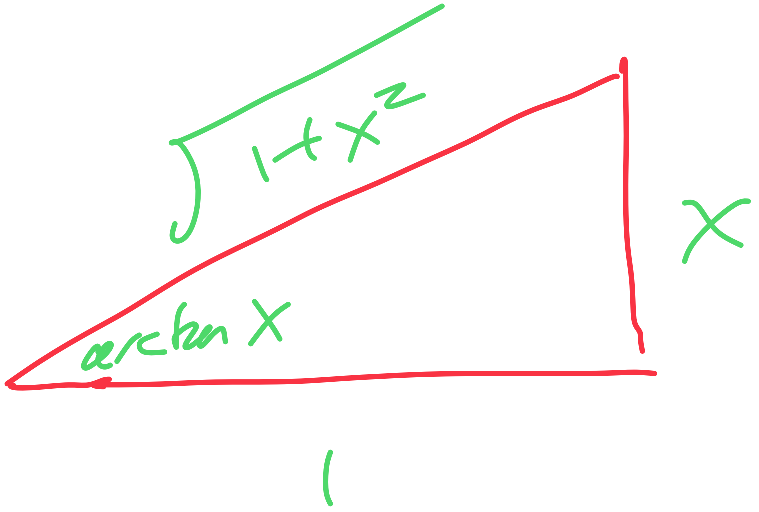
deriv

$$\tan'(\arctan x)\cdot\arctan'(x) = 1$$

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)}$$

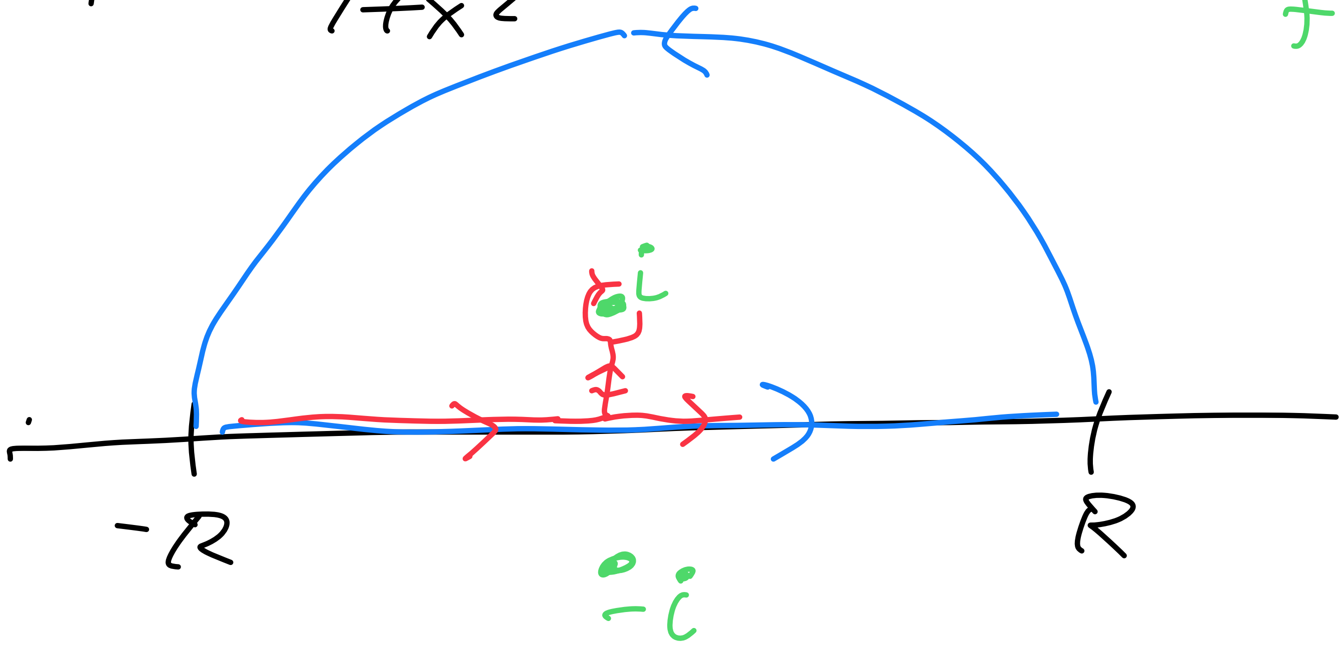
$$= \cos^2(\arctan x)$$

$$= \frac{1}{1+x^2}$$



$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = 2\int_0^{\infty}\frac{1}{1+x^2}\,dx = 2\left(\arctan x\Big|_0^{\infty}\right) = 2\left(\frac{\pi}{2}-0\right) = \pi$$

$$f(x) = \frac{1}{1+x^2}$$



$$f(z) = \frac{1}{1+z^2}$$

$$= \frac{1}{(z-i)(z+i)}$$

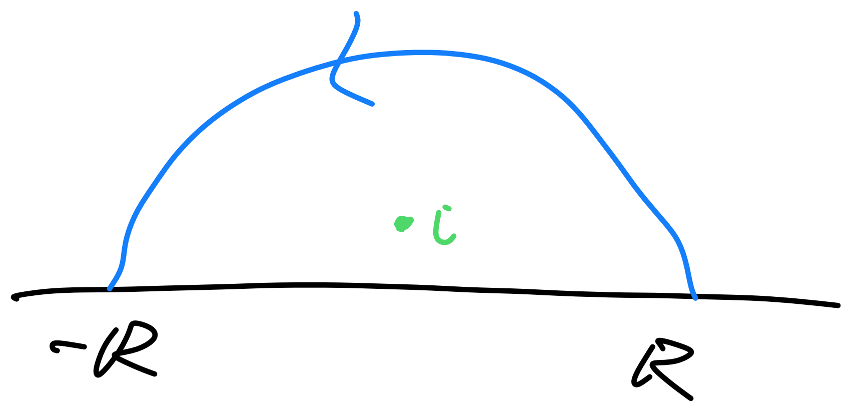
$$= \frac{1/2i}{z-i} - \frac{1/2i}{z+i}$$

$$= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$\int_{-R}^{R} f(x)\,dx + \int_{SC(R)} f(z)\,dz$$

$$+ \int_{-C(i)} f(z)\,dz = 0$$

$$\int_{SC(R)} \frac{1}{1+z^2}\,dz$$

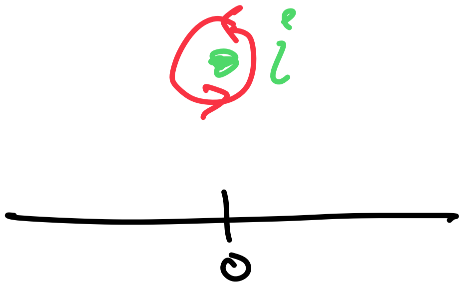


$$\left|\frac{1}{1+z^2}\right| \underset{\substack{\text{on } SC(R)\\ R \text{ large}}}{\leq} \frac{1}{R^2-1}$$

$$\left|\int_{SC(R)} \frac{1}{1+z^2}\,dz\right| \leq \underbrace{\frac{1}{R^2-1}}_{\substack{\text{largest}\\ \text{value}}} * \underbrace{\pi R}_{\substack{\text{length}\\ \text{of curve}}} = \frac{\pi R}{R^2-1} \to 0$$

$$\int_{C(\varepsilon)} \frac{1}{1+z^2}\,dz = \int_{C(\varepsilon)} \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

$$= \frac{1}{2i}\left[\int_{C(\varepsilon)} \frac{1}{z-i}\,dz - \int_{C(\varepsilon)} \frac{1}{z+i}\,dz\right]$$



$C(\varepsilon) \to C_i(\varepsilon)$

$z = i + \zeta$

$dz = d\zeta$

Zero: fn is hol in region, has primitive!

$C_0(\varepsilon) \to C(\varepsilon)$

$$\frac{1}{2i}\int_{C(\varepsilon)} \frac{1}{\zeta}\,d\zeta$$

$$\frac{1}{2i}\,2\pi i = \pi$$

Avoiding Partial fractions

$$f(z) = \frac{1}{(z-i)(z+i)} = \sum_{n=-N}^{\infty} a_n (z-i)^n$$

$$\int_{C_i(\varepsilon)} f(z)\,dz = \sum_{n=-N}^{\infty} a_n \int_{C_i(\varepsilon)} (z-i)^n\,dz$$

$$= a_{-1} \int_{C_i(\varepsilon)} \frac{1}{z-i}\,dz$$

$$= 2\pi i\, a_{-1}$$

Need -1^{st} term in Taylor Series of $\dfrac{1}{(z-i)(z+i)}$

$$f(z) = \frac{1}{z-i}\left(\frac{1}{z+i}\right)$$

only need constant term here at $z=i$, which is $1/2i$

need $z-i$

$$= \frac{1}{z-i}\left(\frac{1}{z-i+i+i}\right)$$

$$= \frac{1}{z-i}\left(2i+(z-i)\right)^{-1}$$

$$\frac{1}{1-r} = 1+r+r^2+\dots$$

$$r = -\frac{z-i}{2i}$$

$$= \frac{1}{z-i}\,\frac{1}{2i}\left(\frac{1}{1-\frac{-(z-i)}{2i}}\right) = \frac{1}{2i}\,\frac{1}{z-i}\left[1-\frac{z-i}{2i}+\left(\frac{z-i}{2i}\right)^2\dots\right.$$

see $a_{-1} = 1/2i$

Ponder: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^n}\,dx$ n pos integer

$$f(z) = \frac{1}{(1+z^2)^n} = \frac{1}{(z-i)^n}\,\frac{1}{(z+i)^n}$$

just need $(n-1)^{st}$ term from second factor

Ponder: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^{2n}}\,dx$ or $\displaystyle\int_{0}^{\infty} \frac{1}{1+x^n}\,dx$

Consider $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$ ($\int$ by parts)

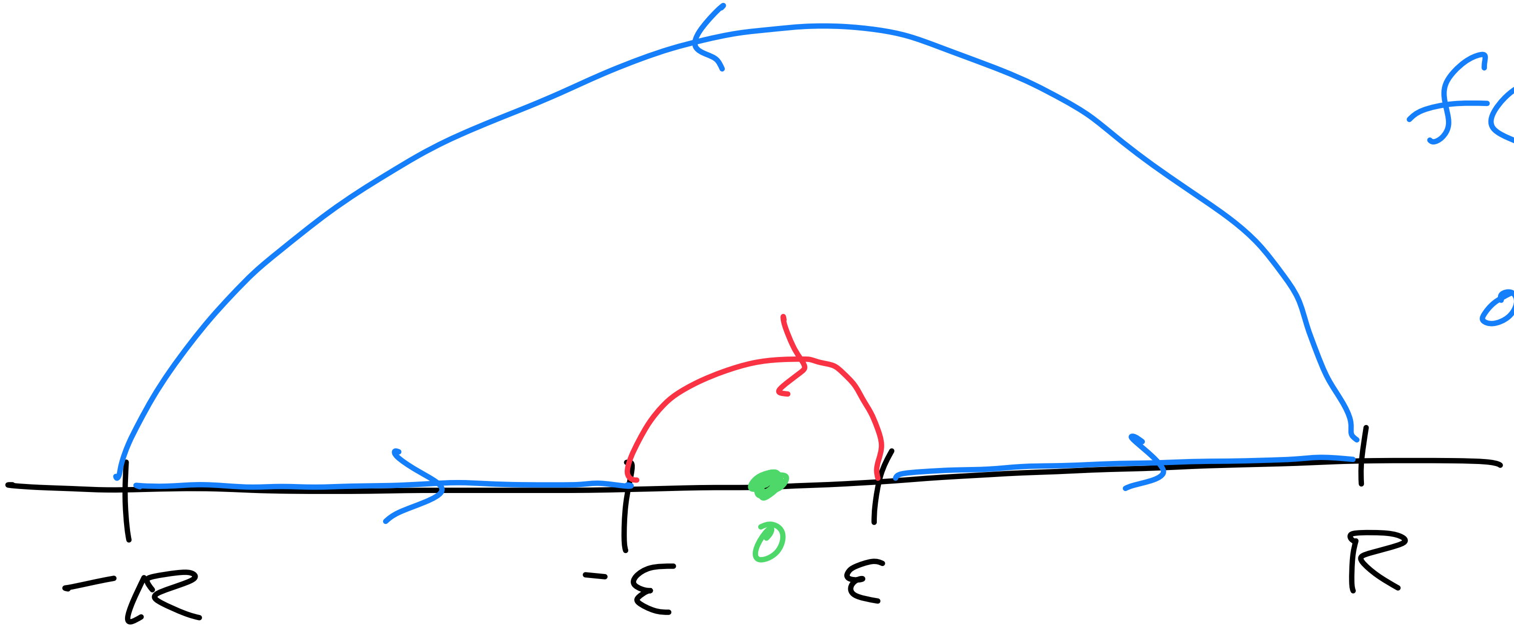
(1) $\sin z = (e^{iz} - e^{-iz})/2i$

(2) $\sin x = \mathrm{Im}(e^{ix})$

Study $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx$, take imaginary part!

$$f(z) = \frac{e^{iz}}{z}$$

only pole at 0



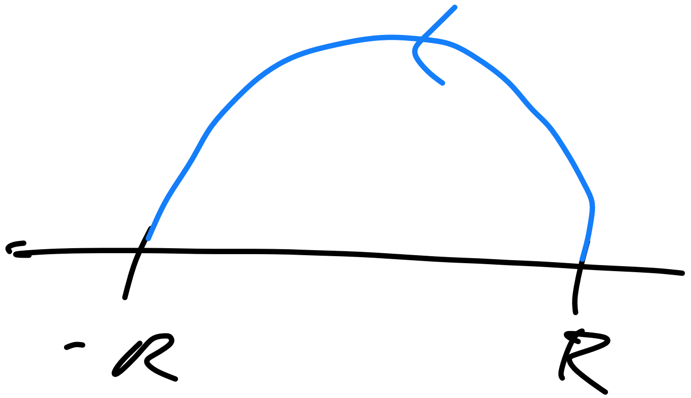
$$0 = \underbrace{\int_{-R}^{-\varepsilon} f(x)\,dx + \int_{\varepsilon}^{R} f(x)\,dx}_{\text{as } R\to\infty \text{ this goes to } i\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx} + \int_{SC(R)} f(x)\,dx + \int_{-C_0(\varepsilon)} f(x)\,dx$$

$$f(z) = \frac{e^{iz}}{z}$$

$z = x+iy$

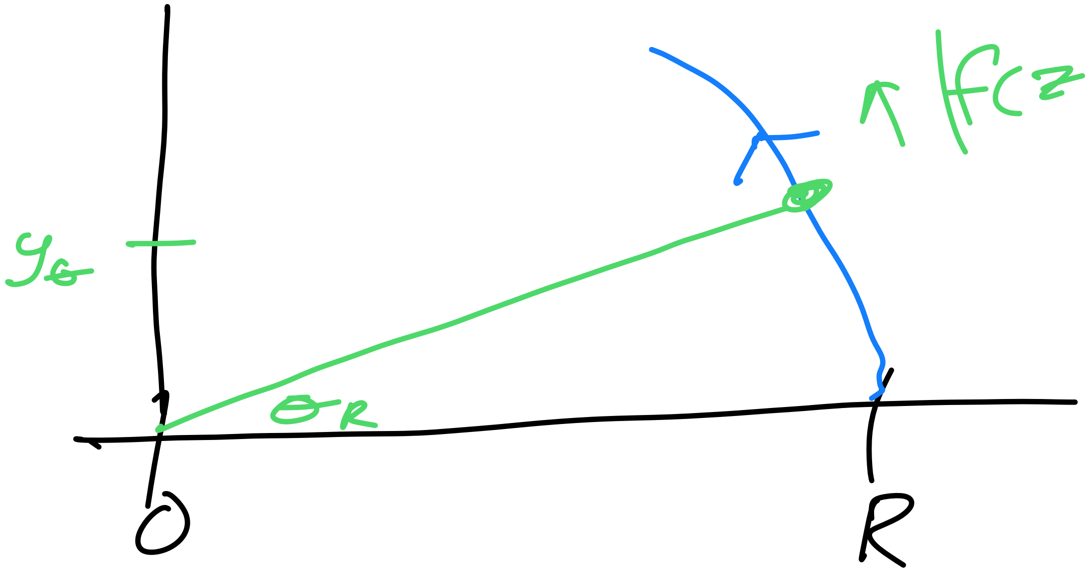
$y \geq 0$

$|z| = R$

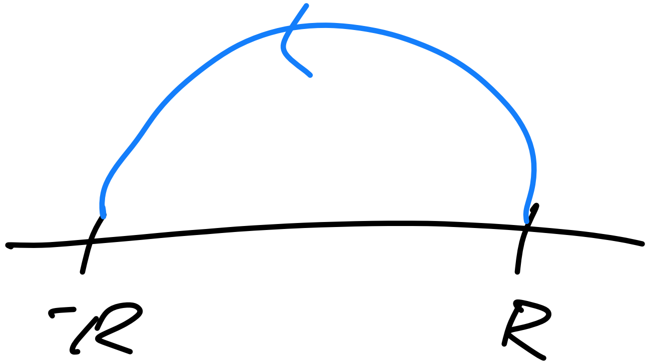


$$\left|\int_{SC(R)} f(z)\,dz\right| \leq \max|f(z)| * \pi R = \frac{1}{R}\pi R = \pi$$

$$e^{iz} = e^{ix}\underbrace{e^{-y}}_{\text{decays!}} \qquad y>0$$



$|f(z)| \le e^{-y_0 R}/R$



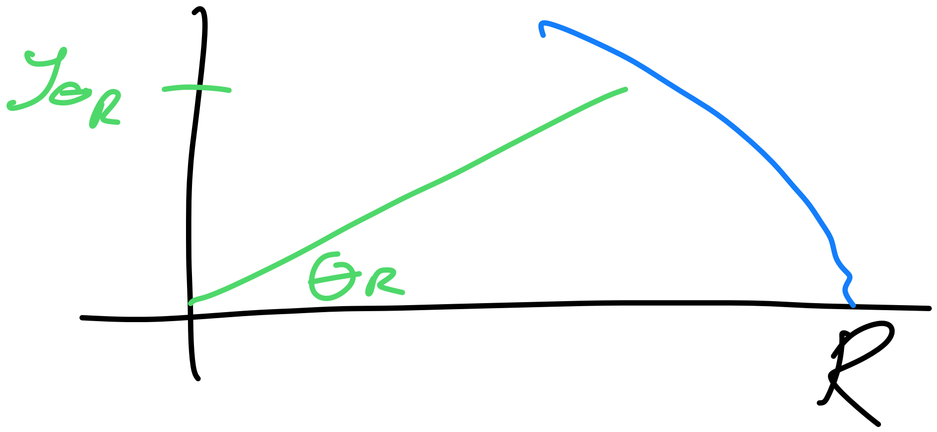
$$\underbrace{\int_{\theta=\theta_R}^{\pi-\theta_R} f(z)\,dz}_{\substack{\text{abs value} \\ \le \frac{e^{-y_0 R}}{R}\cdot\pi}} + \underbrace{\int_{\theta=0}^{\theta_R} f(z)\,dz}_{\le \frac{1}{R}\theta_R\cdot R} + \underbrace{\int_{\theta=\pi-\theta_R}^{\pi} f(z)\,dz}_{\le \frac{1}{R}\theta_R\cdot R}$$

win if $y_0 R \to \infty$

win if $\frac{R\theta_R}{R} \to 0$

Find θ_R such that

(1) $y_{\theta_R} \to \infty$ $\qquad (z)^{R\theta_R}/R \to 0$



$$y_{\theta_R} = R\sin\theta_R \geq R\frac{\theta_R}{2}$$

$$\approx R\,\theta_R$$

$$\theta_R = \frac{1}{R^\delta} \qquad \delta \in (0,1)$$

$$\text{or } \theta_R = \frac{1}{\log R}$$

Note: In the video there is a mistake. The length of the arc as θ goes from 0 to θ_R is NOT θ_R but $R \cdot \theta_R$.

So we need the angle to go to zero, but the corresponding arc length goes to ∞, just much slower than the length of the semi-circle, which is πR.

Thus we need $\frac{R\theta_R}{R} \to 0$ so $\theta_R \to 0$. It cannot go to zero too rapidly or $e^{-\gamma \theta_R} \approx e^{-R\theta_R}$ would not go to zero. Thus $\theta_R \sim 1/R$ is too fast, $\theta_R \sim 1/R^2$ is even worse, but $\theta_R \sim R^{-\delta}$ for $0<\delta<1$ works, as does $\theta_R \sim 1/\log R$.

Lectures 08, 09 and 10:
Lectures from Math 150: Multivariable Calculus: Sequences and Series

Read multivariable calculus (Cain and Herod) and my lecture notes.
Read Intermediate and Mean Value Theorems and Taylor Series (you should know this material already; the main results are stated and mostly proved, subject to some technical results from analysis which we need to rigorously prove the IVT).

- **Twenty-third day lecture: http://youtu.be/aigdKmu-5ow (April 28, 2014: Geometric and Harmonic Series, Memoryless Processes)**
- **Twenty-fourth day lecture: http://youtu.be/6bf9fjwMs2o (May 2, 2014: Comparison Test, Implications of Limits of Terms)**
- **Twenty-fifth lecture:** https://youtu.be/fa4X1AcGFOA **(May 2, 2014: Comparison Test, Implications of Limits of Terms: II)**
- **Twenty-sixth lecture: https://youtu.be/jFgCKfUTOQ8 (Comparison Test, Implications of Limits of Terms: III)**
- **Twenty-seventh day lecture: http://youtu.be/ujJbUpCab6M (May 7, 2014: Root Test, Integral Test)**
- **Twenty-eight day lecture: http://youtu.be/yr01SLw9t4c (May 12, 2014: Taylor Series)**
- **Twenty-ninth day lecture: http://youtu.be/4OcxtpxuSJw (May 14, 2014: Special Series, Alternating Series, Pi formulas, Birthday Problem: Not doing 2018)**

Math 383: Complex Analysis: Fall ‘23 (Williams)

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Lecture 11: 10-02-23: https://youtu.be/3pgsv4hVrPs

Holomorphic is Analytic, Liouville's Theorem, Fundamental Theorem of Algebra, Cauchy Integral

Plan for the day: Lecture 11: October 2, 2023:

- Prove holomorphic and analytic are synonyms.
- Prove the accumulation theorem.
- Prove Liouville’s Theorem.
- Prove the Fundamental Theorem of Algebra.

(don’t get the name fundamental lightly!)

Watch Videos from 2021 By Friday.

- Lecture 08: 9/27/21: Cauchy Formula, Accumulation Theorem, Cauchy-like Integrals: https://youtu.be/wSqTEQ4usno (slides)
- Lecture 09: 9/29/21: Integration Examples: (see for other examples Lecture 09: 9/27/17: Integration Example, Types of Singularities: https://youtu.be/Jz76hM32C80) (slides)

Theorem 4.1 *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \textit{for any point } z \in D.$$

Corollary 4.2 *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for all z in the interior of C.

Corollary 4.3 (Cauchy inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

Theorem 4.4 *Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, and the coefficients are given by $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ *for all $n \geq 0$.*

adding $0 = -z_0 + z_0$

Proof: Key idea is to add zero and then factor and use the geometric series formula:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0}\left(\frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}\right)$$

looks like geometric series: $r = \dfrac{z - z_0}{\zeta - z_0}$

$$f(z) = \frac{1}{2\pi i}\int_R \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad f^{(n)}(\cancel{z}) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - \cancel{z})^{n+1}}\, d\zeta$$

z_0 z_0

$$f(z) = \frac{1}{2\pi i}\int_R \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

z_0 $|\zeta - z| > \delta > 0$

$$= \frac{1}{2\pi i}\int_R \frac{f(\zeta)}{\zeta - z_0}\left(1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots\right) d\zeta$$

$$= \frac{1}{2\pi i}\int_R \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n\, d\zeta$$

Justify $\int \sum = \sum \int$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_R \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right](z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$\blacksquare$

Theorem 4.8 *Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.*

Assume f is not identically zero.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z^*)$$

where $f(z_k) = 0$, $\lim_{k\to\infty} z_k = z^*$

$f(z^*) = 0$ by continuity

Claim f is identically 0 near z^*

Proof: if all $a_n = 0$, done, else at least one $a_n \neq 0$, say a_N is first.

$$f(z) = a_N(z-z^*)^N + a_{N+1}(z-z^*)^{N+1} + \cdots$$

$$= a_N(z-z^*)^N\left(1 + \frac{a_{N+1}}{a_N}(z-z^*) + \cdots\right)$$

as $z \to z^*$, looks like

$$f(z) = a_N(z-z^*)^N(1 + \text{small})$$

Means only point near z^* where f vanishes is z^*, contra.

(1) $f(x) = x^3 \sin(1/x)$

$\Rightarrow z^3 \sin(1/z)$ is not holo.

(2) Say $f(z) = g(z)$ for all z_k with $\lim_{k\to\infty} z_k = z^*$ in region. Then $f = g$

We conclude the proof using the fact that Ω is connected. Let U denote the interior of the set of points where $f(z) = 0$. Then U is open by definition and non-empty by the argument just given. The set U is also closed since if $z_n \in U$ and $z_n \to z$, then $f(z) = 0$ by continuity, and f vanishes in a neighborhood of z by the argument above. Hence $z \in U$. Now if we let V denote the complement of U in Ω, we conclude that U and V are both open, disjoint, and

$$\Omega = U \cup V.$$

Since Ω is connected we conclude that either U or V is empty. (Here we use one of the two equivalent definitions of connectedness discussed in Chapter 1.) Since $z_0 \in U$, we find that $U = \Omega$ and the proof is complete.

Corollary 4.5 (Liouville's theorem) *If f is entire and bounded, then f is constant.*

Corollary 4.3 (Cauchy inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C} |f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

entire: holo on all of $\mathbb{C}$

Proof 1: holo = analytic, all $f^{(n)}(z_0) = 0$ for $n \geq 1$

So Taylor Series is $f(z) = a_0$.

$g(x) = e^{-1/x^2}$ if $x \neq 0$ and 0 otherwise

Proof 2: for all z_0, $f'(z_0) = 0$ so f' identically 0

So f is constant.

What are solutions to polynomials with different spaces of coefficients?

$f(x)$ is $ax+b$ $\quad a,b \in \mathbb{Z} \implies x \in \mathbb{Q}$ not nec $\mathbb{Z}$

$f(x)$ is $r_1 x^2 + r_2$ $\quad r_1, r_2 \in \mathbb{Q} \implies x \in$

$r_1 x^2 + r_2$ with $r_1 > 0$, $r_2 < 0 \implies x \in \mathbb{R}$

$x^2+1 \implies x \in \mathbb{C}$ or $x \in i\mathbb{R}$

$f(z) = a_n z^n + \dots + a_0$ with $a_k \in \mathbb{C}$ then has n roots in $\mathbb{C}$

Executive Summary: $\sqrt{-1} = i$ is enough!

Corollary 4.6 *Every non-constant polynomial* $P(z) = a_n z^n + \cdots + a_0$ *with complex coefficients has a root in* $\mathbb{C}$.

Assume not, $P(z) \neq 0$ for all z

$P(z) = a_n z^n + \cdots + a_0, \quad a_n \neq 0$

$= a_n z^n \left(1 + \left(\frac{a_{n-1}}{a_n}\frac{1}{z} + \frac{a_{n-2}}{a_n}\frac{1}{z^2} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\right)\right)$

as $z \to \infty$, $1/z$, $1/z^2$, ... go to zero

so the sum is at most $n * \max\left|\frac{a_{n-k}}{a_n}\right| * \frac{1}{|z|}$

if $|z|$ is large, this is < 1 in abs value, so

$P(z) \neq 0$ and $|P(z)| \geq \frac{1}{2}|a_n z^n|$ (Liouville)

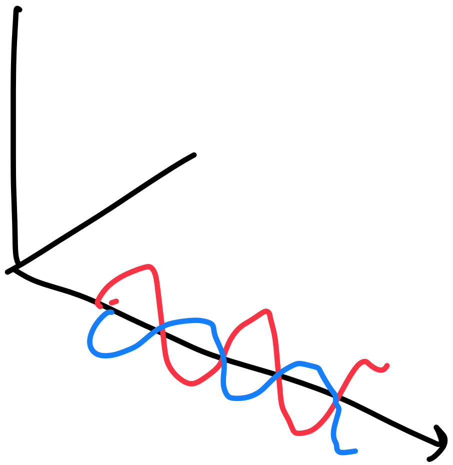
$\frac{1}{P(z)} \leq \frac{2}{|a_n z^n|} \leq B$ for some B. Get $\frac{1}{P(z)}$ is constant

not obs hypothesis

Contradiction

Corollary 4.7 *Every polynomial $P(z) = a_n z^n + \cdots + a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $w_1, \ldots, w_n$, then P can be factored as*

$$P(z) = a_n(z - w_1)(z - w_2) \cdots (z - w_n).$$



Poynting Vector

Math 383: Complex Analysis: Fall ‘23 (Williams)

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Lecture 12: 10-04-23: https://youtu.be/01xjCbFe8_Y

Singularities, Probabilities, Generating Functions, Trig Integrals

Plan for the day: Lecture 12: October 4, 2023:

- Riemann's Removable Singularity Theorem
- Casorati-Weierstrass
- Examples
- Infinities

General items.

- Difference between real and complex
- Finding right object to study

Let f be a function holomorphic in an open set Ω except possibly at one point z_0 in Ω. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a **removable** singularity for f.

Theorem 3.1 (Riemann's theorem on removable singularities) *Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.*

$f(z) = \dfrac{z-3}{z^2-9}$ Candidates for explosions: $z = \pm 3$ $f(z) = \dfrac{z-3}{(z-3)(z+3)} = \dfrac{1}{z+3}$

Surprisingly, we may deduce from Riemann's theorem a characterization of poles in terms of the behavior of the function in a neighborhood of a singularity.

Corollary 3.2 *Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Isolated singularities belong to one of three categories:

- Removable singularities (f bounded near z_0)
- Pole singularities ($|f(z)| \to \infty$ as $z \to z_0$) $\quad a_{-4}\frac{1}{z^4} + a_{-3}\frac{1}{z^3} + \cdots$
  pole of order 4
- Essential singularities.

By default, any singularity that is not removable or a pole is defined to be an **essential singularity**.

$f(z) = e^{1/z^2}$ $\quad w = 1/z$ $\quad e^{w^2}$ nice near $w = 0$ or $z = \infty$

Theorem 3.3 (Casorati-Weierstrass) *Suppose f is holomorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.*

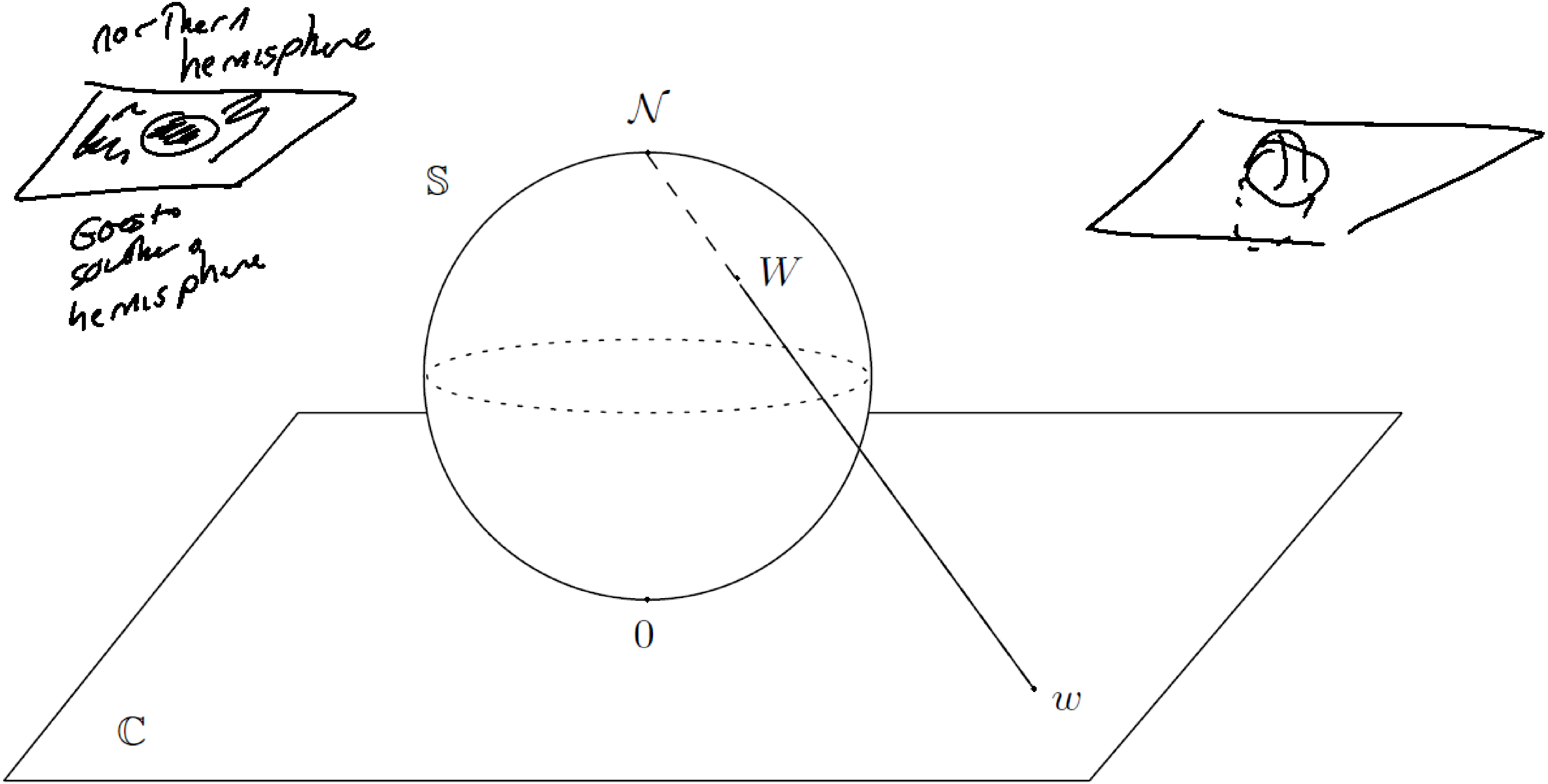


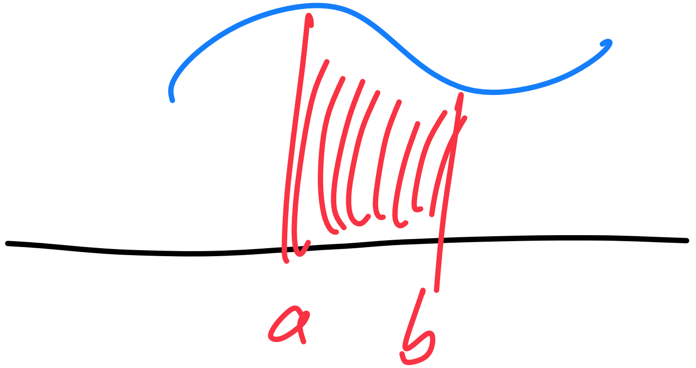
Figure 5. The Riemann sphere $\mathbb{S}$ and stereographic projection

Probability Density

X is a random variable with density f or f_X

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$



(3) $\mathrm{Prob}(a \leq X \leq b) = \int_a^b f(x)\,dx$

Taylor Series: $f(x) \text{ "=" } \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

danger function: $f(x) = e^{-1/x^2}$ as $f^{(n)}(0) = 0$

Note e^x and $1_{x>0} f(x) + e^x$ have same Taylor Series

Moment Generating Functions vs Characteristic Function $\mathbf{E}\left[e^{tX}\right]$ $\mathbf{E}\left(e^{itX}\right)$

n^{th} moment $\mu_n := \mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx$

Hope: moments determine the density

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{n=0}^{\infty} \frac{t^n x^n}{n!}\right) f(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{n!} \Bigg(\underbrace{\int_{-\infty}^{\infty} x^n f(x)\,dx}_{\mu_n}\Bigg) t^n$$

Consider $f(x) = \frac{1}{1+x^2}$ or $\frac{1}{\pi}\frac{1}{1+x^2}$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{1+x^2}\,dx = \begin{cases} \pi & \text{if } t = 0 \\ \infty & \text{if } t \neq 0 \end{cases}$$

Characteristic function: $\mathbb{E}[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$ Always exists

"Fourier Transform"

Consider f(x) = exp(-x) for x non-negative

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} e^{-x}\,dx$$

$$= \int_0^{\infty} e^{-(1-t)x}\,dx$$

need $1-t>0$ else integral diverges

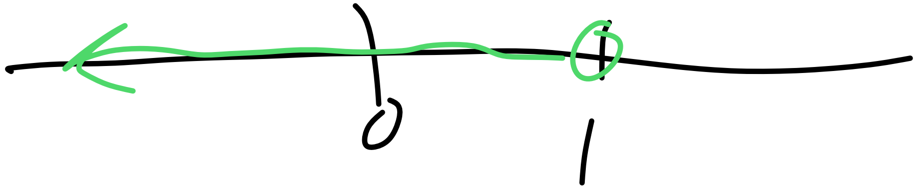
$u=(1-t)x \qquad x: 0 \to \infty$

$du=(1-t)\,dx \qquad u: 0 \to \infty$ if $1-t>0$

$$= \int_{u=0}^{\infty} e^{-u} \frac{du}{1-t} = \frac{1}{1-t}$$

Here $M_X(t)$ is $\frac{1}{1-t}$ if $t<1$

and undefined otherwise



Cauchy: Moment Generating Functions vs Characteristic Functions: $\mathrm{E}[e^{tX}]$ versus $\mathrm{E}(e^{itX})$

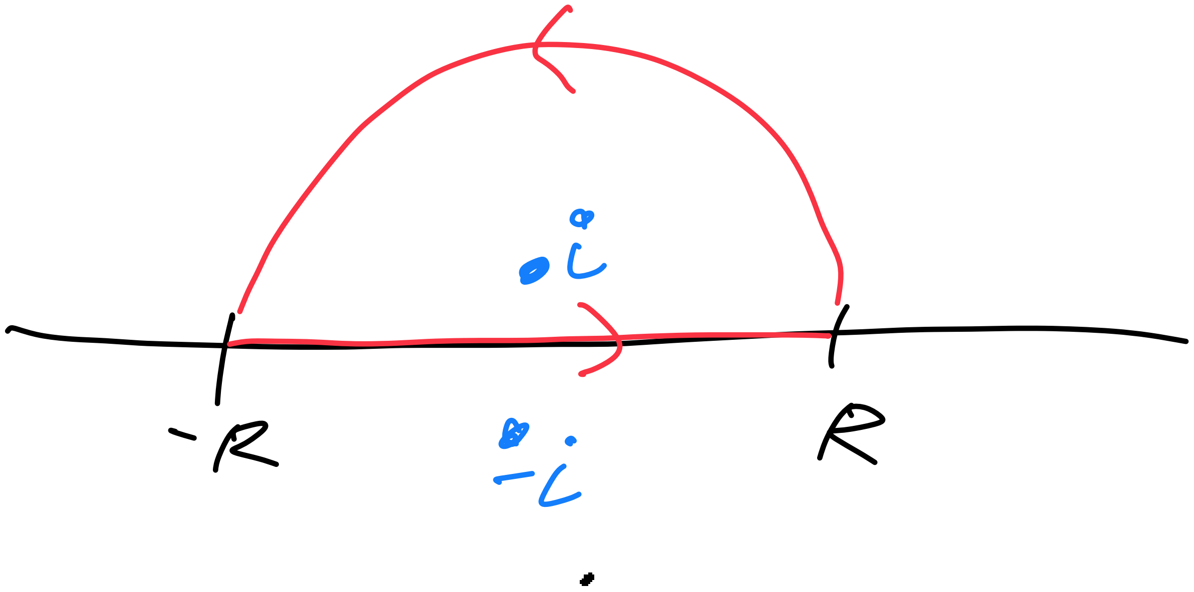
$$E[e^{itX}] = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos(tx) + i\sin(tx)}{1+x^2}\,dx$$

note imaginary part is odd, can drop

$$f(z) = \frac{e^{itz}}{z^2+1} \quad \text{not} \quad \frac{\cos(tz)}{z^2+1}$$

with $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$

Assume $t>0$, if $t<0$ similar



poles of $f(z) = \dfrac{e^{itz}}{z^2+1}$

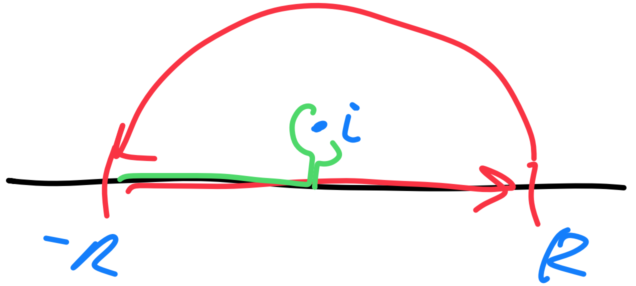
at $z = i$ and $z = -i$

On semicircle

$$|f(z)| \le \frac{|e^{itz}|}{|z^2|-1} = \frac{e^{-ty}}{|z^2|-1}$$

Cauchy: Fourier Transform

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx$$



$$\frac{e^{itz}}{z^2+1} = a_{-1}(z-i)^{-1} + a_0(z-i)^0 + a_1(z-i) + a_2(z-i)^2 + \cdots$$

$$\int_{-R}^{R} f(z)\,dz + \int_{SC(R)} f(z)\,dz = 2\pi i\, a_{-1}(i)$$

$\mathrm{Res}_f(i)$

$$f(z) = \frac{e^{itz}}{z^2+1} = \left(\frac{e^{itz}}{z+i}\right)\left(\frac{1}{z-i}\right)$$

$\hookrightarrow b_0 + b_1(z-i) + b_2(z-i)^2 + \cdots$ here b_0 is just $\dfrac{e^{iti}}{i+i}$ or $e^{-t}/2i$

As $R \to \infty$, get

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx = 2\pi i\,\frac{e^{-t}}{2i} = \pi e^{-t}$$

Reasonable: (1) Get π at $t=0$ (2) Real and positive (3) as $t \to \infty$, goes to 0

Cauchy: Finding Residue

$$f(z) = A(z)\,B(z)$$

if $B(z) = (z-z_0)^{-3} + 2(z-z_0)^{-2} + 8(z-z_0)^{-1} + \ldots$

$$A(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

how do I get $\frac{1}{(z-z_0)}$?

$$a_0 \cdot 8(z-z_0)^{-1} + a_1(z-z_0)\cdot 2(z-z_0)^{-2} + a_2(z-z_0)^2 \cdot (z-z_0)^{-3}$$

Contour Integration: Integrals of Trigonometric Functions

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}$$

could do $\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta} = \frac{1}{\beta}\int_0^{2\pi} \frac{d\theta}{\frac{\alpha}{\beta} + \cos\theta}$

assume $|a| > 1$ so denom $\neq 0$

Integrate[1/(a + Cos[x]), {x, 0, 2 Pi}]

$$\frac{2\, i\, (-1)^{\text{Floor}\left[\frac{-2\,\text{Arg}[-1+a]+\text{Arg}[1-a^2]}{2\,\pi}\right]}\,\pi}{\sqrt{1-a^2}}$$

```
Integrate[1/(a + Cos[x]), {x, 0, 2 Pi},
 Assumptions → {a > 1 && Element[a, Reals]}]
```

$$\frac{2\,\pi}{\sqrt{-1+a^2}}$$

```
Integrate[1/(a + Cos[x]^2), {x, 0, 2 π}, Assumptions → a > 1]
```

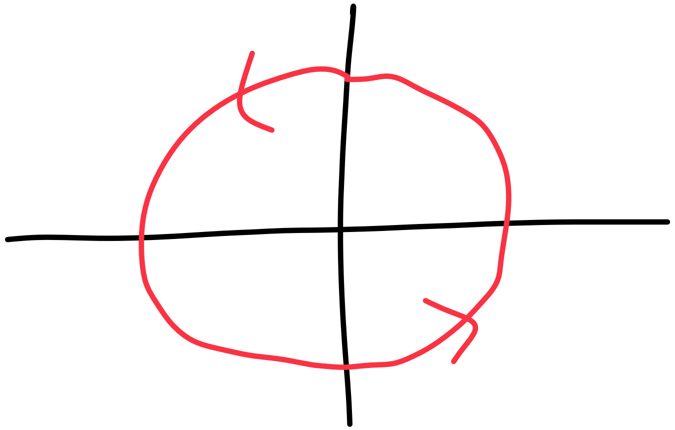
$$\frac{2\,\pi}{\sqrt{a\,(1+a)}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

On the unit circle: $z = e^{i\theta}$ and $e^{-i\theta} = 1/z$, $dz = ie^{i\theta}d\theta$

So on the unit circle: $\cos\theta$ as $(z+1/z)/2$ $\qquad d\theta = -i\frac{1}{z}dz$

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \int_{\text{unit circle}} \frac{(-i/z)\,dz}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)} = -2i \int_{\text{unit circle}} \frac{dz}{z^2+2az+1}$$



Find poles of $\frac{1}{z^2+2az+1}$

or zeros of $x^2+2ax+1$

Math 383: Complex Analysis: Fall '23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

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https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 12: 10-06-23: Lecture 11 from 10/06/21: Meromorphic Functions, Log, Argument Principle, Rouche, Fund Thm Alg, Trig Integral: https://youtu.be/INRdLUT6ckQ (slides)

See also: Lecture 11: 10/02/17: Complex Logarithms, Argument Principle, Rouche's Theorem: https://youtu.be/iyt4EhHvy-s

Plan for the day: Lecture 11 from October 6, 2021 (is lecture 12 in 2023):

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Characterization of Meromorphic Functions
- Complex Logarithms
- Argument Principle
- Rouche's Theorem (and consequences)
- Integration Example (trig)

General items.

- How do we generalize?
- Pavlovian responses
- Continuous discrete functions are…
- Thoreau: Simplify, simplify
- Multiple proofs…
- Deformations

We now turn to functions with only isolated singularities that are poles. A function f on an open set Ω is **meromorphic** if there exists a sequence of points $\{z_0, z_1, z_2, \ldots\}$ that has no limit points in Ω, and such that

(i) the function f is holomorphic in $\Omega - \{z_0, z_1, z_2, \ldots\}$, and

(ii) f has poles at the points $\{z_0, z_1, z_2, \ldots\}$.

f has a **pole at infinity** if $F(z) = f(1/z)$ has a pole at the origin

Theorem 3.4 *The meromorphic functions in the extended complex plane are the rational functions.*

Theorem 3.4 *The meromorphic functions in the extended complex plane are the rational functions.*

Near each pole $z_k \in \mathbb{C}$ we can write $f(z) = f_k(z) + g_k(z)$

$f(1/z) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$

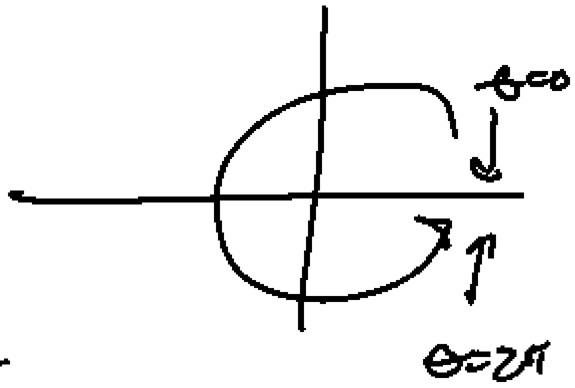
$H = f - f_\infty - \sum_{k=1}^{n} f_k$ is entire and bounded.

$\log f(z)$ is "multiple-valued"

$z = x + iy$ or $re^{i\theta}$

Try $\log(re^{i\theta}) = \log(r) + \log e^{i\theta}$

$= \log(r) + i\theta$



$z = e^{3\pi i/2} = w \qquad zw = (-1) = e^{i\pi}$

$\log(zw) \quad$ vs $\quad \log z + \log w$

$\log e^{i\pi} \quad$ vs $\quad \log e^{3\pi i/2} + \log e^{3\pi i/2}$

$i\pi \quad$ vs $\quad \frac{3\pi i}{2} + \frac{3\pi i}{2} = 3\pi i = \pi i + 2\pi i$

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}, \qquad \frac{\left(\prod_{k=1}^N f_k\right)'}{\prod_{k=1}^N f_k} = \sum_{k=1}^N \frac{f_k'}{f_k}.$$

Consider $\frac{d}{dx} \log[f(x)] = \frac{1}{f(x)} * f'(x) = \frac{f'(x)}{f(x)}$

↳ logarithmic derivative

Pavlovian Response: take the log-deriv of f,
do a contour integral

Similarity: product to sum
have hidden log when study $f'/f = \frac{d}{dx}[\log f(x)]$

$$f(z) = \underbrace{(z-z_0)^n}_{f_1(z)} \underbrace{g(z)}_{f_2(z)}, \qquad \frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + G(z) \qquad g(z) = 1 + a_1(z-z_0) + \cdots$$

$$\frac{f'}{f} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

$$G(z) = \frac{g'(z)}{g(z)}$$

$g(z_0) = 1$ so no pole, thus holo at z_0

$$\frac{1}{2\pi i} \oint_{C_{z_0}} \frac{f'(z)}{f(z)} dz = n = \mathrm{Res}_{f'/f}(z_0)$$

C_{z_0} small circle about z_0

Theorem 4.1 (Argument principle) *Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then* on C

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = (\textit{number of zeros of } f \textit{ inside } C) \textit{ minus}$$
$$(\textit{number of poles of } f \textit{ inside } C),$$

where the zeros and poles are counted with their multiplicities.

Corollary 4.2 *The above theorem holds for toy contours.*

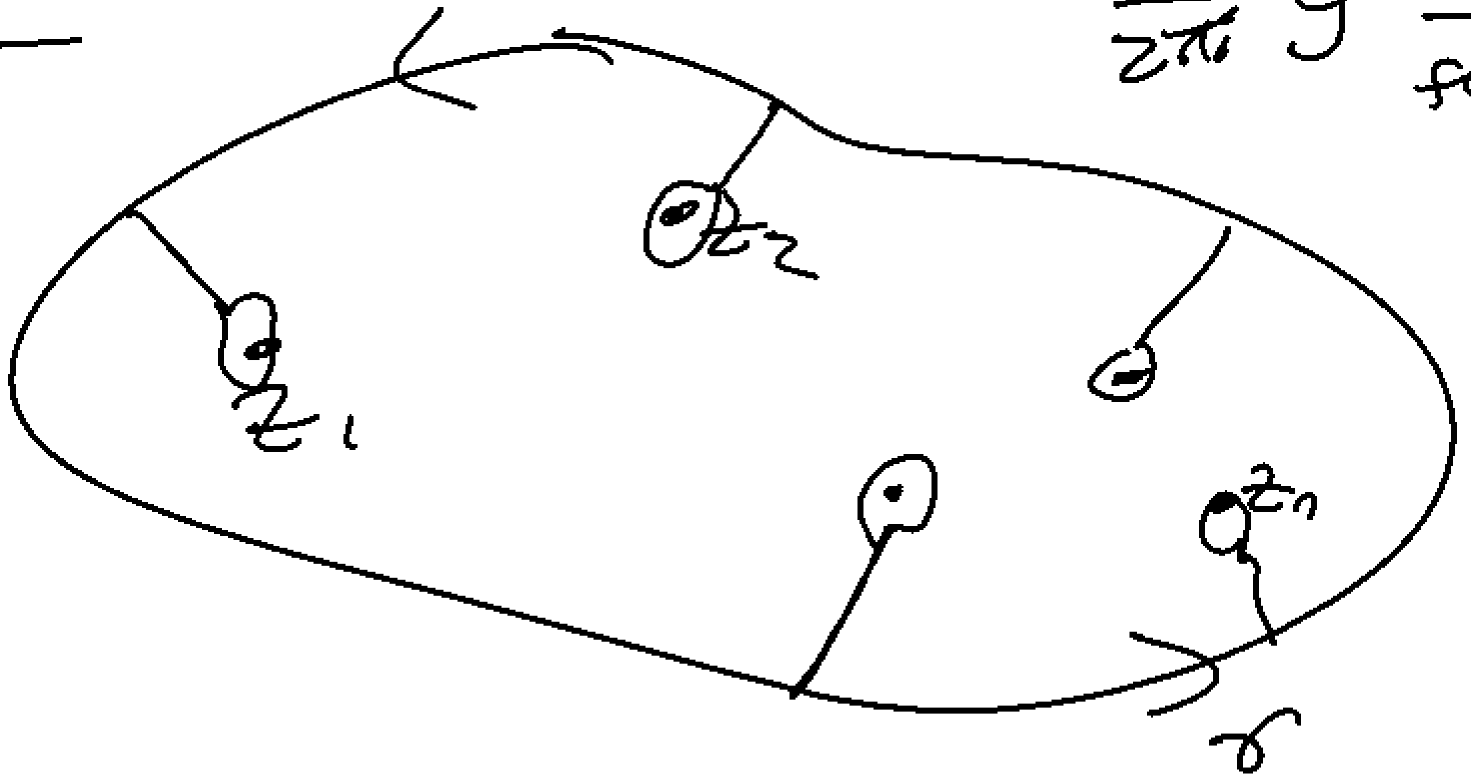
https://www.nbcnews.com/news/us-news/think-commas-don-t-matter-omitting-one-cost-maine-dairy-

An absent "Oxford comma" will cost a Maine dairy company $5 million.

The suit, brought against Oakhurst Dairy by the company's drivers in 2014, sought $10 million in a dispute about overtime payment.

Proof

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\,dz$$



Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \textit{for all } z \in C,$$

then f and $f+g$ have the same number of zeros inside the circle C.

Proof. For $t \in [0,1]$ define $f_t(z) = f(z) + tg(z)$, $n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}\,dz.$

On curve, $|f+g| > 0$ since $|f| > |g|$ on boundary

Can use argument principle for $f_t(z) = f(z) + t g(z)$

(Lin alg: $t\vec{v}_1 + (1-t)\vec{v}_2$ goes from $\vec{v}_2$ to $\vec{v}_1$ as t goes from 0 to 1)

$n_t = \frac{1}{2\pi i}\oint_C \frac{f_t'(z)}{f_t(z)}\,dz = n(t)$ if continuous then $n(0) = n(1)$ done.

integer

Cont: $\frac{1}{2\pi i}\oint_C \frac{f'(z) + t g'(z)}{f(z) + t g(z)}\,dz$ see this is continuous $\boxed{}$

Rouché's theorem implies

A mapping is said to be **open** if it maps open sets to open sets.

Theorem 4.4 (Open mapping theorem) *If f is holomorphic and non-constant in a region Ω, then f is open.*

Theorem 4.5 (Maximum modulus principle) *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Rouché's implies

Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

then f and $f+g$ have the same number of zeros inside the circle C.

Fundamental theorem of algebra, Theorem of equations proved by Carl Friedrich Gauss in 1799. It states that every polynomial equation of degree *n* with complex number coefficients has *n* roots, or solutions, in the complex numbers. BY The Editors of Encyclopaedia Britannica

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad (a_n \neq 0)$

Choose $f(z)$, $g(z)$: either for $f+g$ is $a_n z^n$

Try $f(z) = a_n z^n \qquad g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

$f(z)$ has n zeros in circle of radius R, and $f(z) + g(z) = p(z)$

If C_R is a circle of radius R centered at origin, if R is BIG

Then $|f| > |g|$ on C_R: $|a_n R^n|$ vs $n * \max|a_k| * \max(1, R^{n-1})$

By Rouche, f and $p = f+g$ have the same # of zeros inside C_R!

$$\int_0^{2\pi} \frac{1}{a+\cos^2(x)}\,dx$$

Integrate$\left[\frac{1}{a+\cos^2(x)}, \{x, 0, 2\pi\}, \text{Assumptions} \to a > 1\right]$

$$\frac{2\pi}{\sqrt{a\,(1+a)}}$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

if $|z|=1$ $\quad \cos\theta = \frac{(z+1/z)}{2}$

so $z = e^{i\theta}$

$$\cos^2\theta = \frac{1}{4}\left(z+\frac{1}{z}\right)^2 = \frac{1}{4}\left(z^2+2+z^{-2}\right)$$

$dz = e^{i\theta}\, i\, d\theta$ so $dz = iz\,d\theta$ so $d\theta = \frac{1}{iz}\,dz$

$$\int_0^{2\pi} \frac{1}{a+\cos^2(x)}\,dx = \int_{|z|=1} \frac{(1/iz)\,dz}{a+\frac{1}{4}(z^2+2+z^{-2})}$$

$$\int_0^{2\pi} \frac{1}{a+\cos^2(x)}\,dx$$

<u>Trig</u>: $\cos^2 x$

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\cos^2(x) = \frac{1}{2}(\cos(2x)+1)$$

our integral is $\displaystyle\int_0^{2\pi} \frac{1}{a+\frac{1}{2}+\frac{1}{2}\cos(2x)}\,dx$

$u = 2x \quad du = 2dx$

$x: 0 \to 2\pi$

$u: 0 \to 4\pi$

$$= \int_{u=0}^{4\pi} \frac{1}{a+\frac{1}{2}+\frac{1}{2}\cos(u)} \frac{du}{2} = \int_{u=0}^{4\pi} \frac{du}{(2a+1)+\cos u}$$

$$= 2\int_{u=0}^{2\pi} \frac{du}{(2a+1)+\cos u}$$

https://www.nbcnews.com/news/us-news/think-commas-don-t-matter-omitting-one-cost-maine-dairy-n847151

Think commas don't matter? Omitting one cost a Maine dairy company $5 million.

A Maine dairy company has settled a lawsuit over an overtime dispute that was the subject of a ruling that hinged on the use of the Oxford comma.

Feb. 12, 2018, 4:10 PM EST / Updated Feb. 12, 2018, 4:10 PM EST

By Kalhan Rosenblatt and The Associated Press

An absent "Oxford comma" will cost a Maine dairy company $5 million.

The suit, brought against Oakhurst Dairy by the company's drivers in 2014, sought $10 million in a dispute about overtime payment.

A federal appeals court decided to keep the drivers' lawsuit, concerning an exemption from Maine's overtime law, alive last year.

Court documents filed Thursday show that the company and the drivers settled for $5 million.

Related: Oxford Comma defenders, rejoice! Judge bases ruling on punctuation

"For want of a comma, we have this case," U.S. Court of Appeals for the First Circuit Judge David Barron said in March, 2017.

The sentence at the heart of the dairy drivers' case referred to Maine's overtime law and whom it doesn't apply to: "The canning, processing, preserving, freezing, drying, marketing, storing, packing for shipment or distribution of:

"(1) Agricultural produce;

"(2) Meat and fish products; and

"(3) Perishable foods."

The disagreement stemmed from the lack of a comma after the word "shipment."

The use of the Oxford comma – also called a serial comma – delineates the final item on a list. For example: "Milk, cheese, and yogurt."

Proponents of the punctuation argue it helps to differentiate subjects, while opponents say it's cumbersome.

Different style guides have different rules about the Oxford comma, which gets its name because it was preferred by Oxford University Press editors.

In 2017, Judge David Barron reasoned that the law's punctuation made it unclear if "packing for shipping or distribution" is one activity or if "packing for shipping" is separate from "distribution."

The five drivers who led the lawsuit will receive $50,000 each from the settlement fund, according to the Portland Press Herald.

The other 127 drivers who are eligible to file a claim will be paid a minimum of $100 or the amount of overtime they were owed based on their work from May 2008 to August 2012, the Press Herald reported.

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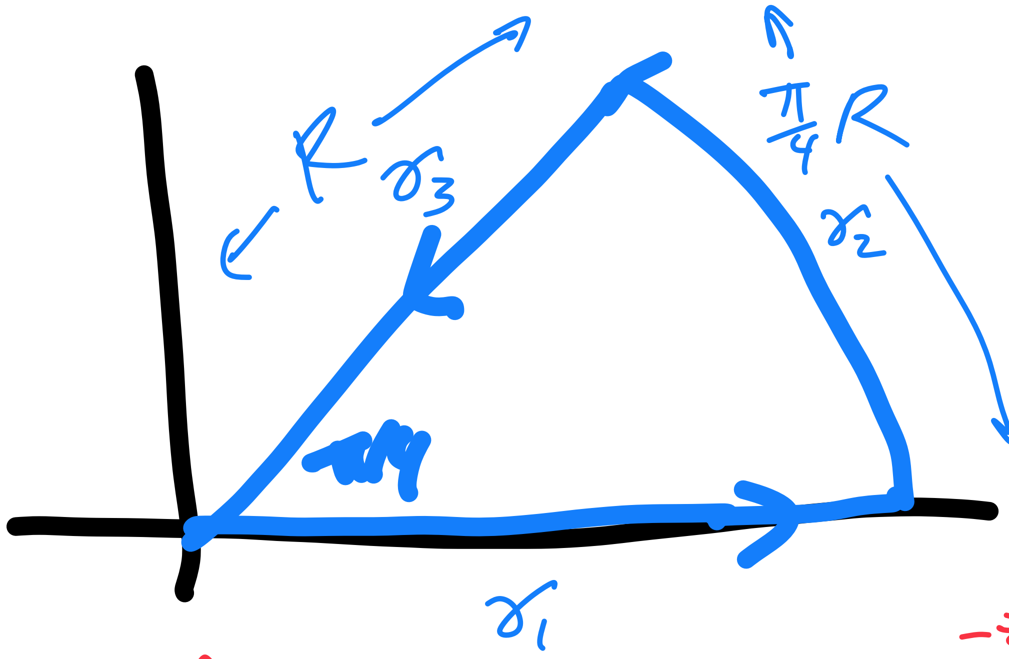
Lecture 13: 10/06/23: (lecture from 10/4/21): Meromorphic Functions, Log, Argument Principle, Rouche, Fund Thm Alg, Trig Integral: https://youtu.be/INRdLUT6ckQ (slides from 2021).

Lecture 14: 10/09/23: NO CLASS (Columbus Day)

Lecture Bonus: 10-10-23: Fresnel Integrals

https://youtu.be/R5lCHdPVESQ

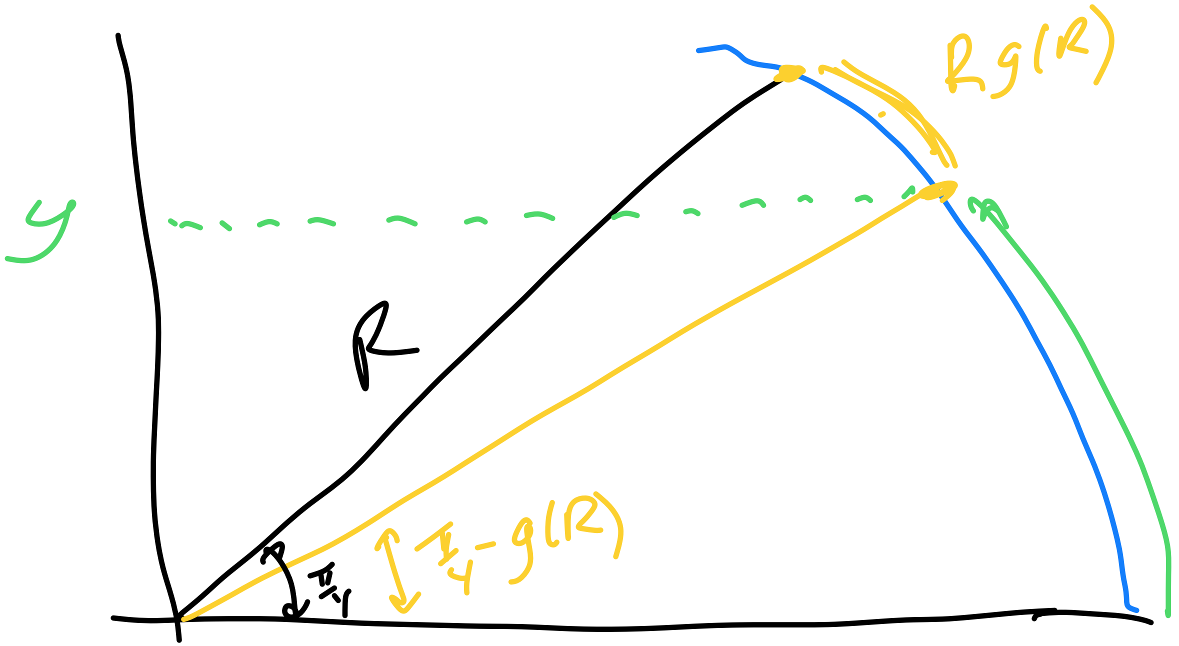


$$f(z) = e^{-z^2}$$

angle is $\pi/4$

Given: $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

$$z^2 = (x^2 - y^2) + 2ixy \qquad e^{-z^2} = e^{y^2 - x^2} e^{-2ixy}$$



Contribution from small arc is at most arc length * |max value|

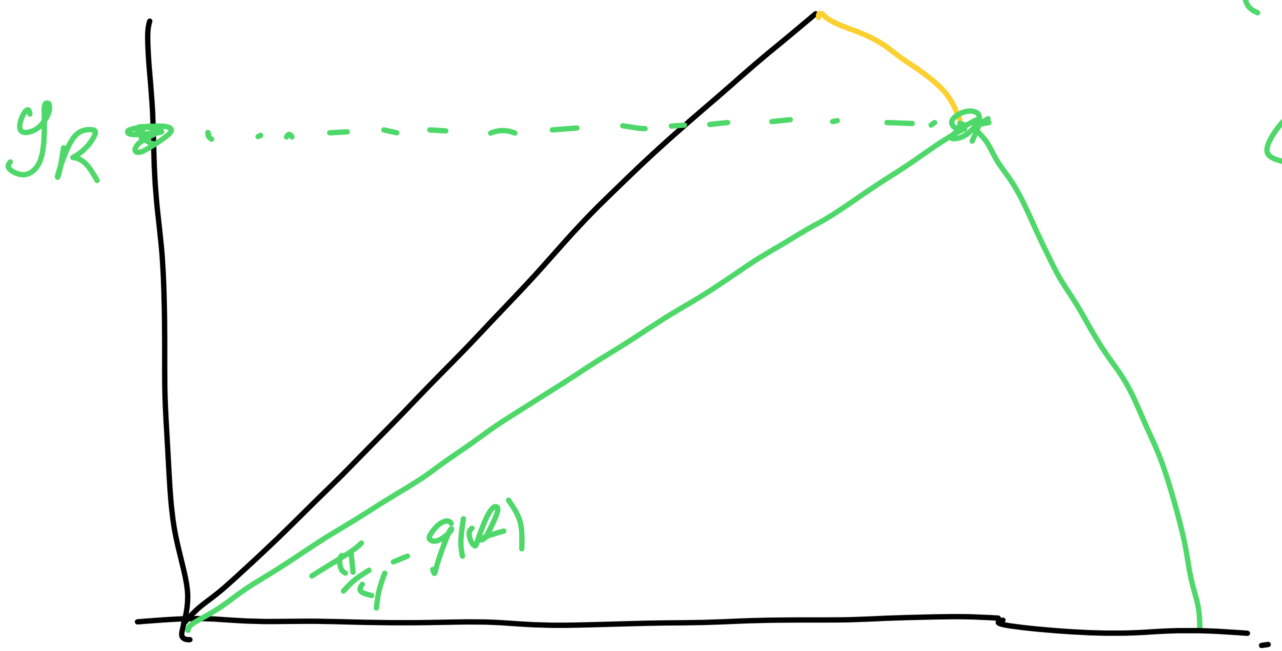
$\leq R\,g(R)\cdot 1$

$\Rightarrow Rg(R) \to 0$

Could try $g(R) = \dfrac{1}{R\log R}$

$g(R) = \dfrac{1}{R^{1+h}}$

h small positive number



$$|f(z)| = e^{y^2 - x^2}$$

Contribution is at most

arclength * |max value|

$$\leq R\frac{\pi}{4} * e^{y_R^2 - x_R^2}$$

$$X_R = R\cos\left(\frac{\pi}{4} - g(R)\right) \qquad y_R = R\sin\left(\frac{\pi}{4} - g(R)\right)$$

Could do $y_R^2 - X_R^2 = (y_R - X_R)(y_R + X_R)$

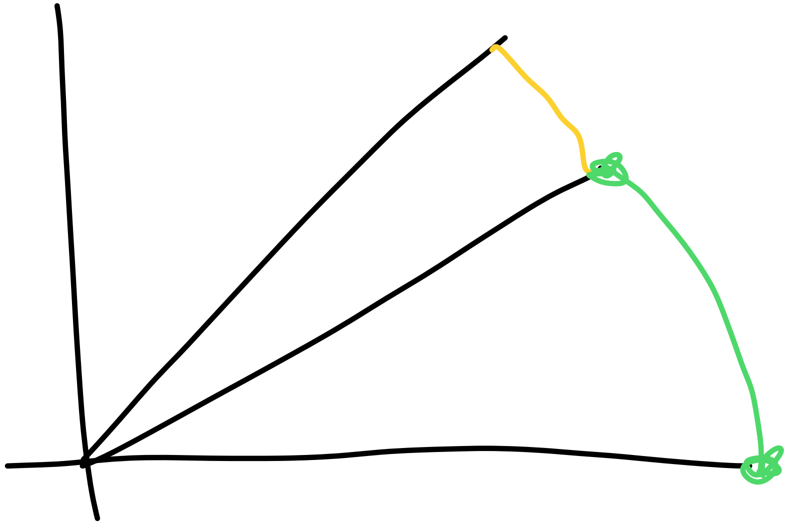
$$= R^2\left[\underbrace{\cos\left(\frac{\pi}{4} - g(R)\right) - \sin\left(\frac{\pi}{4} - g(R)\right)}_{X_R - Y_R}\right]\underbrace{\left[\cos\left(\frac{\pi}{4} - g(R)\right) + \sin\left(\frac{\pi}{4} - g(R)\right)\right]}_{\text{approx } \sqrt{2}}$$

$$= \cos\frac{\pi}{4}\cos g(R) + \sin\frac{\pi}{4}\sin g(R)$$

$$\underline{- \sin\frac{\pi}{4}\cos g(R) + \cos\frac{\pi}{4}\sin g(R)}$$

$$\sqrt{2}\sin g(R) \sim \sqrt{2}\left[g(R) - \frac{g(R)^3}{3!} + \dots\right]$$

$$e^{y_R^2 - X_R^2} \le e^{(-\sqrt{2}g(R) + \sqrt{2}g(R)^3)R^2} \le 2023\, e^{-\sqrt{2}R^2 g(R)}$$



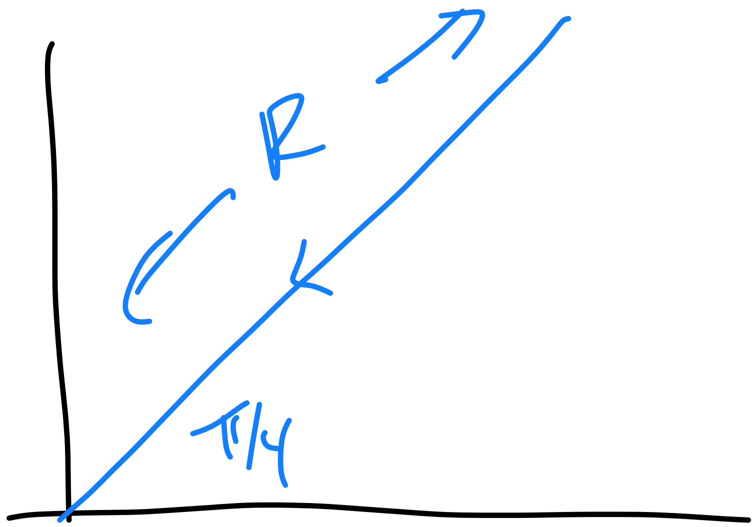
$|\text{Green Arc}|$

$$\le \cancel{2023} e^{-\sqrt{2} R^2 g(R)} \cdot \frac{\pi}{4} R$$

$$\le \text{Const} \frac{R}{e^{\sqrt{2} R^2 g(R)}}$$

If $g(R) = \frac{1}{R \log R}$ or $\frac{1}{R^{1+h}}$ with $h < 1$ <u>win</u>!

Do L'Hôpital, $e^u \le 1 + u + \ldots$ so have $\le \frac{R}{(\sqrt{2} R^2 g(R))^2 / 2!} \to 0$



$$z = x e^{i\theta} \qquad \theta = \pi/4$$

$$x : R \to 0$$

$$dz = e^{i\frac{\pi}{4}} dx = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) dx$$

$$e^{-z^2} = e^{-x^2 e^{2i\pi/4}} = e^{-ix^2} = \cos(x^2) - i\sin(x^2)$$

$$\frac{\sqrt{2}}{2} \int_{x=R}^{0} \left[\cos(x^2) - i\sin(x^2)\right](1+i)\,dx + \frac{\sqrt{\pi}}{2} + 0 = 0$$

(in the limit)   in the limit

$$\cancel{\frac{\sqrt{2}}{2}} \int_{x=0}^{R} \left(\cos(x^2) - i\sin(x^2)\right)(1+i)\,dx = \frac{\sqrt{\pi}}{\cancel{2}} \quad \frac{1}{\sqrt{2}} \qquad R = \text{"}\infty\text{"}$$

$$\int_0^R \left[\cos(x^2) + \sin(x^2)\right] dx + i\left[\int_0^R \left(\cos(x^2) - \sin(x^2)\right) dx\right] = \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$A = \int_0^\infty \cos(x^2)\,dx \qquad B = \int_0^\infty \sin(x^2)\,dx$$

$$A + B = \sqrt{\frac{\pi}{2}} \qquad \Longrightarrow A = B = \frac{\sqrt{\pi/2}}{2}$$

$$i(A - B) = 0 \Longrightarrow A = B \qquad = \frac{\sqrt{2\pi}}{4}$$

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Proof: Let $I = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$

$$I^2 = \int_{x=-\infty}^{\infty} e^{-x^2/2}\,dx \int_{y=-\infty}^{\infty} e^{-y^2/2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy$$

Polar: $r^2 = x^2+y^2 \quad dx\,dy = dr\cdot r\,d\theta = r\,dr\,d\theta$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2/2}\,r\,dr\,d\theta = 2\pi \int_{r=0}^{\infty} e^{-r^2/2}\,r\,dr$$

$$= 2\pi\left[e^{-r^2/2}\Big|_{\infty}^{0}\right] = 2\pi \quad \text{so } I = \sqrt{2\pi} \text{ as } I > 0$$

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Lecture 15: 10-11-23: https://youtu.be/touNug0_fLw

Rouche, Open Mapping and Maximum Modulus

Plan for the day: Lecture 15: October 11, 2023:

REVIEW:

- Argument Principle
- Rouche's Theorem

CONSEQUENCES:

- Open Mapping Theorem
- Maximum Modulus

OTHER:

- Real Analysis Review
- Differences b/w Real and Complex

Theorem 4.1 (Argument principle) *Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = (\textit{number of zeros of } f \textit{ inside } C) \textit{ minus}$$
$$(\textit{number of poles of } f \textit{ inside } C),$$

where the zeros and poles are counted with their multiplicities.

Corollary 4.2 *The above theorem holds for toy contours.*

$f(z) = a_n(z - z_0)^n B(z)$

$f(z) = \underbrace{z\, z \cdots z}_{n\text{ times}}\, B(z)$

$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{B'(z)}{B(z)}$

$\frac{f'(z)}{f(z)} = \frac{1}{z} + \cdots + \frac{1}{z} + \frac{B'(z)}{B(z)}$

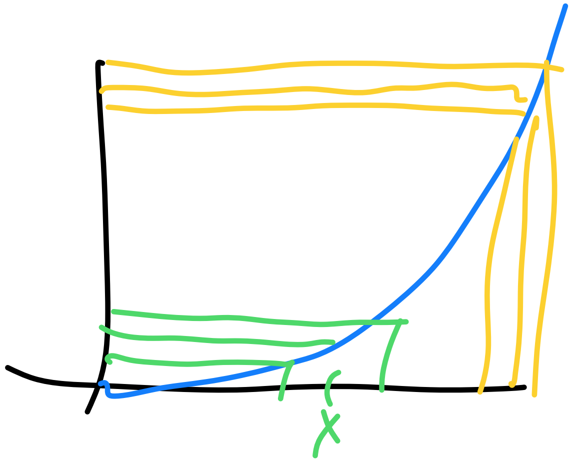
Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

then f and $f+g$ have the same number of zeros inside the circle C.

Proof. For $t \in [0,1]$ define $f_t(z) = f(z) + tg(z)$, $n_t = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f_t'(z)}{f_t(z)}\,dz.$

$$n(t_1) - n(t_2) = \frac{1}{2\pi i}\oint_C \left[\frac{f_{t_1}'(z)}{f_{t_1}(z)} - \frac{f_{t_2}'(z)}{f_{t_2}(z)}\right] dz$$ Substitute, show continuous



$$\oint_C \left[\frac{A(t_1,x) - A(t_2,x)}{t_1 - t_2}\right](t_1 - t_2)\,dx$$

if $\partial A/\partial t$ exists and is bounded, done

Rouché's theorem implies

A mapping is said to be **open** if it maps open sets to open sets.

Theorem 4.4 (Open mapping theorem) *If f is holomorphic and non-constant in a region Ω, then f is open.*

Real Range: f: open $\to$ not open

$\sin: (-2\pi, 2\pi) \to [-1, 1]$

$x^2: (-1,1) \to [0,1)$

$f(z) = z^2$

$z = re^{i\theta}$

$z^2 = r^2 e^{2i\theta}$

Theorem 4.5 (Maximum modulus principle) *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Theorem 4.4 (Open mapping theorem) *If f is holomorphic and non-constant in a region Ω, then f is open.*

Proof. Let w_0 belong to the image of f, say $w_0 = f(z_0)$. We must prove that all points w near w_0 also belong to the image of f.

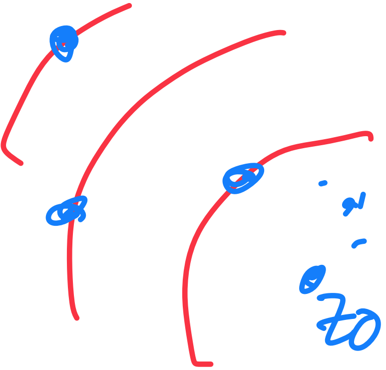
Define $g(z) = f(z) - w$ and write

$$\begin{aligned} g(z) &= (f(z) - w_0) + (w_0 - w) = f(z) - w \\ &= F(z) + G(z). \end{aligned}$$

F has at least one zero

Show F and g have same # of zeros

Now choose $\delta > 0$ such that the disc $|z - z_0| \leq \delta$ is contained in Ω and $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$. We then select $\epsilon > 0$ so that we have $|f(z) - w_0| \geq \epsilon$ on the circle $|z - z_0| = \delta$. Now if $|w - w_0| < \epsilon$ we have $|F(z)| > |G(z)|$ on the circle $|z - z_0| = \delta$, and by Rouché's theorem we conclude that $g = F + G$ has a zero inside the circle since F has one.

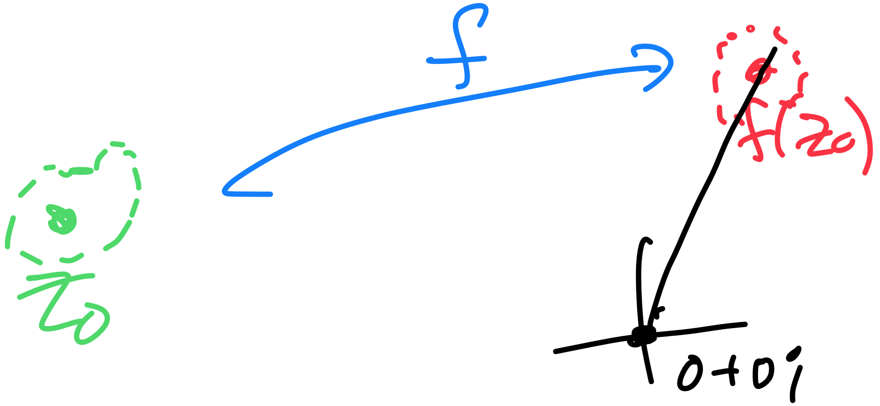


If always can find a point on circle with $f(z) = w_0$, accumulates, and thus f is constant

Why is $|F(z)| > |G(z)|$ on boundary?
Cont fn on compact set attains max/min
Min of $|F|$ is > 0 as not 0 on boundary
Choose $\varepsilon < \frac{1}{2} \min F$

Theorem 4.5 (Maximum modulus principle) *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Proof. Suppose that f did attain a maximum at z_0. Since f is holomorphic it is an open mapping, and therefore, if $D \subset \Omega$ is a small disc centered at z_0, its image $f(D)$ is open and contains $f(z_0)$. This proves that there are points in $z \in D$ such that $|f(z)| > |f(z_0)|$, a contradiction.



<u>Real Diffeat</u>

$f(x) = \sin x \quad I = (-2\pi, 2\pi)$

$f(x) = \frac{1}{1+x^2}$

$f(x) = e^{-x^2}$

Real Analysis

Thm: If f is cont on a compact set, f is bounded.

Proof: Assume not, so $\forall n \; \exists x_n$ st $|f(x_n)| > n$.

$x_1, x_2, x_3, \ldots$ has a convergent subseq $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$

converges to x^*, Since f is cont, $\forall \varepsilon > 0 \; \exists \delta$ st

if $|x - x^*| < \delta$ then $|f(x) - f(x^*)| < \varepsilon$.

Contradiction as $|f(x_{n_k})| \to \infty$

but has to be within ε of $f(x^*)$

Proof II: Use uniform continuity

Given ε, $\exists \delta$ st if $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \varepsilon$

Note δ is indep of x_1, x_2, only depends on ε.

Let $\varepsilon = 1$, so $\exists \delta$ st $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < 1$

Consider $\Omega \subset \bigcup_{x \in \Omega} B_x(1)$ Ball of radius 1 center x

As Ω compact, finite subcover, $\Omega \subset \bigcup_{k=1}^{n} B_{x_{n_k}}(1)$

everything within 1 of $f(\text{center})$, take max of $f(\text{center}) + 1$

Math 383: Complex Analysis: Fall '23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 16: 10-13-23: Mountain Day

Lecture 17: 10-16-23: https://youtu.be/Uz42EoM6yLs

Review of Logarithms, Comments from 2.5 videos to watch before class:

Watch the following videos before class / read the book (better both!):

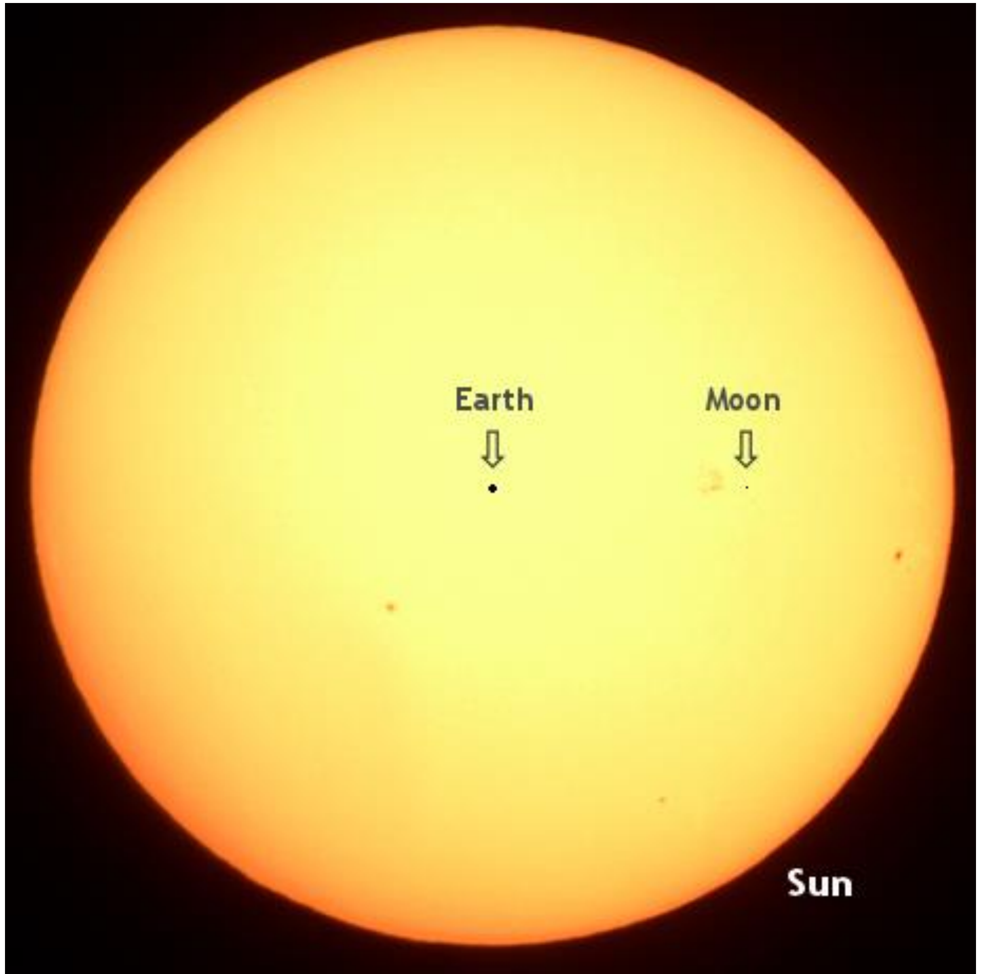
- Another approach to proving open mapping theorem without using Rouche: watch from 28 minutes till end: https://youtu.be/-vuwc6irob4?t=1685
- Complex Logarithms, Earlier Material, Functions with Prescribed Zeros/Values: https://youtu.be/CR-sRChcID4 (slides)
- Writing functions as a product over zeros: https://youtu.be/AoiyKD17aKM

Introduction to Logarithms

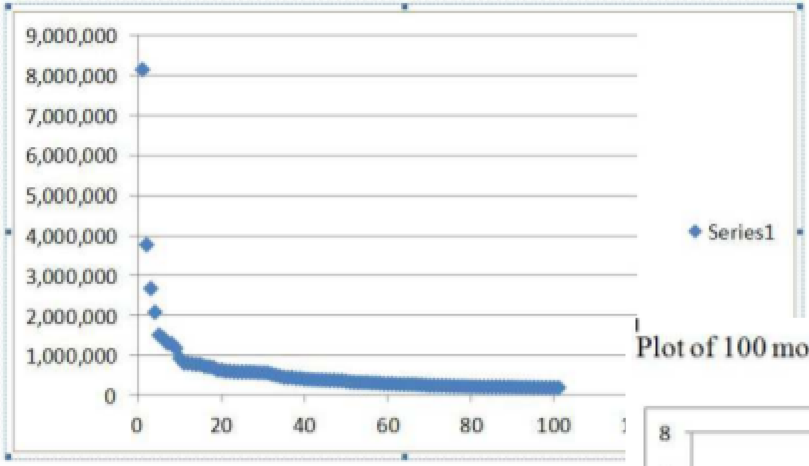
Steven Miller, Williams College
sjm1@Williams.edu

Why do we care about Logarithms

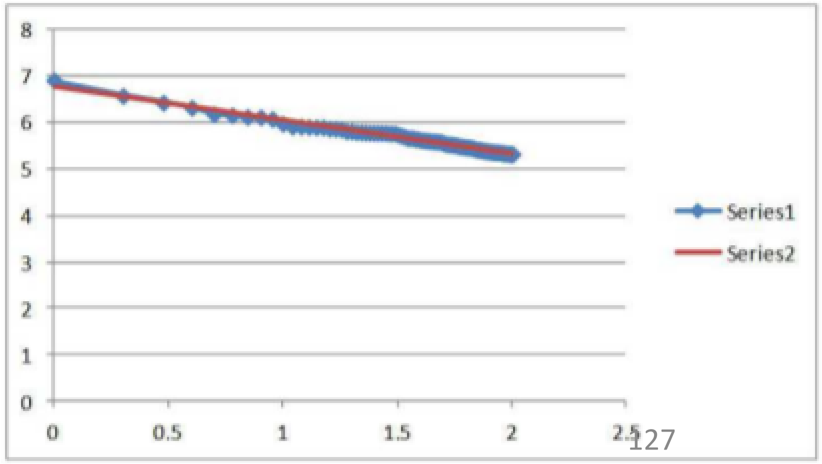
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities



Plot of 100 most populous cities: log-log plot



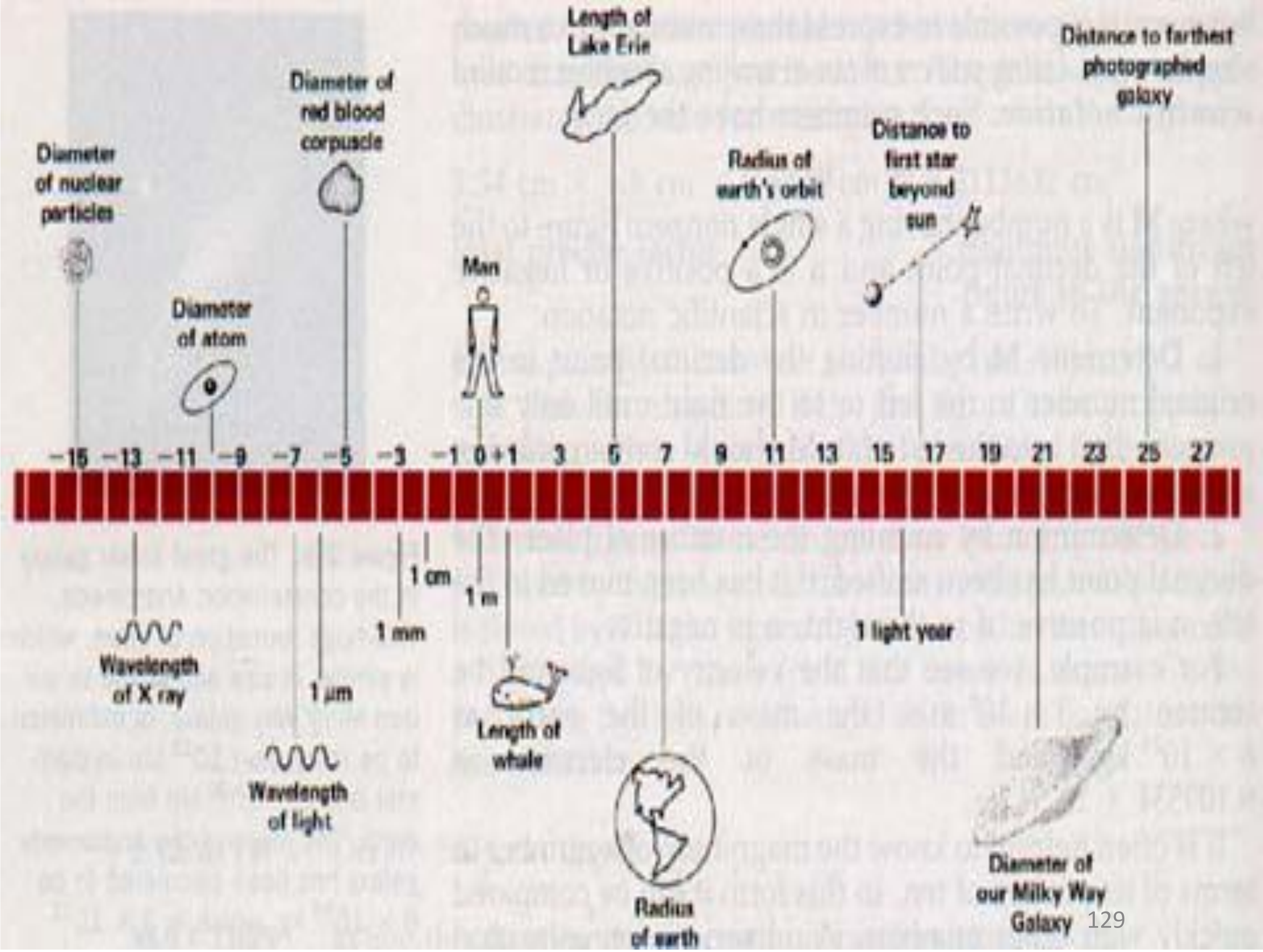
Definition of Logarithms

- **If $\boldsymbol{x = b^y}$ then $\boldsymbol{\log_b x = y}$.**
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x, which is its logarithm base e (e is approximately 2.71828).

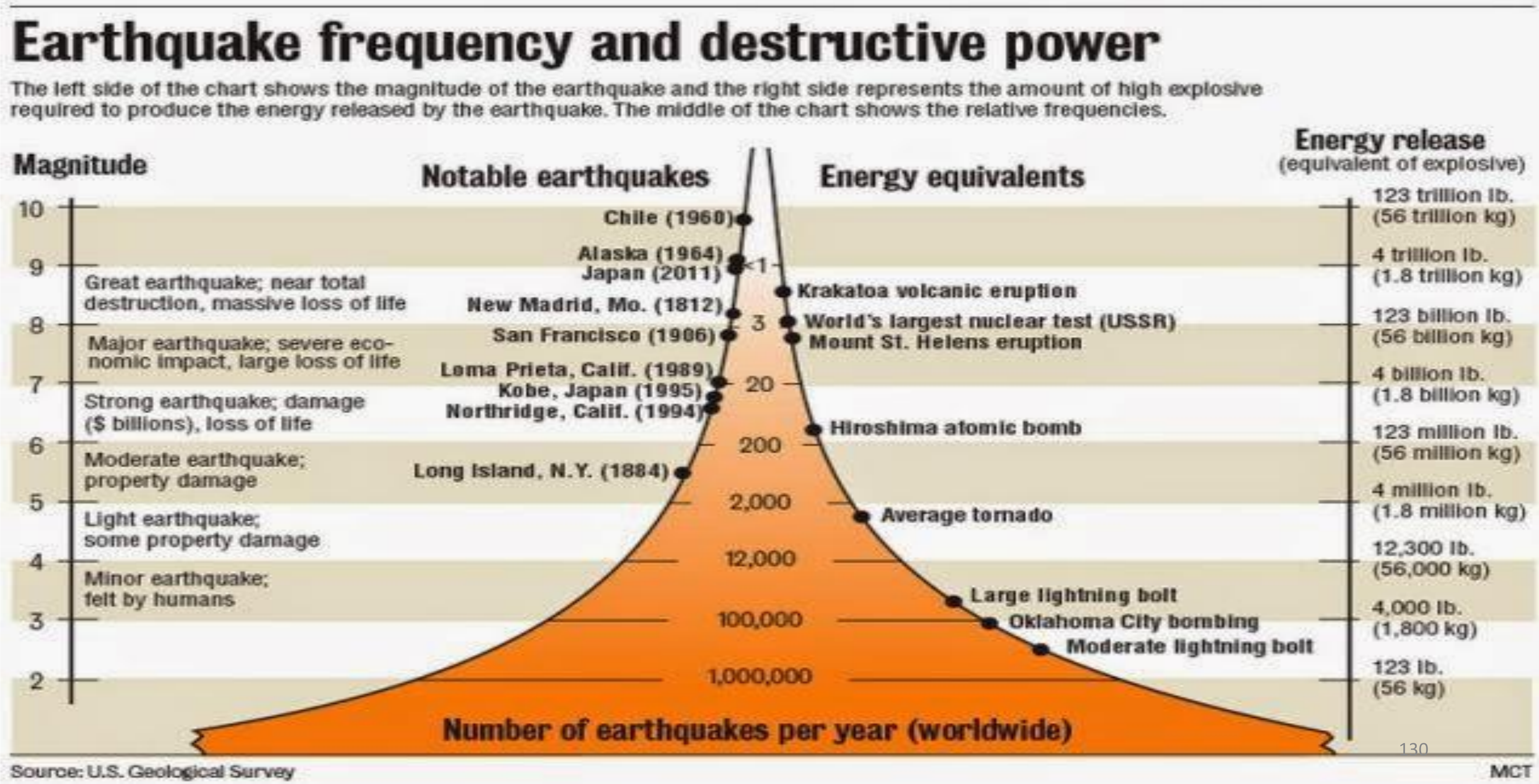
- **Examples: $\boldsymbol{\log_b x = y}$ means we need $\boldsymbol{y}$ powers of $\boldsymbol{b}$ to get $\boldsymbol{x}$.**
  - $100 = 10^2$ becomes $\log_{10} 100 = 2.$ In base e it is about 4.6.
  - $1 = 10^0$ becomes $\log_{10} 1 = 0.$ In base e it is still 0.
  - $.001 = 10^{-3}$ becomes $\log_{10} .001 = -3.$ In base e it is about -6.9.

Examples of Logarithms

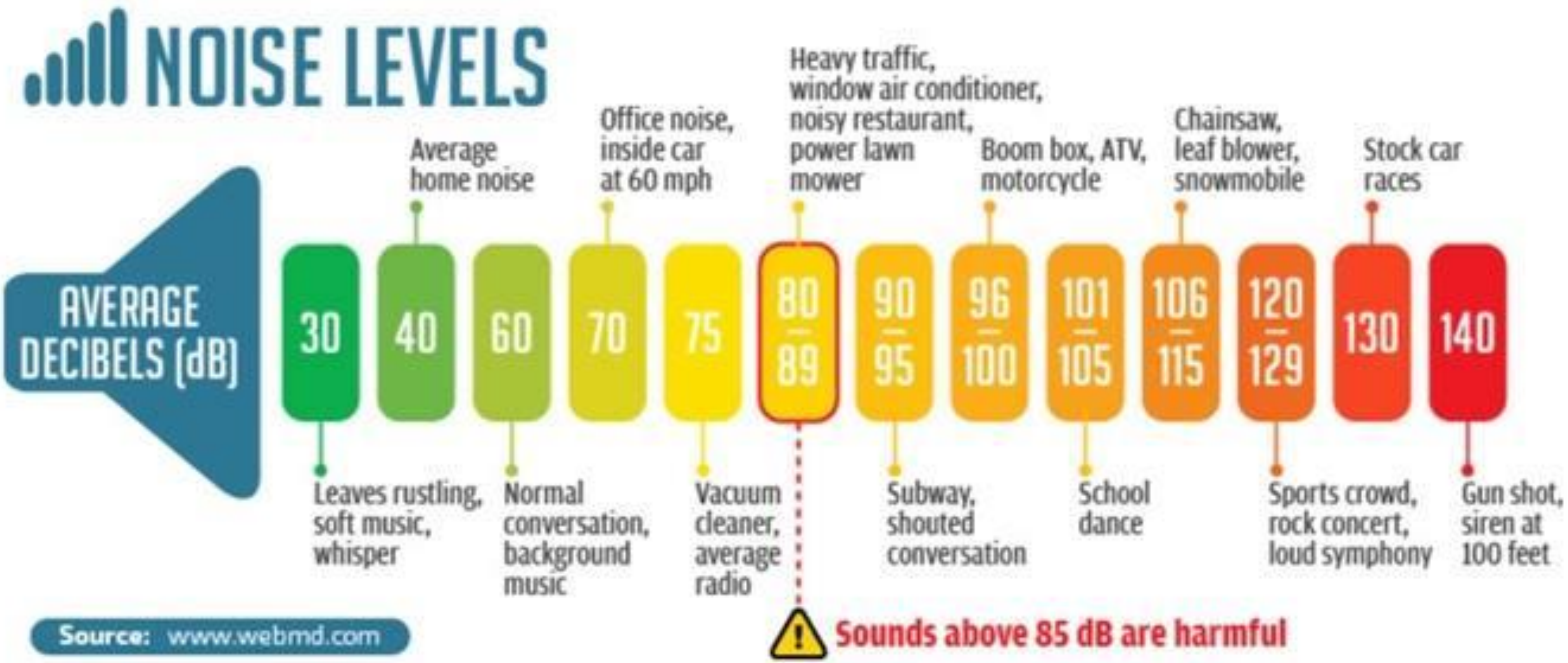
Order of Magnitude of some Lengths	
LENGTH	**meters**
radius of proton	10^{-15}
radius of atom	10^{-10}
radius of virus	10^{-7}
radius of amoeba	10^{-4}
height of human being	10^{0}
radius of earth	10^{7}
radius of sun	10^{9}
earth-sun distance	10^{11}
radius of solar system	10^{13}
distance of sun to nearest star	10^{16}
radius of milky way galaxy	10^{21}
radius of visible Universe	10^{26}



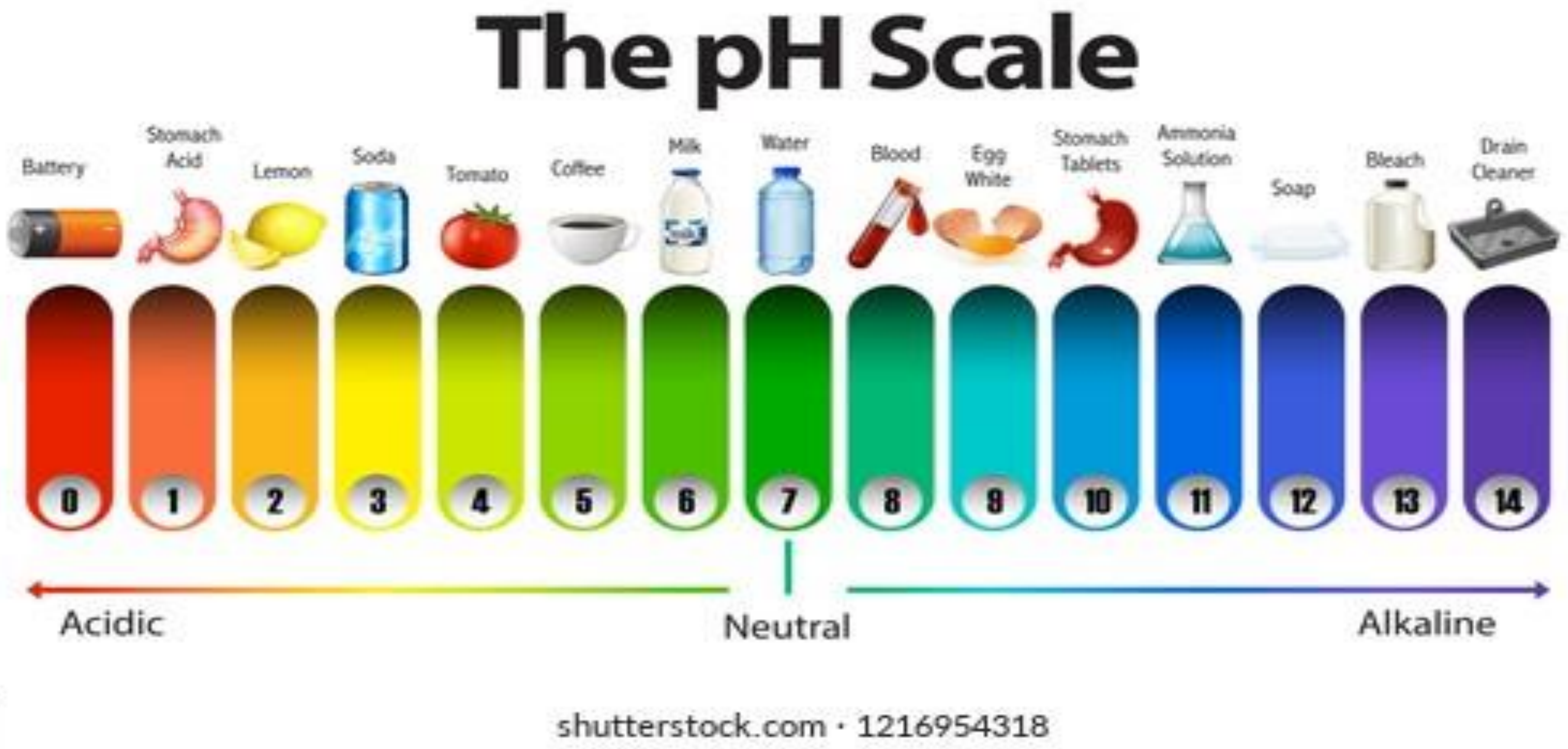
Examples of Logarithms



Examples of Logarithms

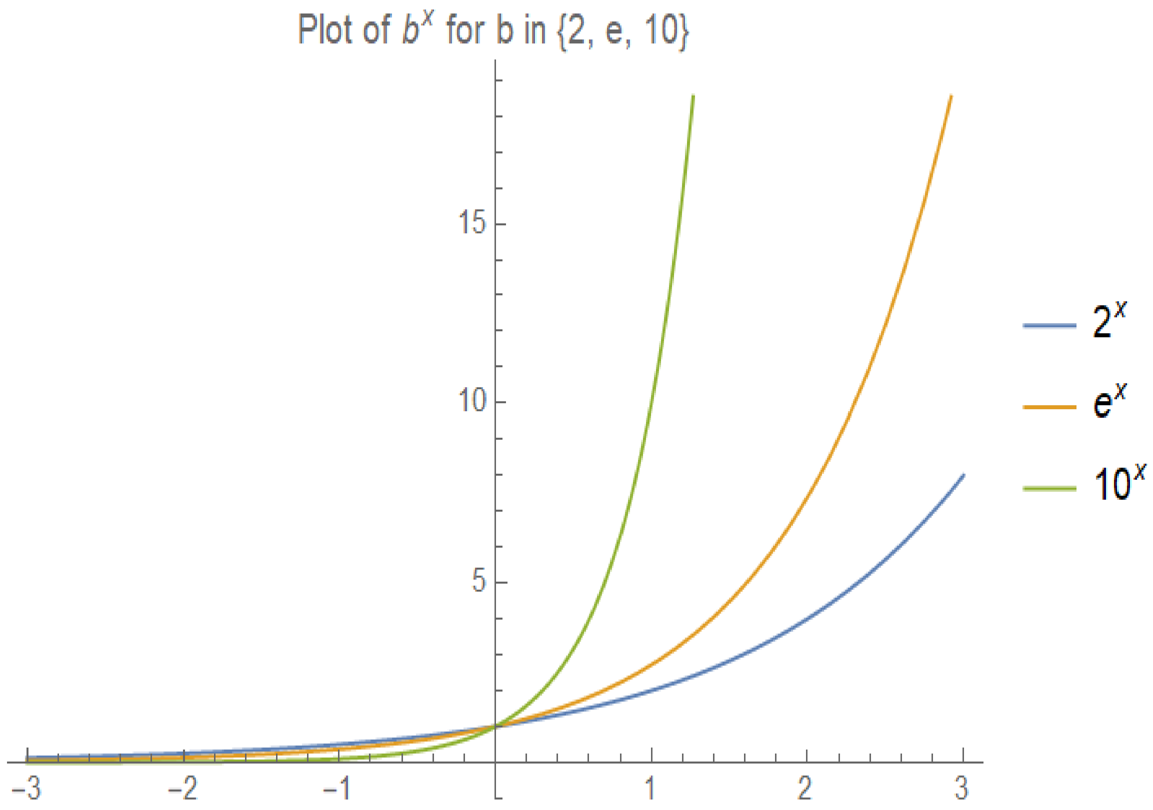


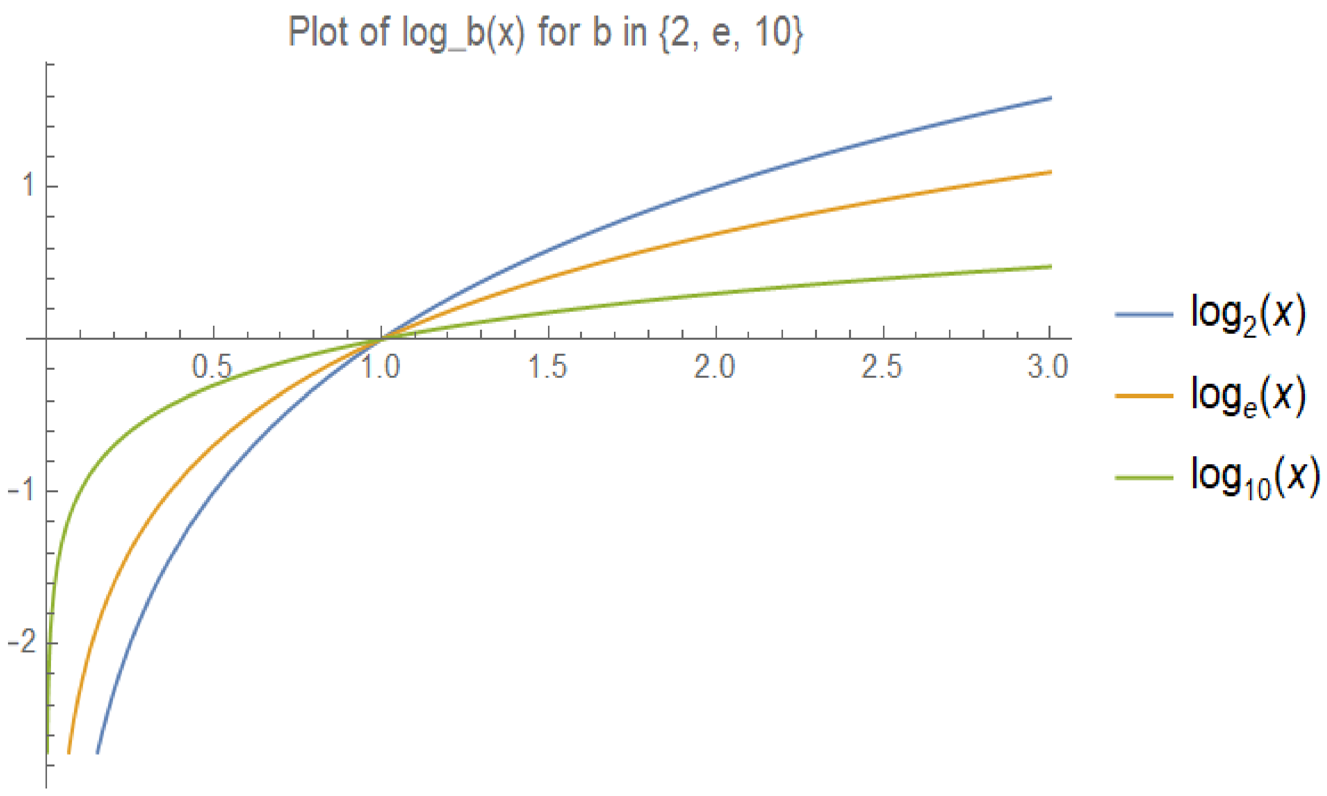
Examples of Logarithms



Plots of Exponentiation and Logarithms

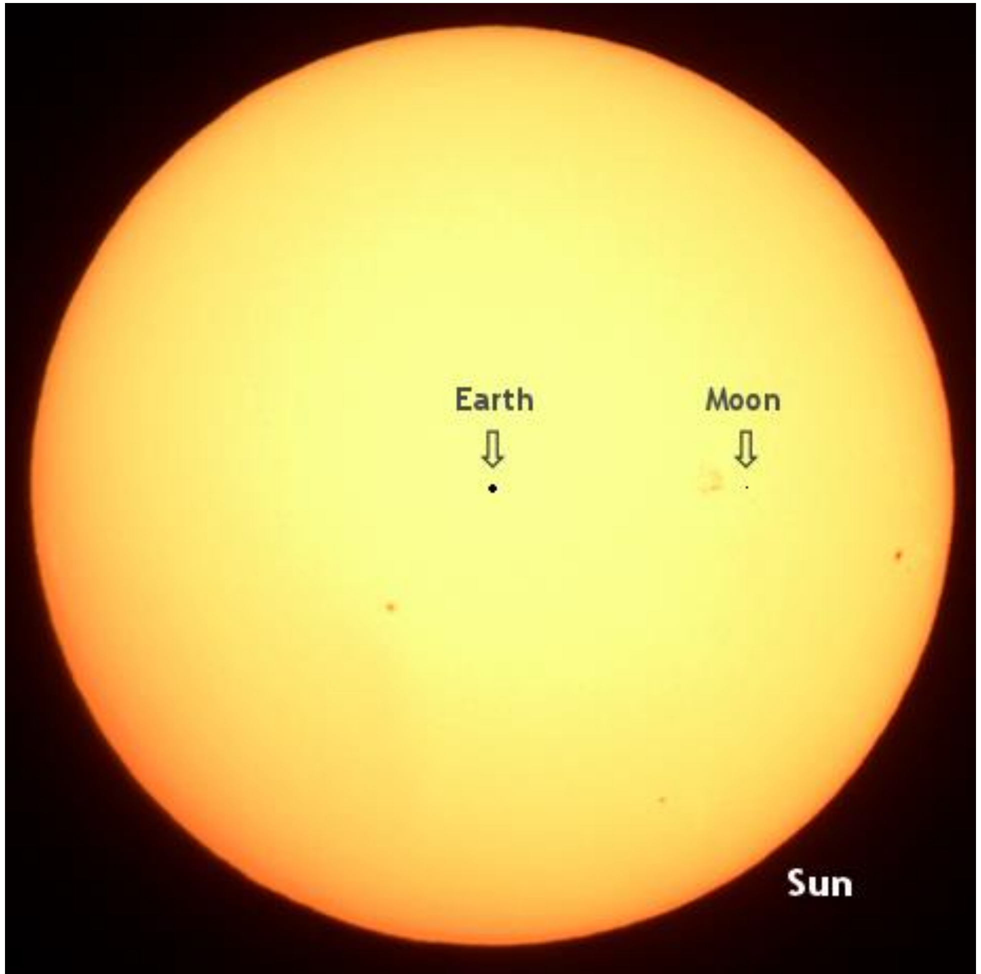
- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y.



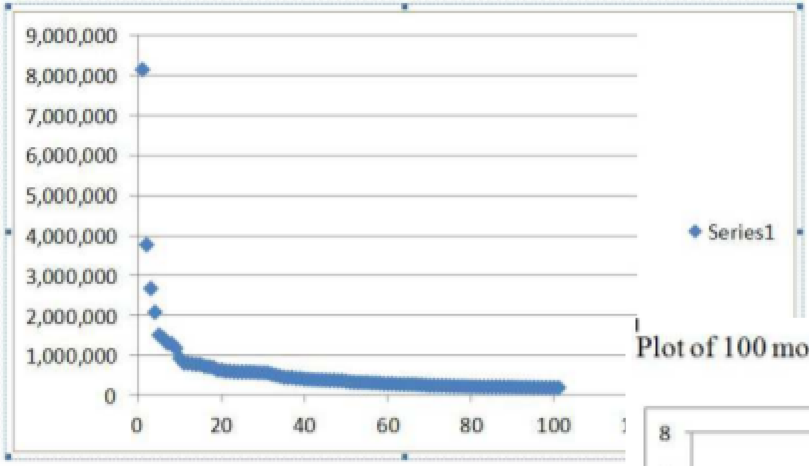


Why do we care about Logarithms

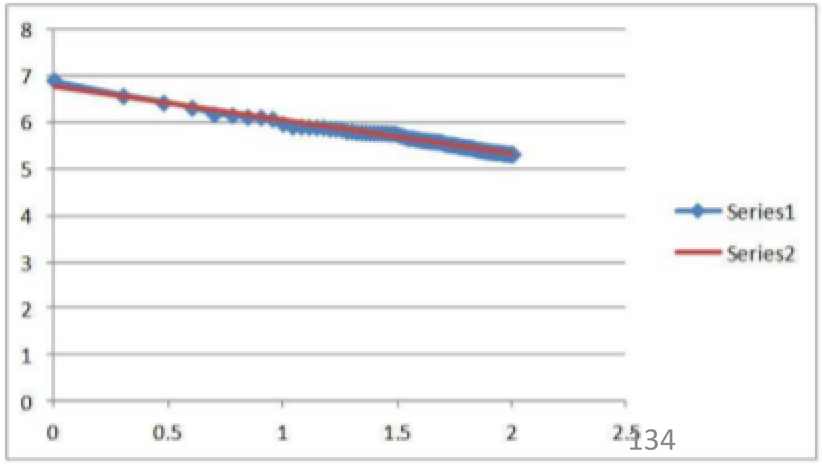
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities

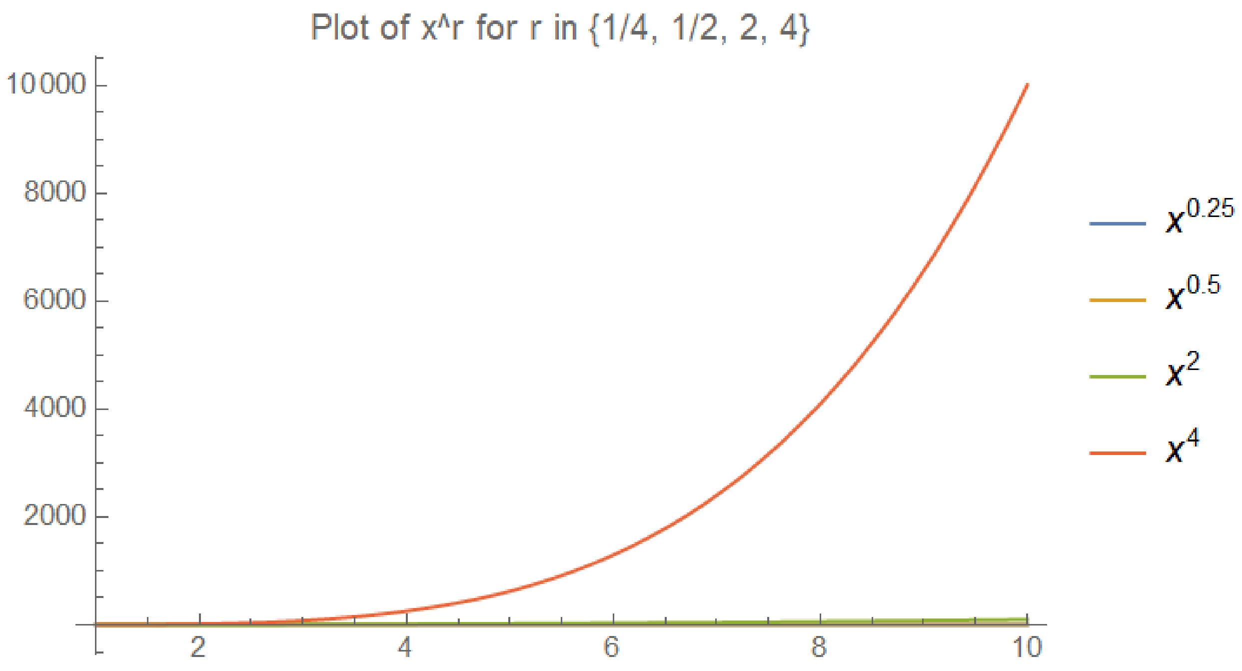


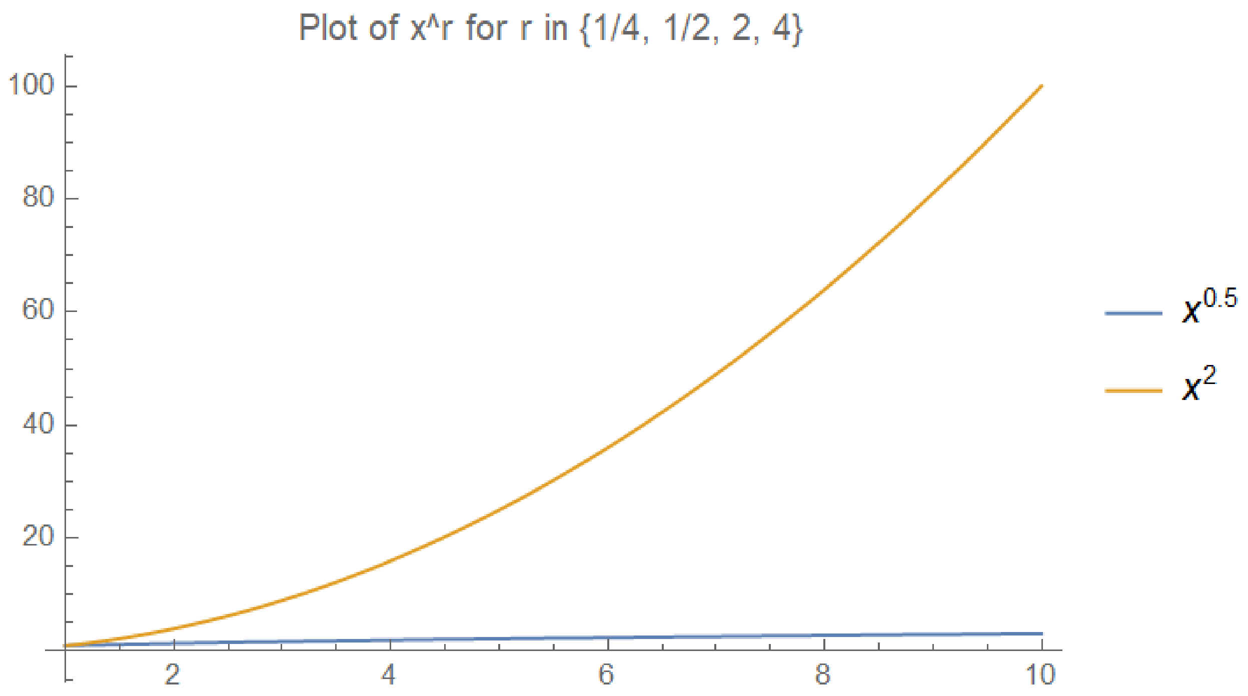
Plot of 100 most populous cities: log-log plot



Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).

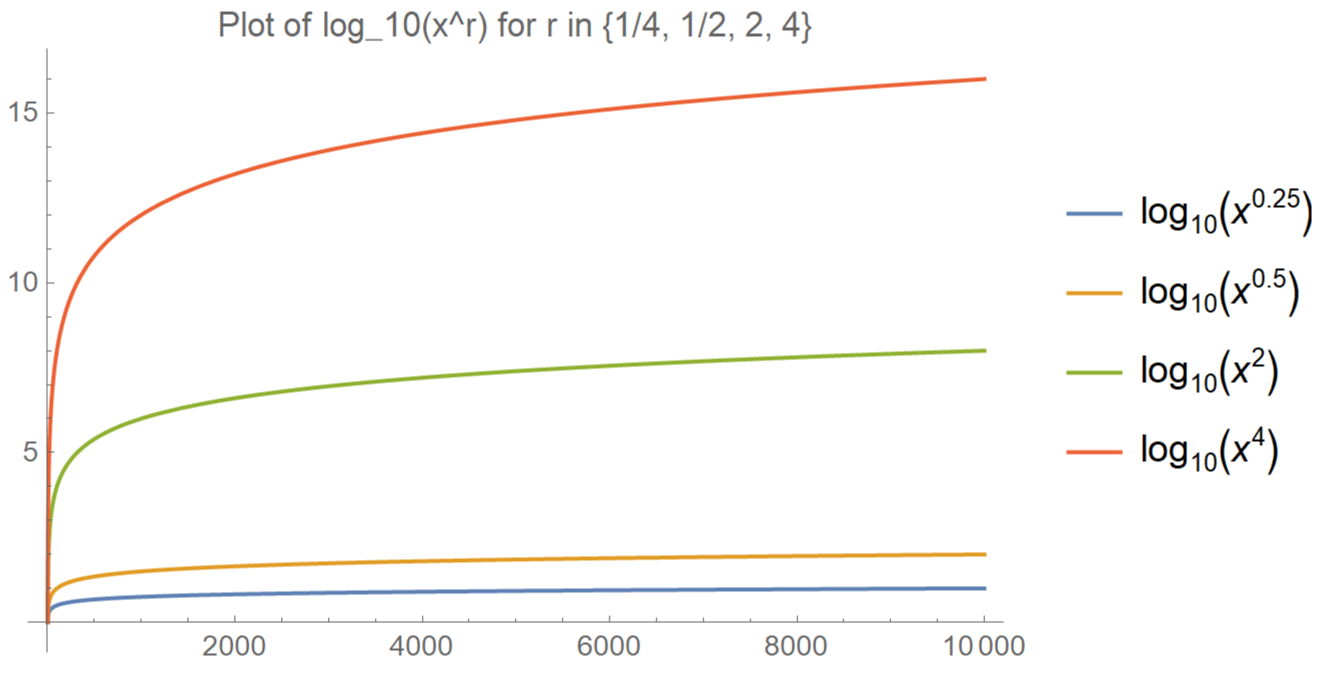


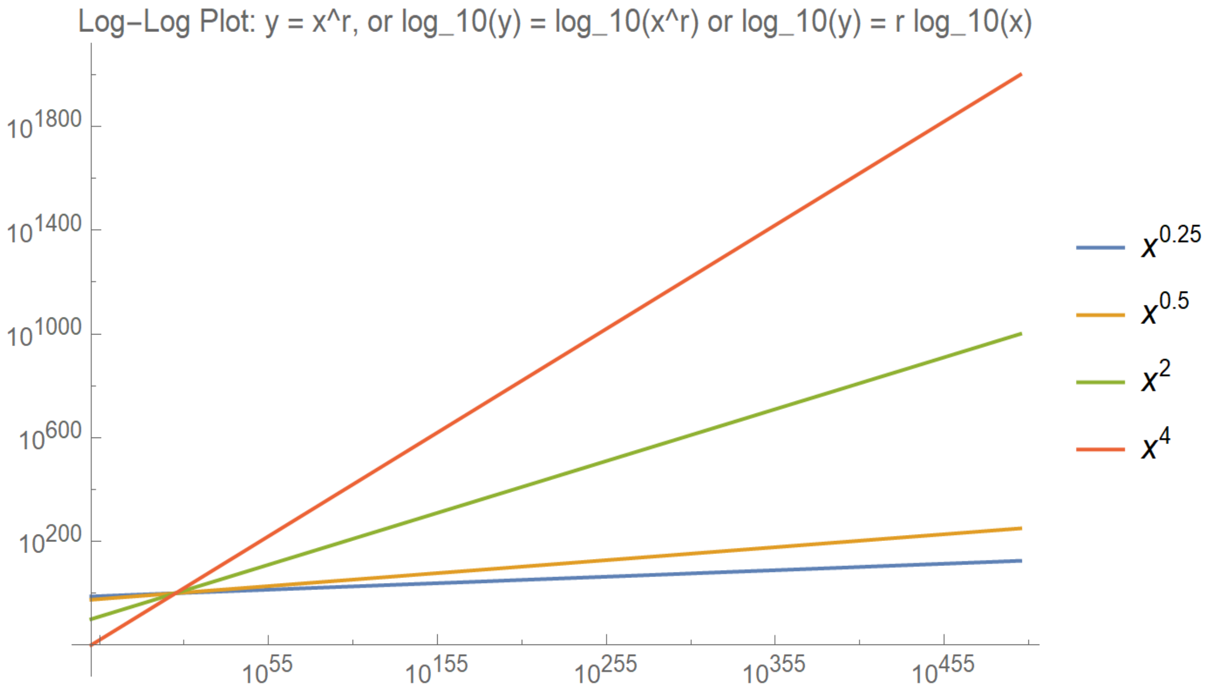


Notice that even on a small range, from 1 to 10, the polynomial of highest degree drowns out the others and can barely see.

Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).





Left: Semi-log plot: $y = \log x^r$. Right: log-log plot: $\log y = \log x^r$.
Note that we can now see the four functions on one plot, and the log-log plot now has linear relations.

Review: Exponent Laws

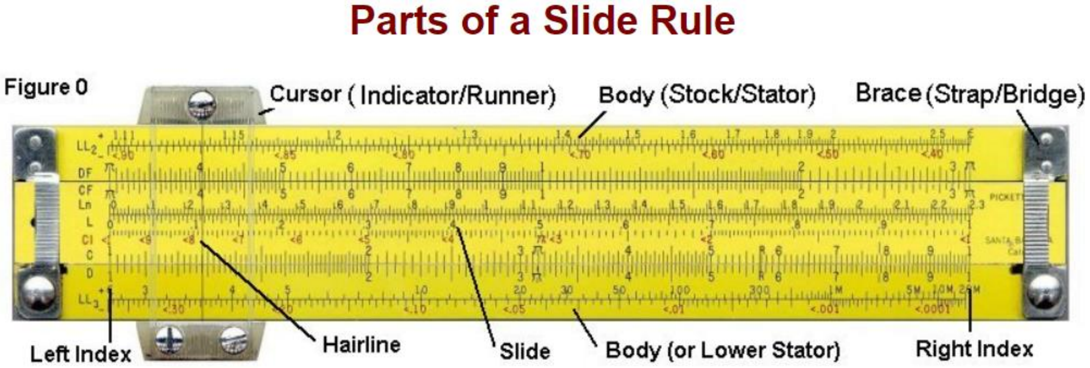
Laws

- $b^m\ b^n = b^{m+n}$
- $b^m\ /\ b^n = b^{m-n}$
- $(b^m)^n = b^{mn}$

Examples

- $10^3 10^2 = (10 * 10 * 10) * (10 * 10) = 10^5$
- $10^3/10^2 = (10 * 10 * 10)/(10 * 10) = 10^1$
- $(10^3)^2 = 103 * 103 = (10 * 10 * 10) * (10 * 10 * 10) = 10^6$

Logarithm Laws



Remember if $x = b^y$ then $\log_b x = y$.

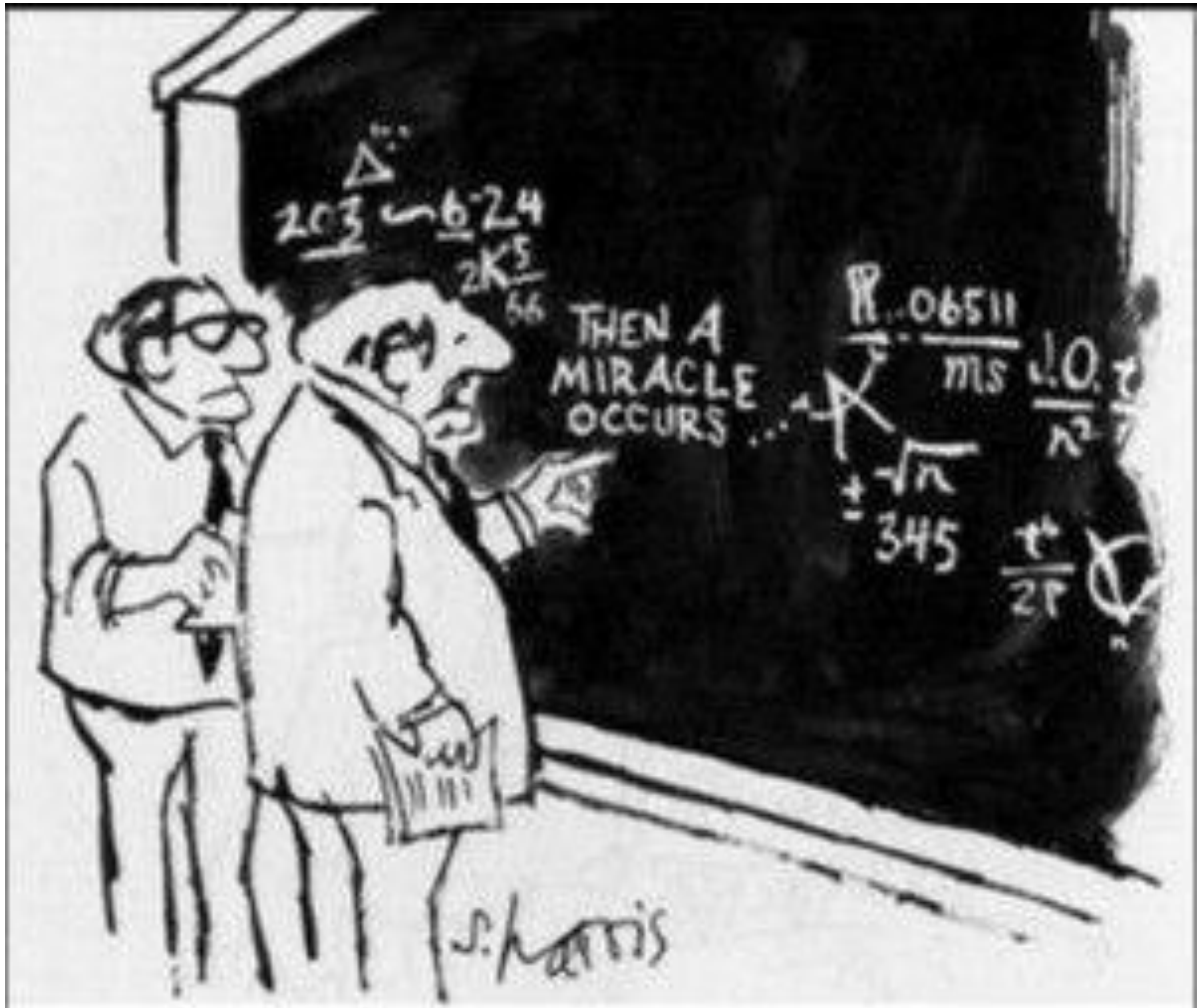
Below assume $\log_b x1 = y_1$ and $\log_b x2 = y_2$.

These allow us to simplify computations with logarithms.

THEOREM

- $\log_b(xn) = \text{n} \log_b x$. Log of a power is that power times the log.
- $\log_b(x1 \; x2) = \log_b(x1) + \log_b(x2)$. Log of a product is the sum of the logs.
- $\log_b(x1 / x2) = \log_b(x1) - \log_b(x2)$. Log of a quotient is the difference of the logs.
- $\log_b x = \log_c x / \log_c b$. If know logs in one base, know in all.

PROOFS OF THE LOG LAWS



"I think you should be more explicit here in step two."

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\boldsymbol{\log_b(x^n) = n \log_b x.}$ **Log of a power is that power times the log.**

Proof:

- $\log_b x = y$ means $x = b^y$.
- Thus $x^n = (b^y)^n = b^{ny}$.
- Taking logarithms: $\log_b(x^n) = ny = n \log_b x$. ■

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\mathbf{\log_b(x_1\ x_2) = \log_b(x_1) + \log_b(x_2)}$. **Log of a product is the sum of the logs.**

Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.
- Thus $x_1\ x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.
- Therefore $\log_b(x_1\ x_2) = y_1 + y_2 = \log_b x_1 + \log_b x_2$. $\blacksquare$

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

- $\boldsymbol{\log_b x = \log_c x / \log_c b}$. **Know logs in one base, know in all.**

Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^y = (c^v)^y = \cancel{c^{vy}}$. c^{vy}
- As also have $x = c^u$ we have $u = vy$ or $y = u/v$.
- Substituting gives $\log_b x = \log_c x / \log_c b$. $\blacksquare$

Example: Factorial Function:
Number ways to order $\boldsymbol{n}$ objects when order matters:

$$\boldsymbol{n}! = \boldsymbol{n} * (\boldsymbol{n}-\mathbf{1}) * \bullet\bullet\bullet * \mathbf{3} * \mathbf{2} * \mathbf{1}.$$

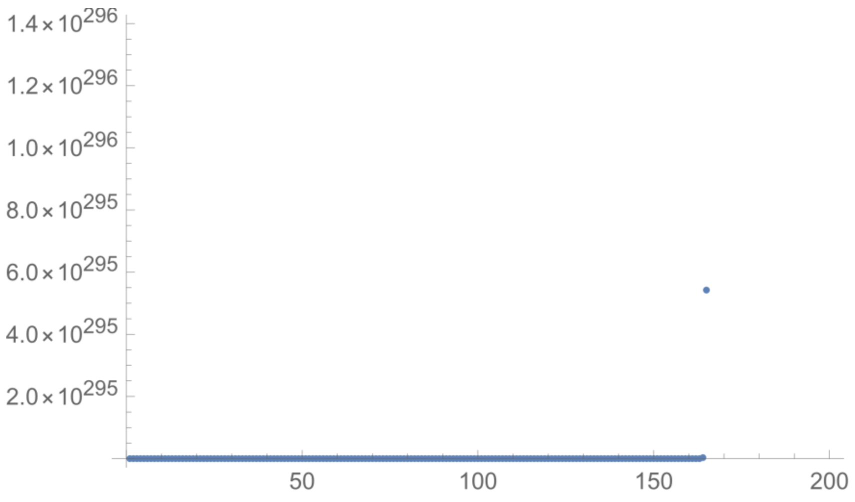
```
list = {}; semiloglist = {}; logloglist = {};
For[n = 1, n <= 200, n++,
 {
  list = AppendTo[list, {n, n!}];
  semiloglist = AppendTo[semiloglist, {n, Log[n!]}];
  logloglist = AppendTo[logloglist, {Log[n], Log[n!]}];
  }];
Print[ListPlot[list]]; Print[ListPlot[semiloglist]]; Print[ListPlot[logloglist]];
```

Example: Factorial Function:
Number ways to order $\boldsymbol{n}$ objects when order matters:

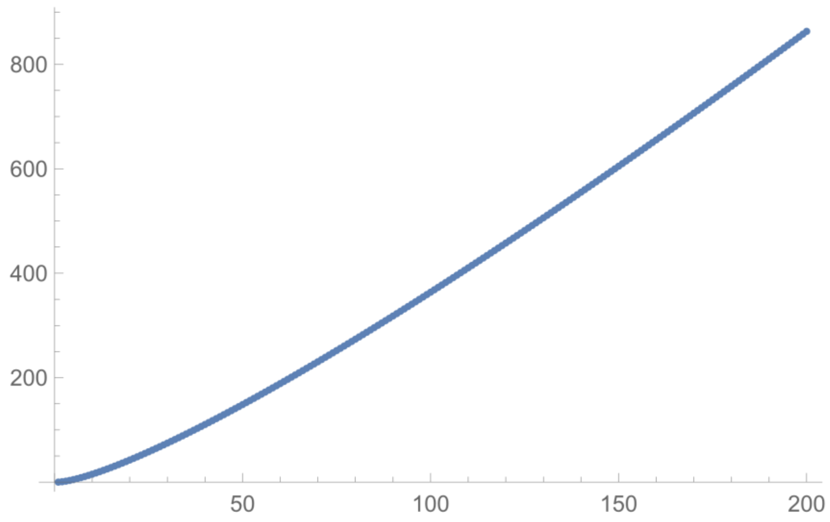
$$n! = n * (n-1) * \bullet\bullet\bullet * 3 * 2 * 1.$$

$n! \approx \left(\frac{n}{e}\right)^n (2\pi n)^{1/2}$

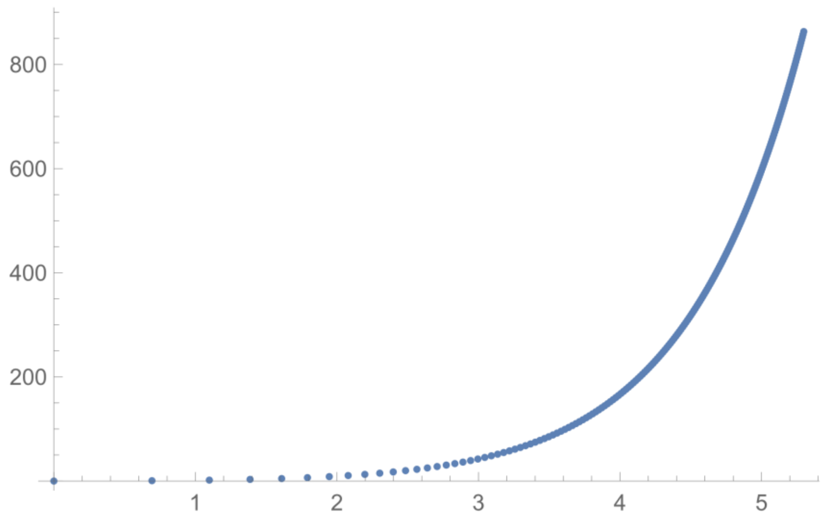
$\log(n!) \approx n \ln\left(\frac{n}{e}\right) + \frac{1}{2}\ln(2\pi n)$



Normal Plot



Semi-log Plot



Log-Log Plot

$n \ln\left(\frac{n}{e}\right)$

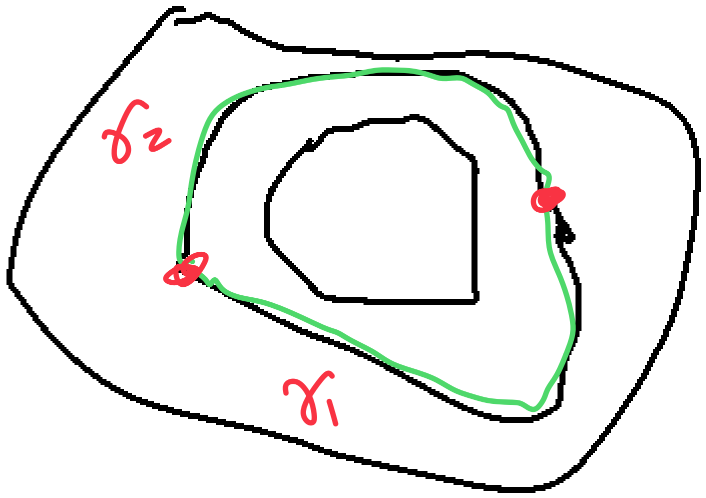
For large n, have $n! \approx n^n e^{-n}\sqrt{2\pi n}$, so $\log n! \approx \frac{n}{e}\log n$ (plus a much smaller term).

Simply connected space

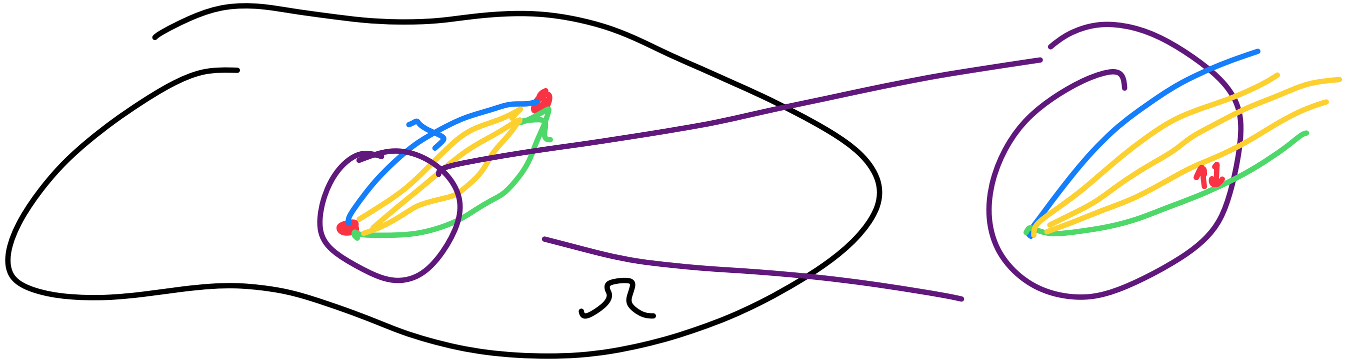
https://en.wikipedia.org/wiki/Simply_connected_space

From Wikipedia, the free encyclopedia

In topology, a topological space is called **simply connected** (or **1-connected**, or **1-simply connected**[1]) if it is path-connected and every path between two points can be continuously transformed (intuitively for embedded spaces, staying within the space) into any other such path while preserving the two endpoints in question. The fundamental group of a topological space is an indicator of the failure for the space to be simply connected: a path-connected topological space is simply connected if and only if its fundamental group is trivial.



$$t\cdot\gamma_1(z) + (1-t)\,\gamma_2(z)$$



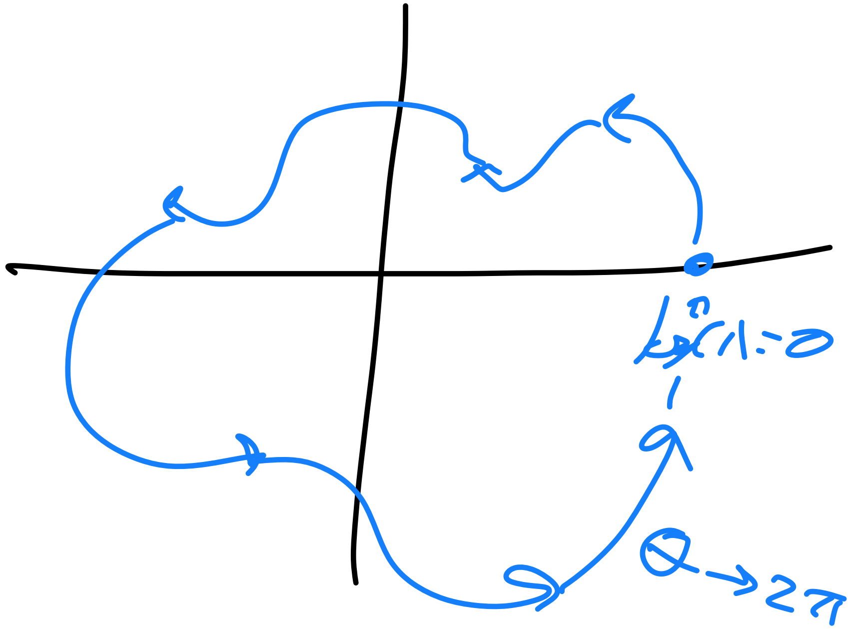
Branch cuts [edit]

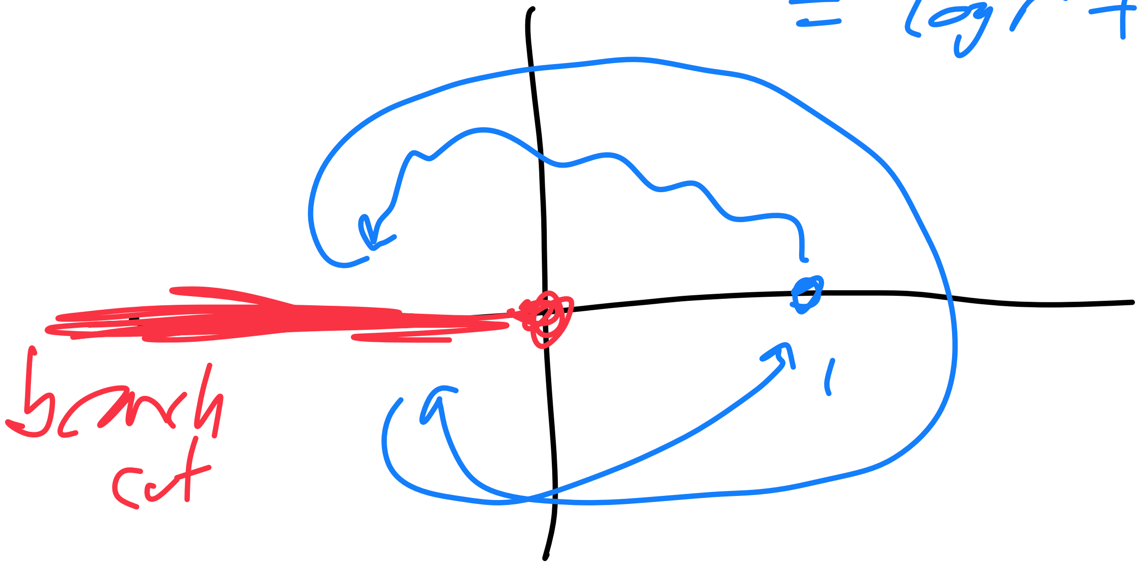
Roughly speaking, branch points are the points where the various sheets of a multiple valued function come together. The branches of the function are the various sheets of the function. For example, the function $w = z^{1/2}$ has two branches: one where the square root comes in with a plus sign, and the other with a minus sign. A **branch cut** is a curve in the complex plane such that it is possible to define a single analytic branch of a multi-valued function on the plane minus that curve. Branch cuts are usually, but not always, taken between pairs of branch points.

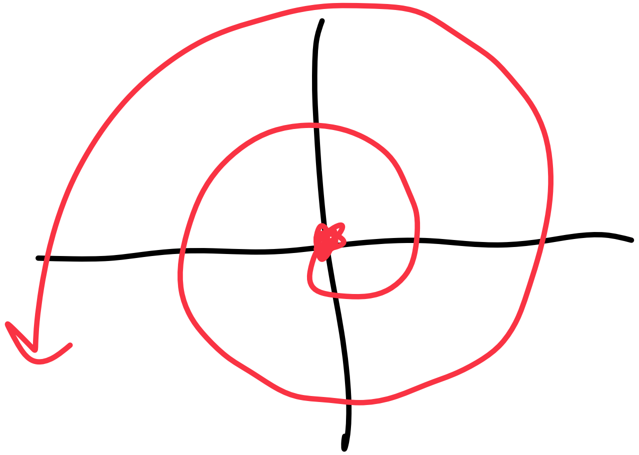
$z = x$, $\log(z) = \log(x)$ if $x>0$

$z = re^{i\theta}$ $\quad \log z \text{ "=" } \text{"}\log r + i\theta\text{"}$

$$\log(r e^{(2\pi-\varepsilon) i}) = \log r + (2\pi - \varepsilon) i$$





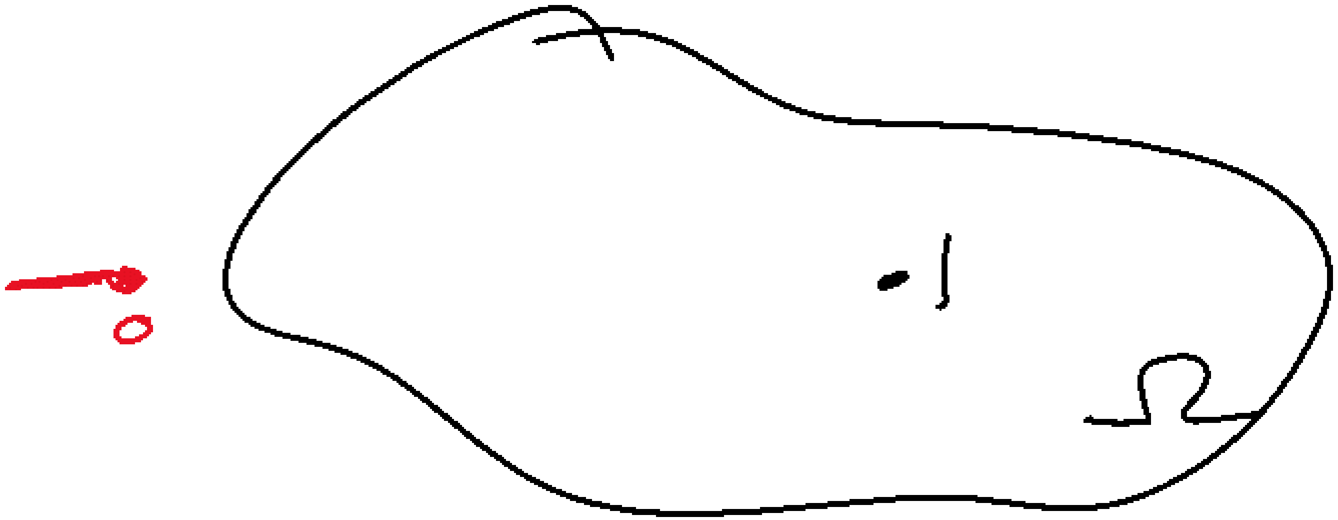


Theorem 6.1 *Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that*

(i) *F is holomorphic in Ω,*

(ii) *$e^{F(z)} = z$ for all $z \in \Omega$,*

(iii) *$F(r) = \log r$ whenever r is a real number and near 1.*



Calculus!

$\log(x)$, $\frac{d}{dx} \log x = \frac{1}{x}$

so $\log x = \int_1^x \frac{1}{t}\, dt$

$\exp(\log x) = x$

Theorem 6.1 *Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that*

(i) *F is holomorphic in Ω,*

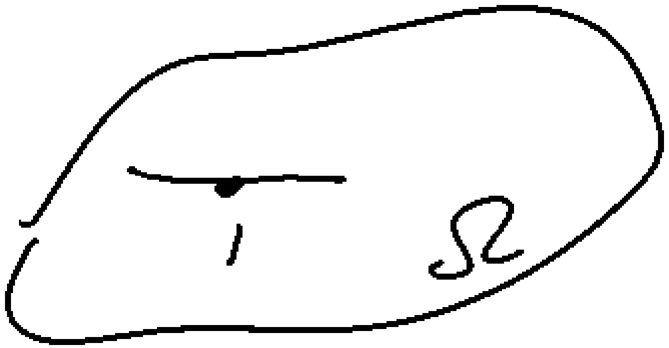
(ii) *$e^{F(z)} = z$ for all $z \in \Omega$,*

(iii) *$F(r) = \log r$ whenever r is a real number and near 1.*

Proof. We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω. We define

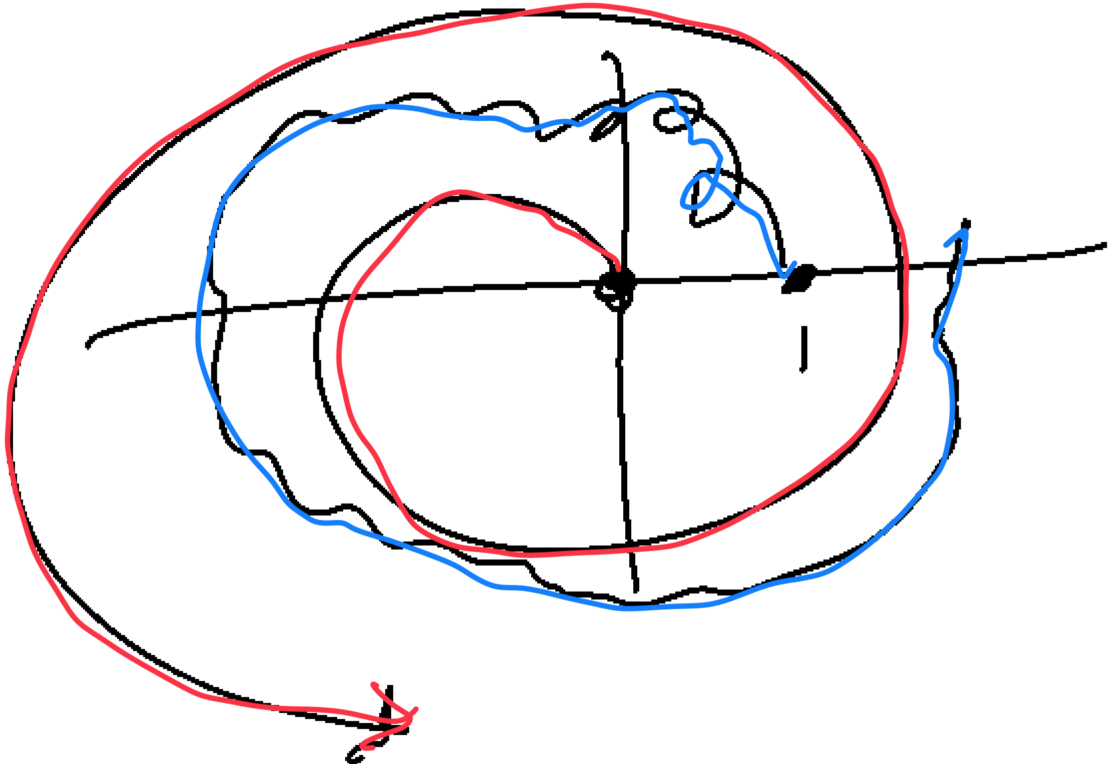
$$\log_\Omega(z) = F(z) = \int_\gamma f(w)\, dw\,,$$

Need $F(r) = \log r$ *whenever* r *is a real number and near* 1.



$z = x \in \mathbb{R}$ near 1

$F(z) = F(x) = \int_1^x \frac{1}{t}\,dt = \log x$



Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

Proof. Fix a point z_0 in Ω, and define a function

$$g(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)}\, dw + c_0,$$

well defined as $f(z) \neq 0$ in Ω

$g(z_0) = c_0$ choose st $f(z_0) = e^{c_0}$

morally it is $\log f(z)$ up to a constant

~~log~~ $g'(z) = f'(z)/f(z)$

Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

The function $g(z)$ in the theorem can be denoted by $\log f(z)$, and determines a "branch" of that logarithm.

Proof. Fix a point z_0 in Ω, and define a function

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)}\, dw + c_0,$$

where γ is any path in Ω connecting z_0 to z, and c_0 is a complex number so that $e^{c_0} = f(z_0)$. This definition is independent of the path γ since Ω is simply connected. Arguing as in the proof of Theorem 2.1, Chapter 2, we find that g is holomorphic with

$$g'(z) = \frac{f'(z)}{f(z)},$$

and a simple calculation gives

$$\frac{d}{dz}\left(f(z)e^{-g(z)}\right) = 0\,,$$

so that $f(z)e^{-g(z)}$ is constant. Evaluating this expression at z_0 we find $f(z_0)e^{-c_0} = 1$, so that $f(z) = e^{g(z)}$ for all $z \in \Omega$, and the proof is complete.

Exponential Function: Properties and Relation to Trig.

$$e^z = \sum_{n=0}^{\infty} z^n/n! \qquad e^{ix} = \cos x + i\sin x$$

$$\cos x = 1 - x^2/2! + x^4/4! - \dots \qquad \sin x = x - x^3/3! + x^5/5! - \dots$$

Thm: $e^z e^w = e^{z+w}$

Proof: $e^z e^w = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!}$

Fubini
Sum abs values converges

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{k!\, z^l w^{k-l}}{k!\, l!\,(k-l)!} \qquad \binom{k}{l}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left[\sum_{l=0}^{k} \binom{k}{l} z^l w^{k-l}\right] = \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = e^{z+w}$$

$\blacksquare$

$$e^{ix} = \cos x + i\sin x$$

$$e^{ix}e^{-ix} = e^{0} = 1$$

$$(\cos x + i\sin x)(\cos x - i\sin x) = 1 \implies \cos^2 x + \sin^2 x = 1$$

$$e^{ix}e^{iy} = e^{i(x+y)}$$

$$(\cos x + i\sin x)(\cos y + i\sin y) = \cos(x+y) + i\sin(x+y)$$

$$\left[\cos x\cos y - \sin x\sin y\right] + i\left[\sin x\cos y + \cos x\sin y\right]$$

Much of math is about solving equations.

Example: polynomials:

- $ax + b = 0$, root $x = -b/a$.
- $ax^2 + bx + c = 0$, roots $(-b \pm \sqrt{b^2 - 4ac})/2a$.
- Cubic, quartic: formulas exist in terms of coefficients; not for quintic and higher.

In general cannot find exact solution, how to estimate?

Cubic: For fun, here's the solution to $ax^3 + bx^2 + cx + d = 0$

```
Solve[a x^3 + b x^2 + c x + d == 0, x]
```

$$\left\{\left\{x \to -\frac{b}{3 a} - \frac{2^{1/3}\left(-b^2+3 a c\right)}{3 a \left(-2 b^3+9 a b c-27 a^2 d+\sqrt{4\left(-b^2+3 a c\right)^3+\left(-2 b^3+9 a b c-27 a^2 d\right)^2}\right)^{1/3}} + \frac{\left(-2 b^3+9 a b c-27 a^2 d+\sqrt{4\left(-b^2+3 a c\right)^3+\left(-2 b^3+9 a b c-27 a^2 d\right)^2}\right)^{1/3}}{3 \times 2^{1/3} a}\right\},\right.$$

$$\left\{x \to -\frac{b}{3 a} + \frac{\left(1+i\sqrt{3}\right)\left(-b^2+3 a c\right)}{3 \times 2^{2/3} a \left(-2 b^3+9 a b c-27 a^2 d+\sqrt{4\left(-b^2+3 a c\right)^3+\left(-2 b^3+9 a b c-27 a^2 d\right)^2}\right)^{1/3}} - \frac{\left(1-i\sqrt{3}\right)\left(-2 b^3+9 a b c-27 a^2 d+\sqrt{4\left(-b^2+3 a c\right)^3+\left(-2 b^3+9 a b c-27 a^2 d\right)^2}\right)^{1/3}}{6 \times 2^{1/3} a}\right\},$$

$$\left.\left\{x \to -\frac{b}{3 a} + \frac{\left(1-i\sqrt{3}\right)\left(-b^2+3 a c\right)}{3 \times 2^{2/3} a \left(-2 b^3+9 a b c-27 a^2 d+\sqrt{4\left(-b^2+3 a c\right)^3+\left(-2 b^3+9 a b c-27 a^2 d\right)^2}\right)^{1/3}} - \frac{\left(1+i\sqrt{3}\right)\left(-2 b^3+9 a b c-27 a^2 d+\sqrt{4\left(-b^2+3 a c\right)^3+\left(-2 b^3+9 a b c-27 a^2 d\right)^2}\right)^{1/3}}{6 \times 2^{1/3} a}\right\}\right\}$$

One of four solutions to quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

```
Solve[a x^4 + b x^3 + c x^2 + d x + e == 0, x]
```

$$
\begin{aligned}
\Big\{\Big\{x \to {} & -\frac{b}{4a} - \frac{1}{2}\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \right.} \\
& \left(2^{1/3}\left(c^2-3bd+12ae\right)\right)\Big/\left(3a\left(2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4\left(c^2-3bd+12ae\right)^3+\left(2c^3-9bcd+27ad^2+27b^2e-72ace\right)^2}\right)^{1/3}\right) + \\
& \left.\frac{1}{3\times 2^{1/3}a}\left(2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4\left(c^2-3bd+12ae\right)^3+\left(2c^3-9bcd+27ad^2+27b^2e-72ace\right)^2}\right)^{1/3}\right) - \frac{1}{2}\sqrt{\left(\frac{b^2}{2a^2} - \frac{4c}{3a} - \right.} \\
& \left(2^{1/3}\left(c^2-3bd+12ae\right)\right)\Big/\left(3a\left(2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4\left(c^2-3bd+12ae\right)^3+\left(2c^3-9bcd+27ad^2+27b^2e-72ace\right)^2}\right)^{1/3}\right) - \\
& \frac{1}{3\times 2^{1/3}a}\left(2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4\left(c^2-3bd+12ae\right)^3+\left(2c^3-9bcd+27ad^2+27b^2e-72ace\right)^2}\right)^{1/3} - \\
& \left(-\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}\right)\Big/\left(4\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \left(2^{1/3}\left(c^2-3bd+12ae\right)\right)\Big/\right.}\right. \\
& \quad \left(3a\left(2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4\left(c^2-3bd+12ae\right)^3+\left(2c^3-9bcd+27ad^2+27b^2e-72ace\right)^2}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}a} \\
& \quad \left.\left.\left(2c^3-9bcd+27ad^2+27b^2e-72ace+\sqrt{-4\left(c^2-3bd+12ae\right)^3+\left(2c^3-9bcd+27ad^2+27b^2e-72ace\right)^2}\right)^{1/3}\right)\right)\Big\},
\end{aligned}
$$

Prescribe a fn that vanishes at z_1, z_2, z_3

$$A(z - z_1)(z - z_2)(z - z_3)$$

$$f(z_1)\frac{(z - z_2)(z - z_3)}{(z_1 - z_2)(z_1 - z_3)} + f(z_2)\frac{(z - z_1)(z - z_3)}{(z_2 - z_1)(z_2 - z_3)} + f(z_3)\frac{(z - z_1)(z - z_2)}{(z_3 - z_1)(z_3 - z_2)}$$

Can do for $n = 3, 4, 5, \ldots$ so long as n is

FINITE!

Mathematica Programs

Links to LaTeX and Mathematica tutorials / templates:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/handouts/latex.htm

Program on plotting zeros of a sequence of functions:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/mathematicaprograms/PlotZerosExpApprox.nb
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/mathematicaprograms/PlotZerosExpApprox.pdf

Math 383: Complex Analysis: Fall '23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 18: 10-16-23: https://youtu.be/WzJhBEwPwpA

Watch the following videos before class / read the book (better both!):

- Weierstrass Products, Conformal Maps: https://youtu.be/clP3ZO5HpV4
- Introduction to Conformal Maps: https://youtu.be/kzPm-0X_HW8 (slides)

Topic: Zeros of functions, Weierstrass products

Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

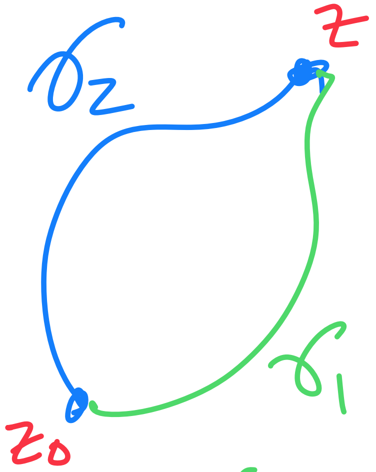
$f(x) = e^{g(x)}$

take logs:

$g(x) = \ln f(x)$

$g'(x) = \frac{1}{f(x)} f'(x)$

The function $g(z)$ in the theorem can be denoted by $\log f(z)$, and determines a "branch" of that logarithm.



Proof. Fix a point z_0 in Ω, and define a function

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)}\, dw + c_0,$$

where γ is any path in Ω connecting z_0 to z, and c_0 is a complex number so that $e^{c_0} = f(z_0)$. This definition is independent of the path γ since Ω is simply connected. Arguing as in the proof of Theorem 2.1, Chapter 2, we find that g is holomorphic with

$\int_{\gamma_2} = \int_{\gamma_1}$

$\int g'(x)\, dx = g(x)$

$$g'(z) = \frac{f'(z)}{f(z)},$$

$x \to z$

and a simple calculation gives

$$\frac{d}{dz}\left(f(z)e^{-g(z)}\right) = 0\,,$$

so that $f(z)e^{-g(z)}$ is constant. Evaluating this expression at z_0 we find $f(z_0)e^{-c_0} = 1$, so that $f(z) = e^{g(z)}$ for all $z \in \Omega$, and the proof is complete.

Jensen's formula

Theorem 1.1 *Let Ω be an open set that contains the closure of a disc D_R and suppose that f is holomorphic in Ω, $f(0) \neq 0$, and f vanishes nowhere on the circle C_R. If $z_1, \ldots, z_N$ denote the zeros of f inside the disc (counted with multiplicities),[1] then*

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta. \tag{1}$$

2 Functions of finite order

Let f be an entire function. If there exist a positive number ρ and constants $A, B > 0$ such that

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all } z \in \mathbb{C},$$

then we say that f has **an order of growth** $\leq \rho$. We define *the* **order of growth of** f as

$$\rho_f = \inf \rho\,,$$

where the infimum is over all $\rho > 0$ such that f has an order of growth $\leq \rho$.

For example, the order of growth of the function e^{z^2} is 2.

Theorem 2.1 *If f is an entire function that has an order of growth $\leq \rho$, then:*

(i) $\mathfrak{n}(r) \leq Cr^\rho$ *for some $C > 0$ and all sufficiently large r.*

(ii) *If $z_1, z_2, \ldots$ denote the zeros of f, with $z_k \neq 0$, then for all $s > \rho$ we have*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty.$$

3 Infinite products

3.1 Generalities

Given a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers, we say that the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if the limit

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)$$

of the partial products exists.

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right) = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \frac{2}{1}\cdot\frac{5}{4}\cdot\frac{10}{9}\cdot\frac{17}{16}$$

$$\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right) = \frac{4}{3}\cdot\frac{9}{8}\cdot\frac{16}{15}\cdot\frac{25}{24}\cdots$$

$$\frac{2\cdot 2}{1\cdot 3}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{5\cdot 5}{4\cdot 6}$$

like $\frac{2}{1}$ × almost 1 $\Rightarrow$ 2

Take logs

$$\log\prod_{n=1}^{N}(1+a_n) = \sum_{n=1}^{N}\log(1+a_n)$$

Taylor series of log(1-x) and Harmonic Series

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad \text{if } |x|<1$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = \log(1-x) \qquad f'(x) = \frac{-1}{1-x} = -(1-x)^{-1} \qquad f''(x) = -1\cdot(1-x)^{-2}$$

$$f'''(x) = -1\cdot 2\,(1-x)^{-3}$$

$$\log(1+x) = \log(1-(-x)) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

If $x>0$, $\log(1+x) > 0$ see above

If $x>0$, $\log(1-x) < 0$ see above

Harmonic Series $a_n = 1/n$

$$\sum_{n=1}^{N} \frac{1}{n}$$ Sum diverges:

Proof: $1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{\geq \frac{1}{2}} + \cdots$ diverges

Proof: Assume converges

$$\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\right) + \underbrace{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)}_{\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)} = S$$

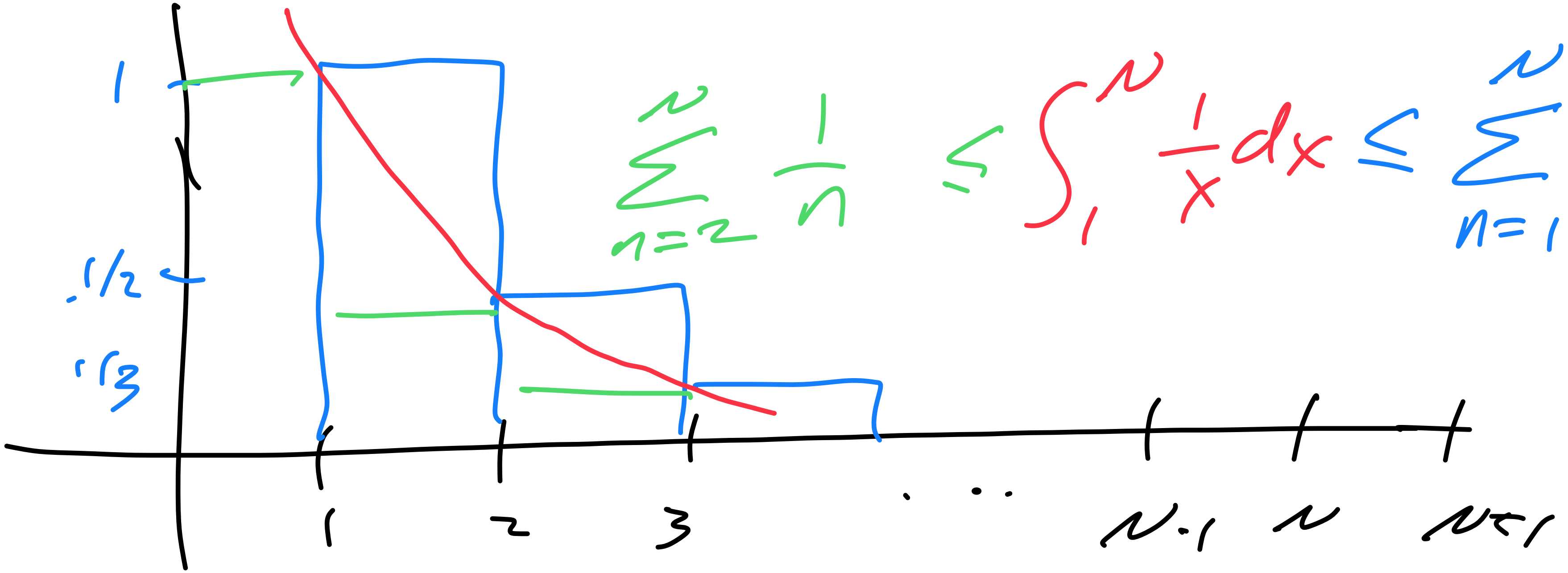
$$(\text{odd reciprocals}) + \frac{1}{2}S = S$$

$$\text{odd reciprocals} = \frac{1}{2}S = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots$$

absurd!

Integral test:

$$\sum_{n=1}^{N} \frac{1}{n} \simeq \int_1^N \frac{1}{x}\,dx = \ln(N) - \ln(1) = \ln(N)$$



$$\sum_{n=2}^{N} \frac{1}{n} \leq \int_1^N \frac{1}{x}\,dx \leq \sum_{n=1}^{N} \frac{1}{n}$$

$$\log(n!) = \sum_{k=1}^{N} \log k \simeq \int_1^N \log t\,dt = \left[t\log t - t\right]_1^N = N\log N - N$$

or $n! \simeq N^N e^{-N}$

Proposition 3.2 *Suppose $\{F_n\}$ is a sequence of holomorphic functions on the open set Ω. If there exist constants $c_n > 0$ such that*

$$\sum c_n < \infty \quad \text{and} \quad |F_n(z) - 1| \le c_n \quad \text{for all } z \in \Omega,$$

then:

(i) *The product $\prod_{n=1}^{\infty} F_n(z)$ converges uniformly in Ω to a holomorphic function $F(z)$.*

(ii) *If $F_n(z)$ does not vanish for any n, then*

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

4 Weierstrass infinite products

We now turn to Weierstrass's construction of an entire function with prescribed zeros.

Theorem 4.1 *Given any sequence* $\{a_n\}$ *of complex numbers with* $|a_n| \to \infty$ *as* $n \to \infty$*, there exists an entire function* f *that vanishes at all* $z = a_n$ *and nowhere else. Any other such entire function is of the form* $f(z)e^{g(z)}$*, where* g *is entire.*

For each integer $k \geq 0$ we define **canonical factors** by

truncation of $-\log(1-z)$

$$E_0(z) = 1 - z \quad \text{and} \quad E_k(z) = (1-z)e^{z+z^2/2+\cdots+z^k/k}, \quad \text{for } k \geq 1.$$

$1-z = e^{\log(1-z)}$

The integer k is called the **degree** of the canonical factor.

Lemma 4.2 *If $|z| \leq 1/2$, then $|1 - E_k(z)| \leq c|z|^{k+1}$ for some $c > 0$.*

Proof: $(1-z)\,e^{z+\cdots+z^k/k} = e^{\log(1-z)+z+\cdots+z^k/k}$

$= e^{-\left(z^{k+1}/k+1 + z^{k+2}/k+2 + \cdots\right)}$

Study exponent: $\left|z^{k+1}\right|\left|\sum_{n=0}^{\infty} \frac{|z|^n}{n+k+1}\right| \leq |z|^{k+1} \sum_{n=0}^{\infty} |z|^n$

$= |z|^{k+1} \frac{1}{1-|z|} \leq 2|z|^{k+1}$

$|1-E_k(z)|$

$\uparrow$

$1+$ Error of size $2|z|^{k+1}$ + small terms

$$\underbrace{e^{-2|z|^{k+1}}}_{1-\frac{2|z|^{k+1}}{1!}+\cdots} \le E_k(z) \le \underbrace{e^{2|z|^{k+1}}}_{1+\frac{2|z|^{k+1}}{1!}+\cdots}$$

If u is small, $|e^u - 1| \le 2u$

$$|1 - E_k(z)| \le 2 \cdot 2|z|^{k+1}$$

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Lecture 19: 10-16-23: No class, take-home exam.

Lecture 20: 10-23-23: https://youtu.be/xftsQF-6yUs

Watch the following videos before class / read the book (better both!):

- Weierstrass Products, Conformal Maps: https://youtu.be/clP3ZO5HpV4
- Introduction to Conformal Maps: https://youtu.be/kzPm-0X_HW8 (slides)

Topic: Zeros of functions, Weierstrass products (Continued)

3 Infinite products

3.1 Generalities

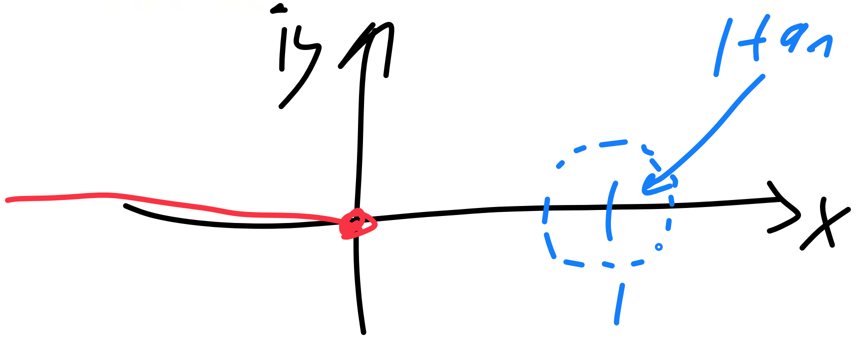
Given a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers, we say that the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if the limit

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)$$

of the partial products exists.



define logarithm useful for all $n \geq N$

$$\log \prod_{n=1}^{\infty}(1+a_n) = \sum_{n=1}^{\infty}\log(1+a_n) = \sum_{n=1}^{\infty} a_n - \underbrace{\sum_{n=1}^{\infty}\frac{a_n^2}{2} + \dots}_{\text{small}}$$

$$\log 1+a_n = a_n + \left(\sum_{k=2}^{\infty}\frac{(-1)^{k+1}a_n^k}{k}\right)$$

$(a_n \to 0)$

Product converges $\iff$ $\sum a_n$ converge

Lagrange Interpolation

Points $P_1, P_2, \ldots, P_n$ with $P_k = (x_k, y_k)$ all x's distinct

$y_k \leftarrow f(x_k)$

Define $f_k(x) := \dfrac{(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$

$$= \prod_{\substack{l=1 \\ l \neq k}}^{n} \frac{x - x_l}{x_k - x_l}$$

$$f_k(x_l) = \begin{cases} 0 & \text{if } l \neq k \\ 1 & \text{if } l = k \end{cases}$$

$$f(x) = \sum_{k=1}^{n} f_k(x) * y_k$$

$$f(x_k) = y_k$$

prescribed zeros

$$\prod_{k=1}^{n} (x - x_k)$$

issues as $n \to \infty$

For each integer $k \geq 0$ we define **canonical factors** by

$$E_0(z) = 1 - z \quad \text{and} \quad E_k(z) = (1-z)e^{z+z^2/2+\cdots+z^k/k}, \quad \text{for } k \geq 1.$$

The integer k is called the **degree** of the canonical factor.

Lemma 4.2 *If* $|z| \leq 1/2$, *then* $|1 - E_k(z)| \leq c|z|^{k+1}$ *for some* $c > 0$.

Moral! $E_k(z) \approx 1$ if k is large

$z - z_k$ rewrite as $z_k - z$ or $1 - \frac{z}{z_k}$

zero at z_k if $z_k \neq 0$

Consider $z^n \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right)$

zeros at 0 non-zero zeros

If f is holo and not constant, zeros go off to infinity

This is $E_0(z/z_k)$

Note $E_k(z/z_k)$ zero only at $z = z_k$

https://en.wikipedia.org/wiki/Elementary_matrix

Elementary matrix

From Wikipedia, the free encyclopedia

In mathematics, an **elementary matrix** is a matrix which differs from the identity matrix by one single elementary row operation. The elementary matrices generate the general linear group $\mathrm{GL}_n(\mathbf{F})$ when $\mathbf{F}$ is a field. Left multiplication (pre-multiplication) by an elementary matrix represents **elementary row operations**, while right multiplication (post-multiplication) represents **elementary column operations**.

Elementary row operations are used in Gaussian elimination to reduce a matrix to row echelon form. They are also used in Gauss–Jordan elimination to further reduce the matrix to reduced row echelon form.

The first type of row operation on a matrix A switches all matrix elements on row i with their counterparts on a different row j. The corresponding elementary matrix is obtained by swapping row i and row j of the identity matrix.

$$T_{i,j} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Suppose that we are given a zero of order m at the origin, and that $a_1, a_2 \ldots$ are all non-zero. Then we define the Weierstrass product by

$$f(z) = z^m \prod_{n=1}^{\infty} E_n(z/a_n).$$

Proof Sketch: $|a_n| \to \infty$ so eventually $|z/a_n| \le 1/2$

(if $|a_n|$ doesn't go to ∞, get accumulation)

Note $E_n\left(\frac{z}{a_n}\right) = 1 + \underbrace{\text{something at most Const.} \left|\frac{z}{a_n}\right|^{n+1}}_{\alpha_n}$

$\prod E_n(z/a_n) = \prod (1+\alpha_n)$ converges iff $\sum \alpha_n$ converges

but $\sum \alpha_n$ is a geometric series with $|r|<1$, converges

Below are the Slides for Lecture 16 from 2021: 10-22-21: https://youtu.be/kzPm-0X_HW8

10/18/17: Introduction to Conformal Maps: https://youtu.be/5klb8gxnQTc

Plan for the day: Lecture 16: October 22, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Review inverse functions: $f(g(z)) = g(f(z)) = z$, application to derivatives (arctan)
- Conformal maps
- Specific conformal maps

General items.

- Differences b/w real and complex
- Seeing what theorems to use

Inverse functions: f(g(z)) = z, get formula for g'(z) (do for exp-log)

$$f(g(z)) = z$$

$$f'(g(z))\, g'(z) = 1$$

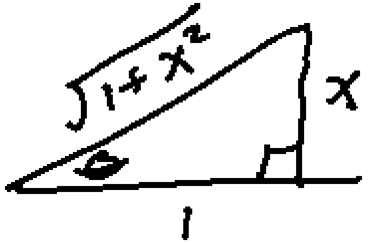
$$\Rightarrow g'(z) = \frac{1}{f'(g(z))}$$

Ex: $\exp(\log x) = x$

$$\log'(x) = \frac{1}{\exp'(\log x)} = \frac{1}{\exp(\log x)} = \frac{1}{x}$$

Ex: $\tan(\arctan x) = x$

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)} \overset{\text{work}}{=} \frac{1}{1+x^2}$$



$$\exp(x) := \sum_{n=0}^{\infty} x^n/n!$$

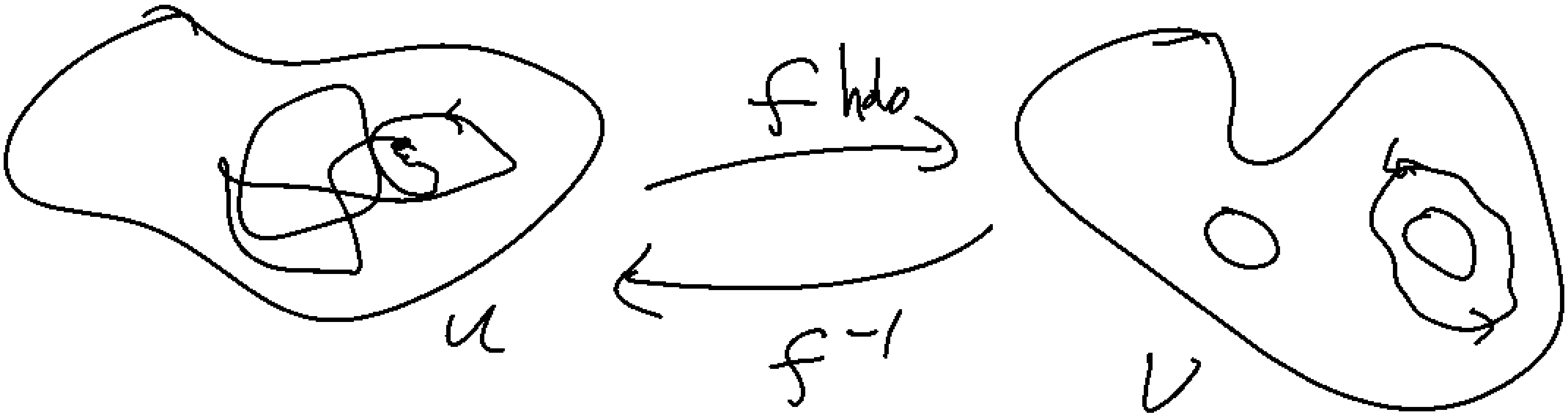
$$\exp'(x) = \sum_{n=0}^{\infty} x^n/n! = \exp(x)$$

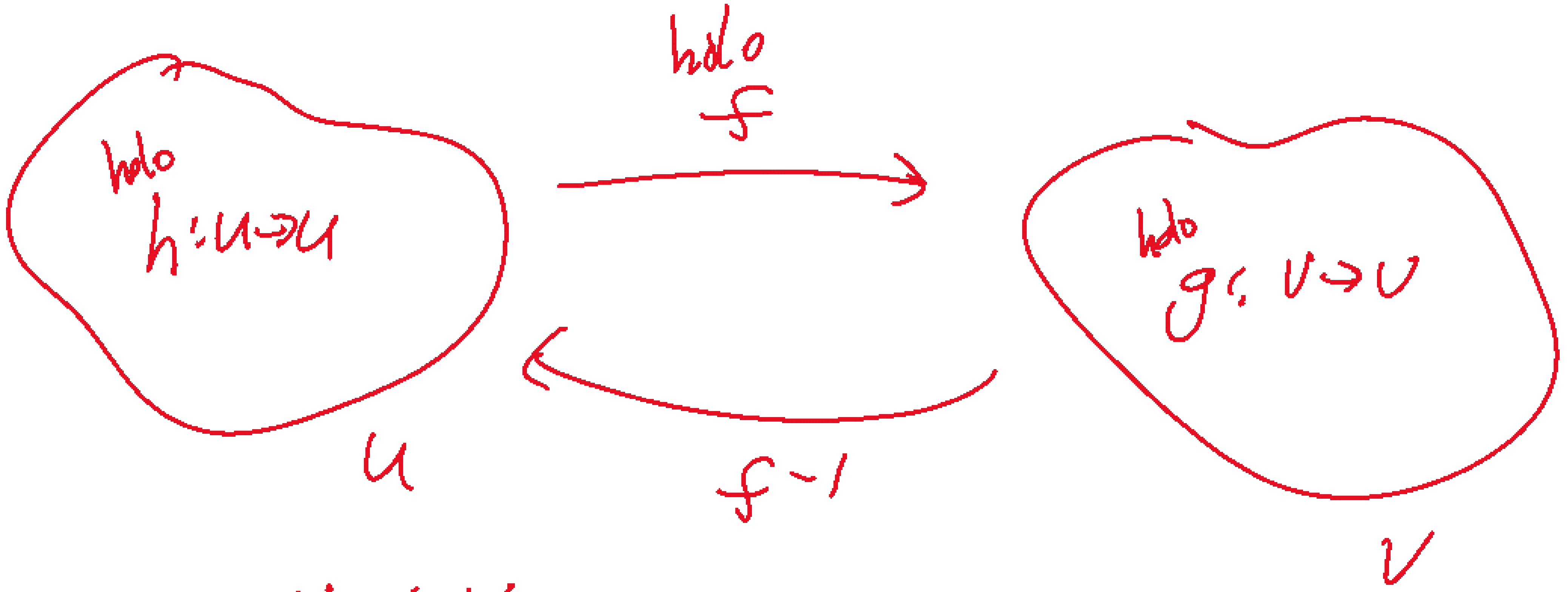
Given two open sets U and V in C, does there exist a holomorphic bijection between them?

Given an open subset Ω of C, what conditions on Ω guarantee that there exists a holomorphic bijection from Ω to D?

1-1 or injective means if $f(z_1) = f(z_2)$ then $z_1 = z_2$

onto or surjective means if $w \in$ "Range" then $\exists z$ st $f(z) = w$





$g : V \to V$

$g \circ f : U \to V$

$f^{-1} \circ g \circ f : U \to U$

<u>hope</u>: 1-1 correspondence

enough to study just one

Proposition 1.1 *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

First prove f'(z) is never zero, then prove its inverse is holomorphic.
Comment: Is this true if f is real analytic?

$f: (-1,1) \to (-1,1)$

f is real analytic and injective

$\hookrightarrow$ must $f'(x) \neq 0$?

$f(x) = x^3 = y$

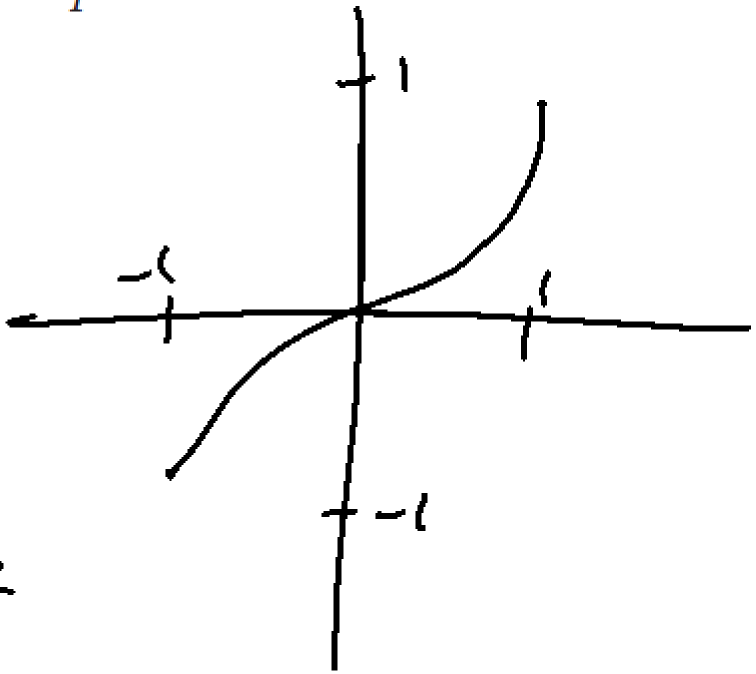
$g(x)$ st $f(g(x)) = x$

$g(x)^3 = x$ so $g(x) = x^{1/3}$

If $x^3 = y$ then $x = y^{1/3}$

$g'(x) = \frac{1}{3} X^{-2/3}$ not diff at $X = 0$

$f(x) = X^3$



$f'(x) = 3X^2$

$f'(0) = 0$

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https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 21: 10-25-23: https://youtu.be/C9JCkmjVrEg

•Conformal Maps, Automorphisms of the unit disk

Lecture from 2021: Lecture 17: 10/25/21: Schwarz Lemma, Automorphisms of the Disk: https://youtu.be/q4eZRrVPGA0 (slides)

Proposition 1.1 *If* $f : U \to V$ *is holomorphic and injective, then* $f'(z) \neq 0$ *for all* $z \in U$*. In particular, the inverse of* f *defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

First prove f'(z) is never zero, then prove its inverse is holomorphic.
Comment: Is this true if f is real analytic?

$f: (-1,1) \to (-1,1)$

f is real analytic and injective

$\hookrightarrow$ must $f'(x) \neq 0$?

$f(x) = x^3 = y$

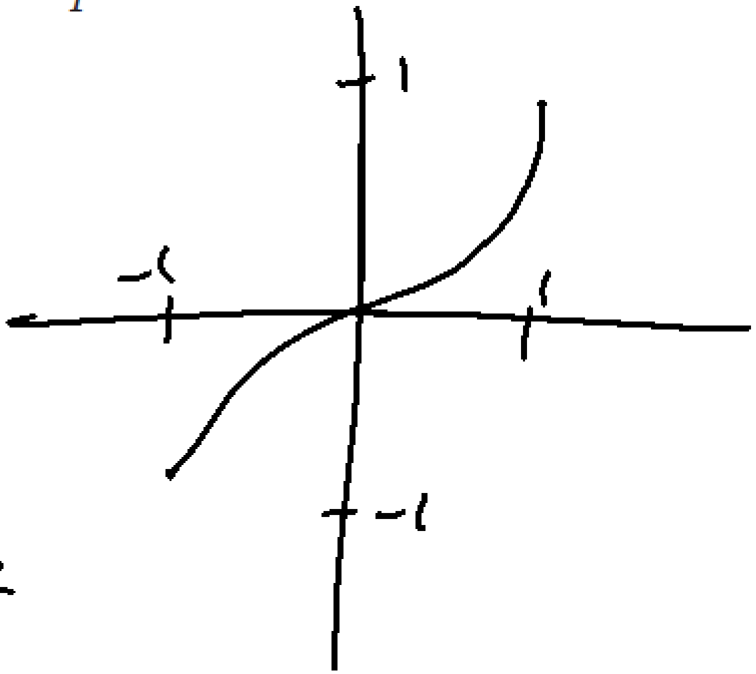
$g(x)$ st $f(g(x)) = x$

$g(x)^3 = x$ so $g(x) = x^{1/3}$

If $x^3 = y$ then $x = y^{1/3}$

$g'(x) = \frac{1}{3} x^{-2/3}$ not diff at $x = 0$

$f(x) = x^3$



$f'(x) = 3x^2$

$f'(0) = 0$

Proposition 1.1 *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

then f and $f + g$ have the same number of zeros inside the circle C.

Proof. We argue by contradiction, and suppose that $f'(z_0) = 0$ for some $z_0 \in U$. Then

$$f(z) \cancel{- f(z_0)} = a(z - z_0)^k + G(z) \quad \text{for all } z \text{ near } z_0,$$

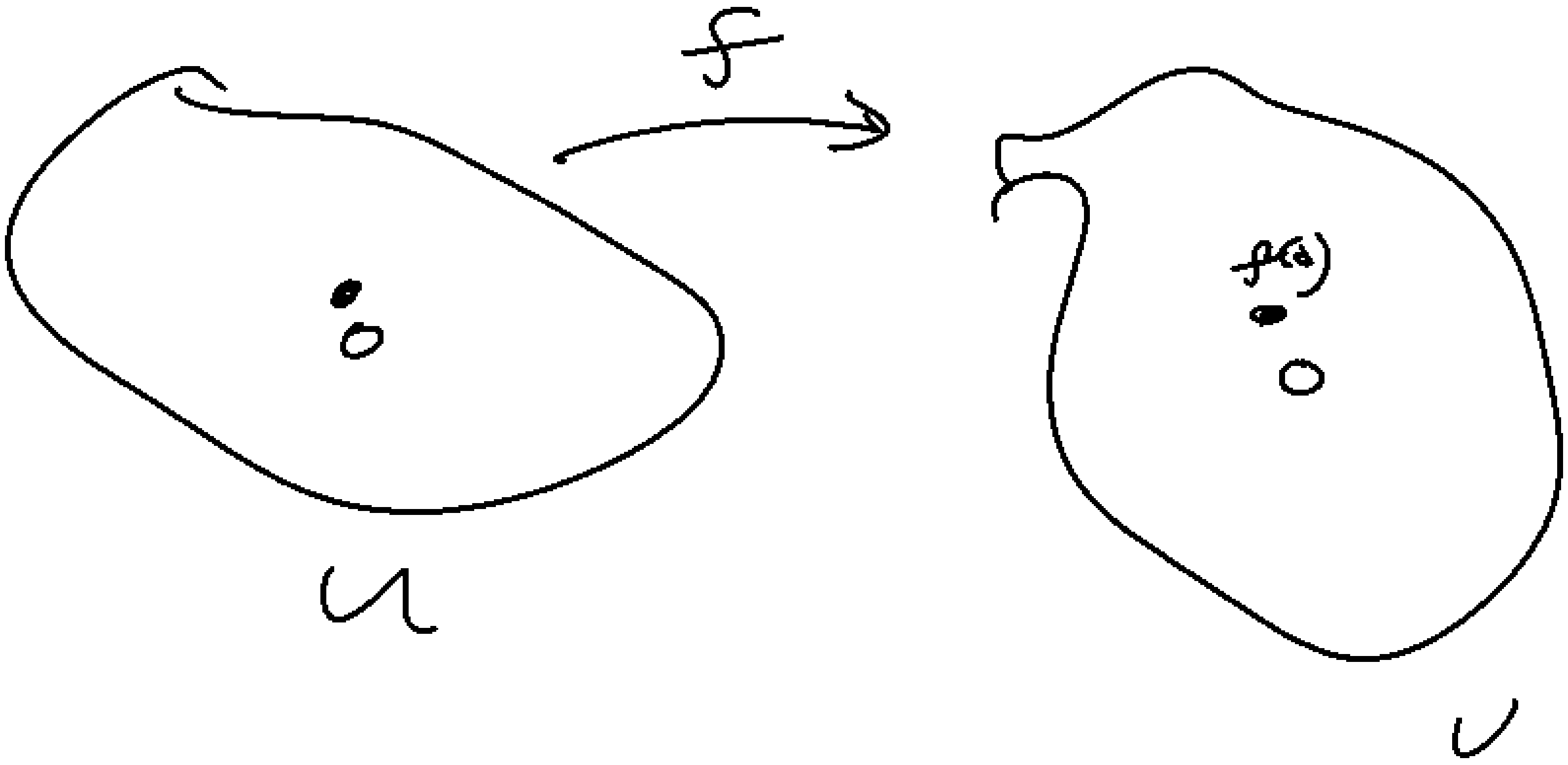
with $a \neq 0$, $k \geq 2$ and G vanishing to order $k+1$ at z_0. For sufficiently small w, we write

$$f(z) \cancel{- f(z_0)} - w = F(z) + G(z), \quad \text{where } F(z) = a(z - z_0)^k - w.$$

Since $|G(z)| < |F(z)|$ on a small circle centered at z_0, and F has at least two zeros inside that circle, Rouché's theorem implies that $f(z) \cancel{- f(z_0)} - w$ has at least two zeros there. Since $f'(z) \neq 0$ for all $z \neq z_0$ but sufficiently close to z_0 it follows that the roots of $f(z) \cancel{- f(z_0)} - w$ are distinct, hence f is not injective, a contradiction.

roots of [illegible]

WLOG adjust st $z_0 = 0$, $f(z_0) = f(0) = 0$



$f(z) = az^2$ f is holo and 1-1 near 0 ...

Clearly not 1-1 near $z=0$

also if $f(z) = az^k$ for $k \geq 3$ problem! $\varepsilon e^{2\pi i/k}$

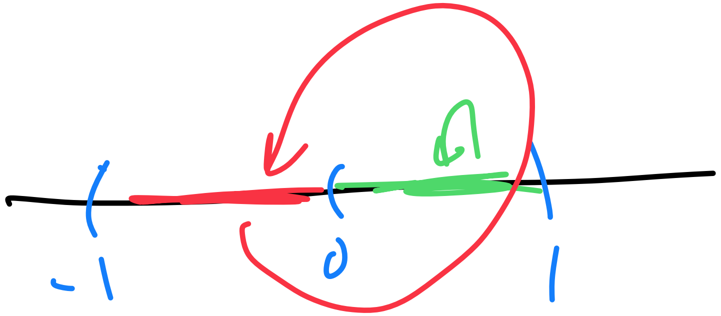
Real vs complex maps: f(x) = x³ on [-1,1] and unit disk

$f(x) = x^3$

$[-1,1] \to [-1,1]$

1-1

onto



$re^{i\theta}$ $\quad 0 \le r \le 1$

$\theta = \pi$

$z \to z^3$

$re^{i\theta} \to r^3 e^{i3\theta}$

as $\theta = \pi$ $\quad e^{i3\pi} = e^{i\pi} = -1$

$f(z) = a_n z^n + \dots$

Issues if $n \geq 2$

wouldn't be injective

Proof Sketch:

if $a_n z^n$ clearly not injective if $n \geq 2$

adding $a_{n+1} z^{n+1} + \dots$ very small if $|z|$ small

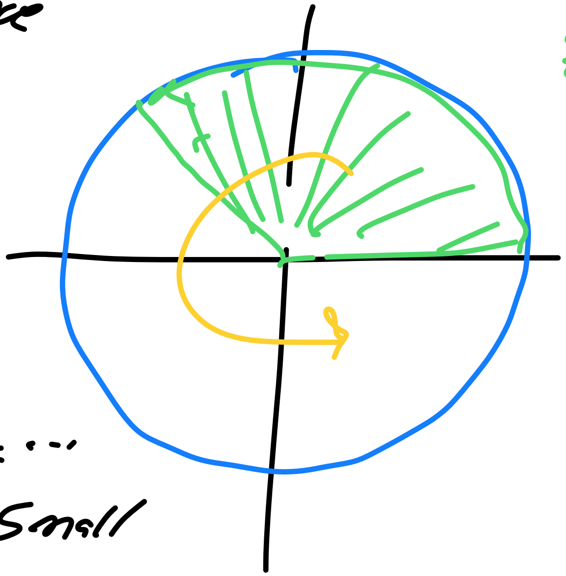
use Rouche

$f(z) = z^3$

unit disk → unit disk

1-1: no $\quad$ 3:1 save origin

onto: yes



$z = re^{i\theta}$

$0 \le r < 1$

$0 \le \theta < 2\pi/3$

Below is the rest of the proof of the theorem, the differentiability of the inverse.

The proof is standard, following from the definition and the fact that the derivative of f is never zero.

Note this is different than the real case, where we can have a real analytic bijection whose derivative vanishes at a point, namely f(x) = x^3.

Now let $g = f^{-1}$ denote the inverse of f on its range, which we can assume is V. Suppose $w_0 \in V$ and w is close to w_0. Write $w = f(z)$ and $w_0 = f(z_0)$. If $w \neq w_0$, we have

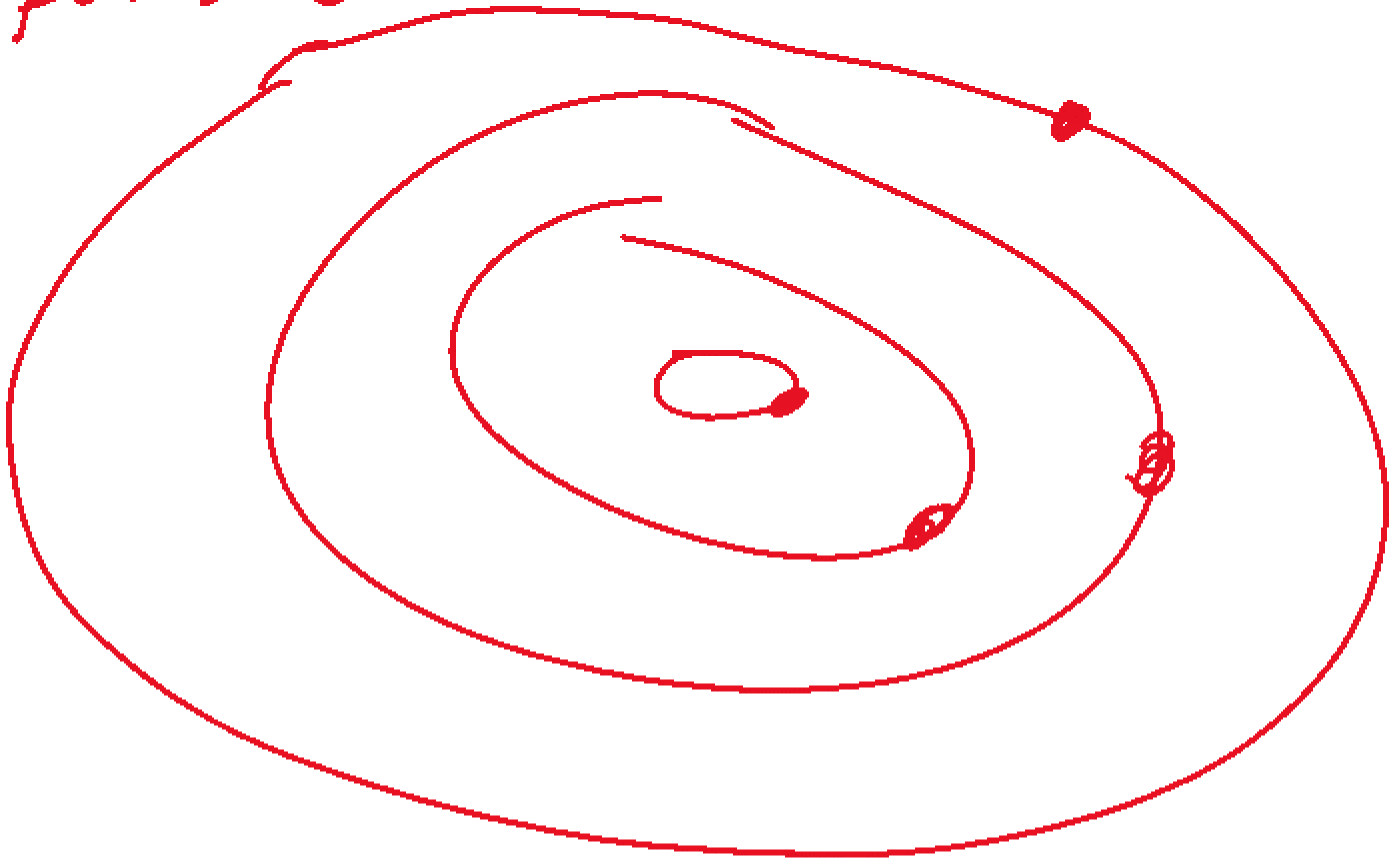
$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w - w_0}{g(w) - g(w_0)}} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}.$$

Since $f'(z_0) \neq 0$, we may let $z \to z_0$ and conclude that g is holomorphic at w_0 with $g'(w_0) = 1/f'(g(w_0))$.

$f'(0) = 0$ by assumption

Claim $f'(z) \neq 0$ if z near 0 but not 0

↳ by points of accumulation



$$F(z) = \frac{i-z}{i+z} \quad \text{and} \quad G(w) = i\frac{1-w}{1+w}.$$

Theorem 1.2 *The map* $F : \mathbb{H} \to \mathbb{D}$ *is a conformal map with inverse* $G : \mathbb{D} \to \mathbb{H}$.

$\{z = x+iy : y>0\}$ $\{z : |z|<1\}$

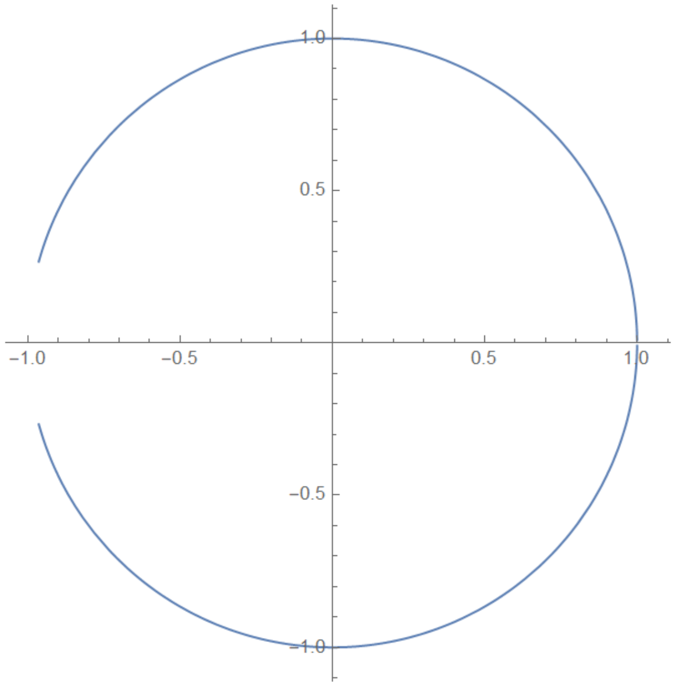
```
F[z_] := (I - z) / (I + z)
f[z_] := {Re[F[z]], Im[F[z]]};
```

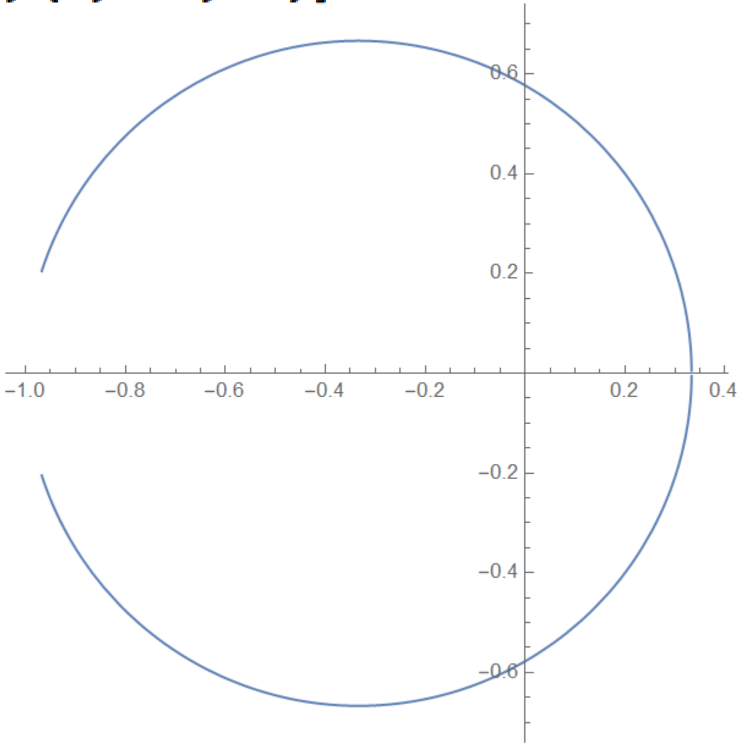
```
Manipulate[ParametricPlot[f[t], {t, -c, c}], {c, .01, 10}]
Manipulate[ParametricPlot[f[t + .5 I], {t, -c, c}], {c, .01, 10}]
```

$$F(t) = \frac{i-t}{i+t}\cdot\frac{-i+t}{-i+t}$$

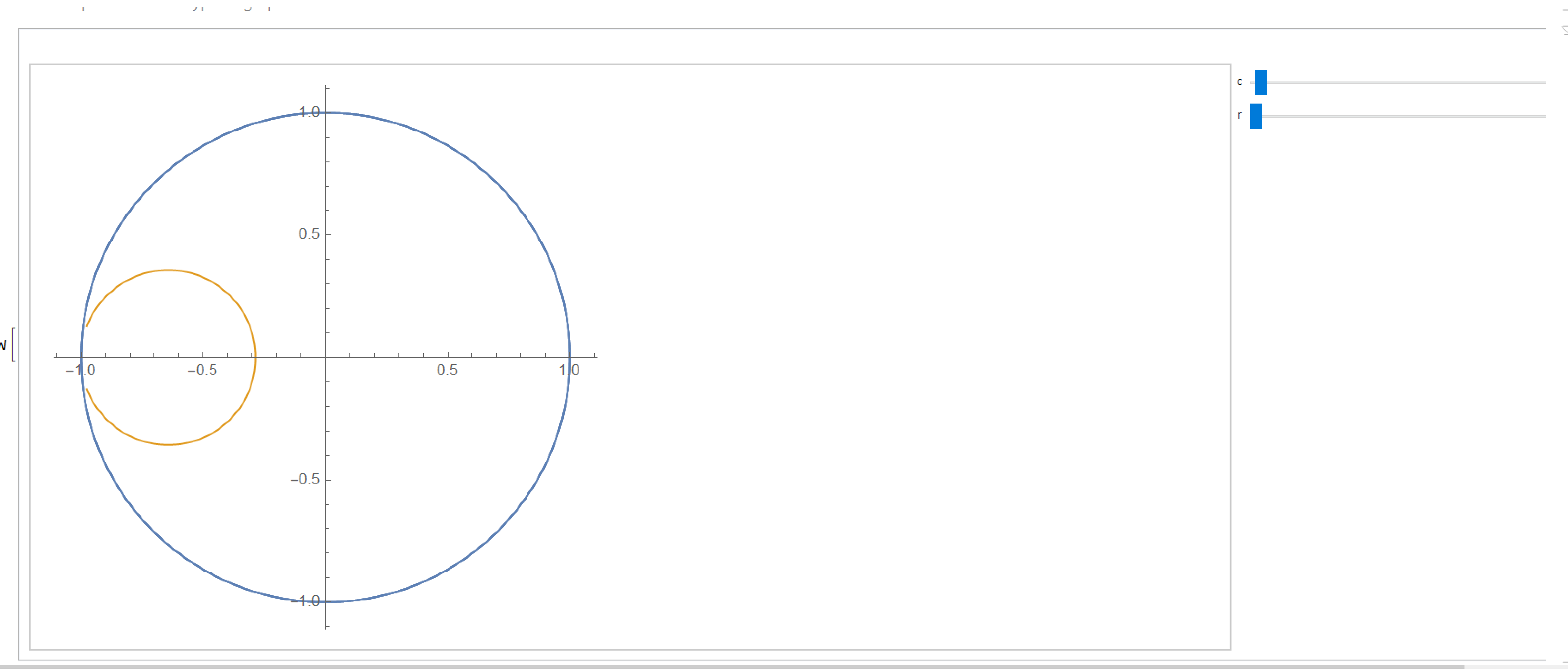
$$= \frac{(1-t^2)+2it}{t^2+1}$$

$$\Rightarrow \left(\frac{1-t^2}{1+t^2}, \frac{2t}{t^2+1}\right)$$





$$\text{Show}[\text{Manipulate}[\text{ParametricPlot}[\{\{r\ \text{Cos}[t], r\ \text{Sin}[t]\}, f[t + c\ I]\}, \{t, -15, 15\}], \{c, 0, 100\}, \{r, 1, 2\}]]$$



$$F(z) = \frac{i-z}{i+z}$$

$w \in \mathbb{D}$

Map horizontal lines to circles?

Solve $F(z) = w$

$$\frac{i-z}{i+z} = w$$

$$(i-z) = w(i+z)$$

$$i - wi = wz + z$$

$$z = \frac{i - wi}{1+w}$$

is This in $\mathbb{H}$?

Show imaginary part is positive

Fractional Linear Transformations

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad z \in \mathbb{C}$ $\quad$ <u>define</u> $\quad \gamma z = \dfrac{az+b}{cz+d}$ $\quad$ If $z \neq -d/c$ and $c, d \neq 0$

$\gamma_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \Rightarrow (\gamma_1 \gamma_2) z \overset{?}{=} \gamma_1 (\gamma_2 z)$ probably not equal $\gamma_2(\gamma_1 z) \overset{?}{=} (\gamma_2 \gamma_1) z$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \longleftrightarrow \frac{az+b}{cz+d}$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} \downarrow \frac{z}{1}$$

Matrix Form of FLT....

$$f_{abcd}(z) = \frac{az+b}{cz+d}$$ think about $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

rescale a, b, c, d by any non-zero number

so, wlog, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$, is 1 or 0

$$f_{abcd}\left(f_{ABCD}(z)\right) =$$

$$= f_{\alpha\beta\gamma\delta}(z)$$

$$\frac{(a_1a_2+b_2c_1)z+(a_2b_1+b_2d_1)}{(a_1c_2+c_1d_2)z+(b_1c_2+d_1d_2)}$$

```
f[z_, a_, b_, c_, d_] := (a z + b) / (c z + d)
Simplify[f[f[z, a1, b1, c1, d1], a2, b2, c2, d2]]
Simplify[f[f[z, a2, b2, c2, d2], a1, b1, c1, d1]]
Simplify[Simplify[f[f[1/2, a1, b1, c1, d1], a2, b2, c2, d2]] -
  Simplify[f[f[1/2, a2, b2, c2, d2], a1, b1, c1, d1]]]
```

$$\frac{a2\,b1 + b2\,d1 + a1\,a2\,z + b2\,c1\,z}{b1\,c2 + d1\,d2 + a1\,c2\,z + c1\,d2\,z}$$

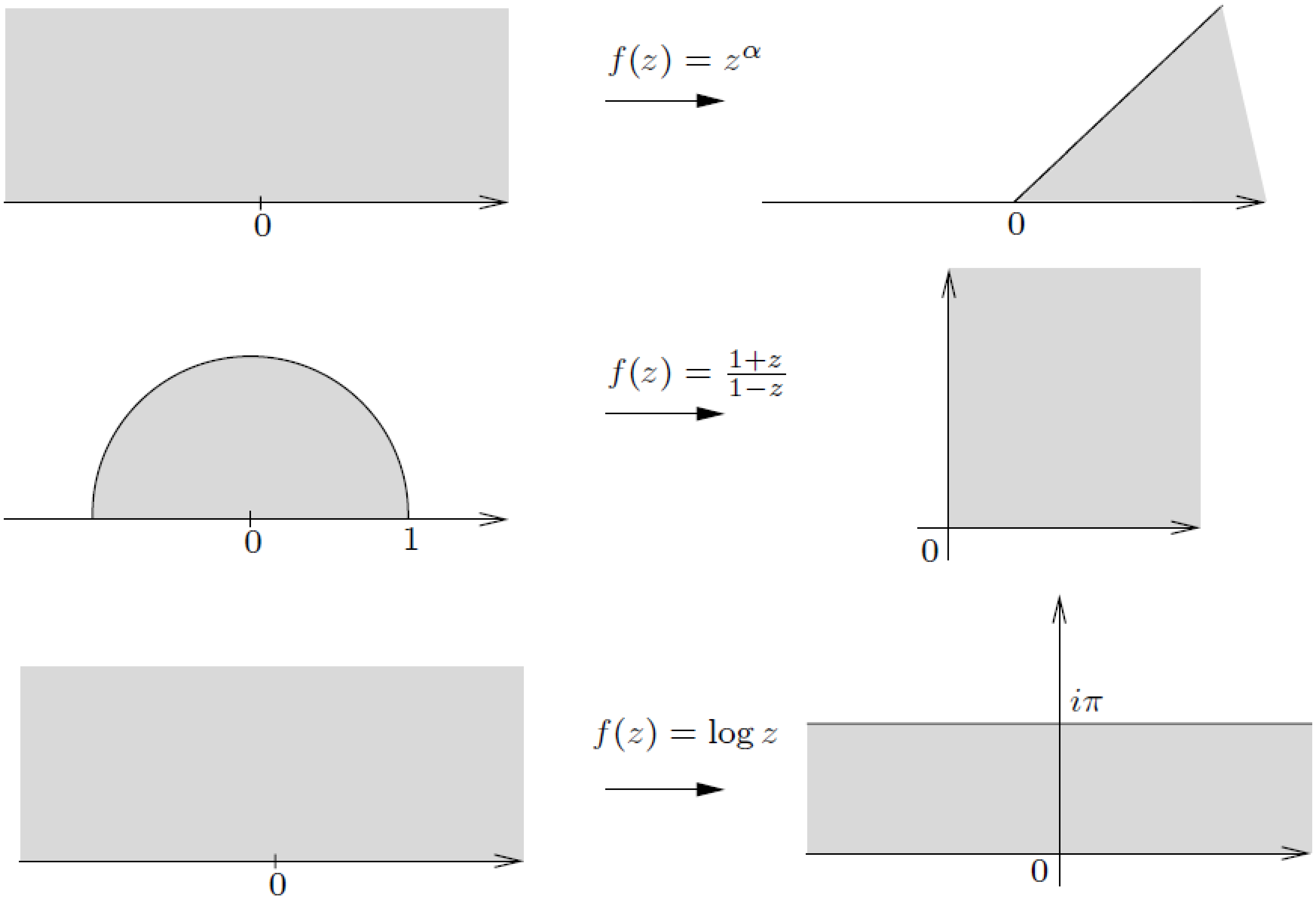
$$\frac{a1\,b2 + b1\,d2 + a1\,a2\,z + b1\,c2\,z}{b2\,c1 + d1\,d2 + a2\,c1\,z + c2\,d1\,z}$$

$$-\frac{a1\,a2 + 2\,a1\,b2 + b1\,c2 + 2\,b1\,d2}{a2\,c1 + 2\,b2\,c1 + c2\,d1 + 2\,d1\,d2} + \frac{a1\,a2 + 2\,a2\,b1 + b2\,c1 + 2\,b2\,d1}{a1\,c2 + 2\,b1\,c2 + c1\,d2 + 2\,d1\,d2}$$

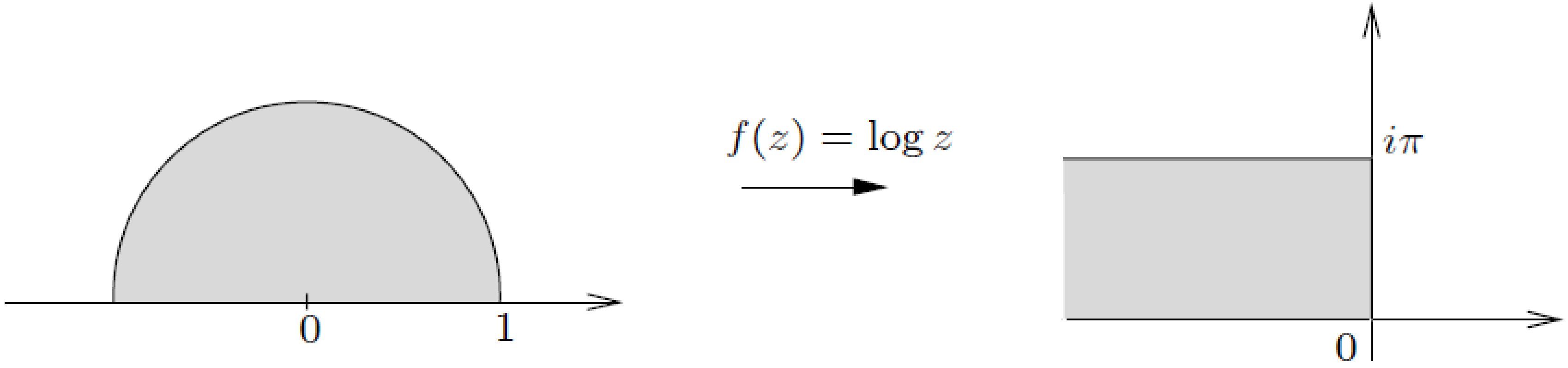
```
Simplify[f[f[z, a1, b1, c1, d1], a2, b2, c2, d2]]
```

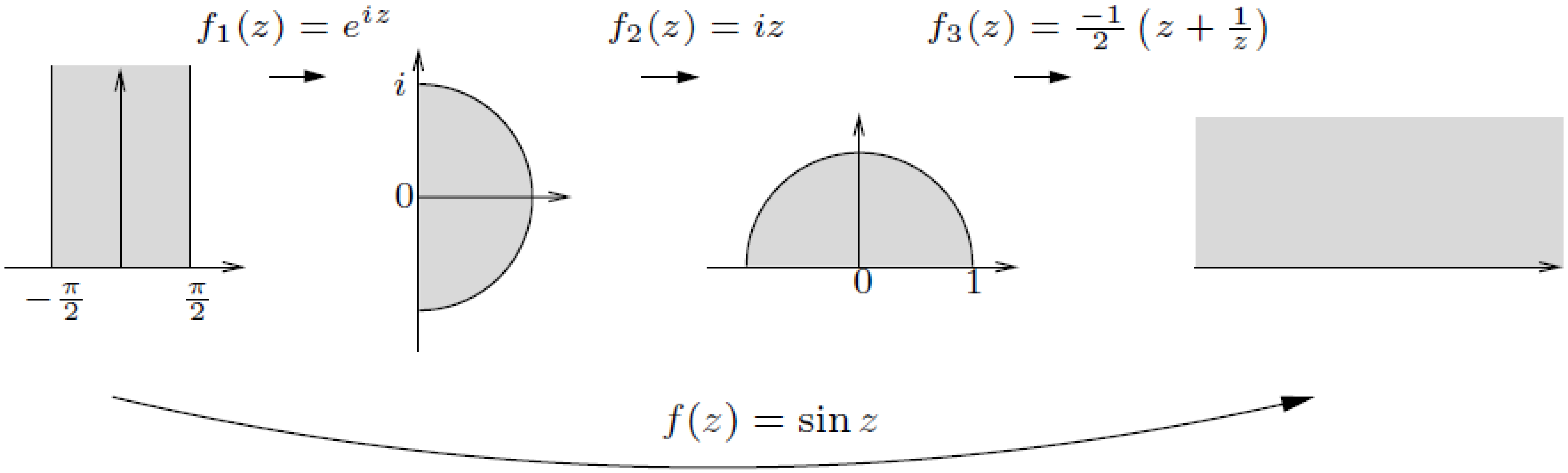
$$\frac{a2\,b1 + b2\,d1 + a1\,a2\,z + b2\,c1\,z}{b1\,c2 + d1\,d2 + a1\,c2\,z + c1\,d2\,z}$$

FLT!



$f(z) = z^{\alpha}$
0
0
$f(z) = \frac{1+z}{1-z}$
0
1
0
$f(z) = \log z$
$i\pi$
0
0


$f(z) = \log z$
0
1
$i\pi$
0


$f_1(z) = e^{iz}$
$f_2(z) = iz$
$f_3(z) = \frac{-1}{2}\left(z + \frac{1}{z}\right)$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
i
0
0
1
$f(z) = \sin z$

$$F(z) = \frac{i - z}{i + z} \quad \text{and} \quad G(w) = i\frac{1 - w}{1 + w}.$$

Theorem 1.2 *The map* $F : \mathbb{H} \to \mathbb{D}$ *is a conformal map with inverse* $G : \mathbb{D} \to \mathbb{H}$.

$\{z = x+iy : y>0\}$ $\{z : |z|<1\}$

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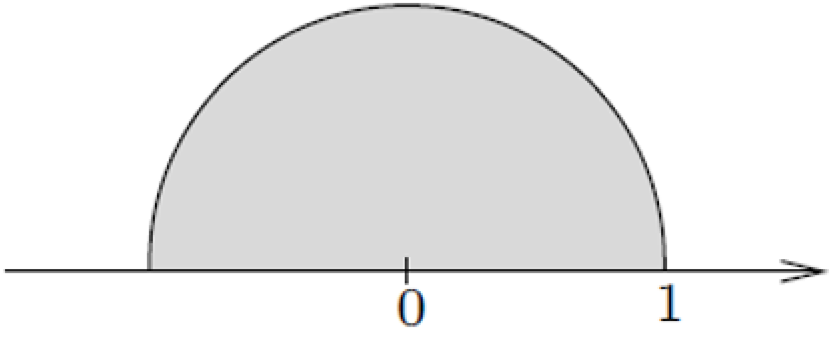
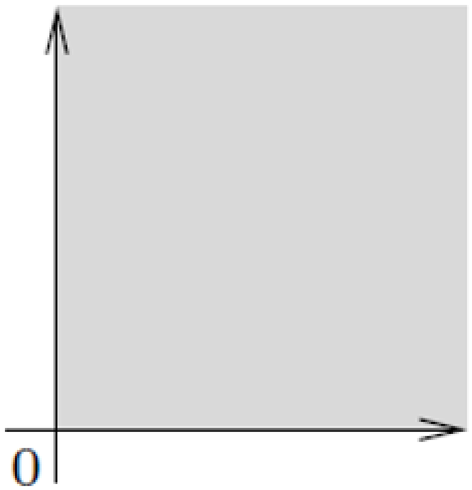
F[z_] := (I - z) / (I + z)
f[z_] := {Re[F[z]], Im[F[z]]};
```

```
Manipulate[ParametricPlot[f[t], {t, -c, c}], {c, .01, 10}]
Manipulate[ParametricPlot[f[t + .5 I], {t, -c, c}], {c, .01, 10}]
```



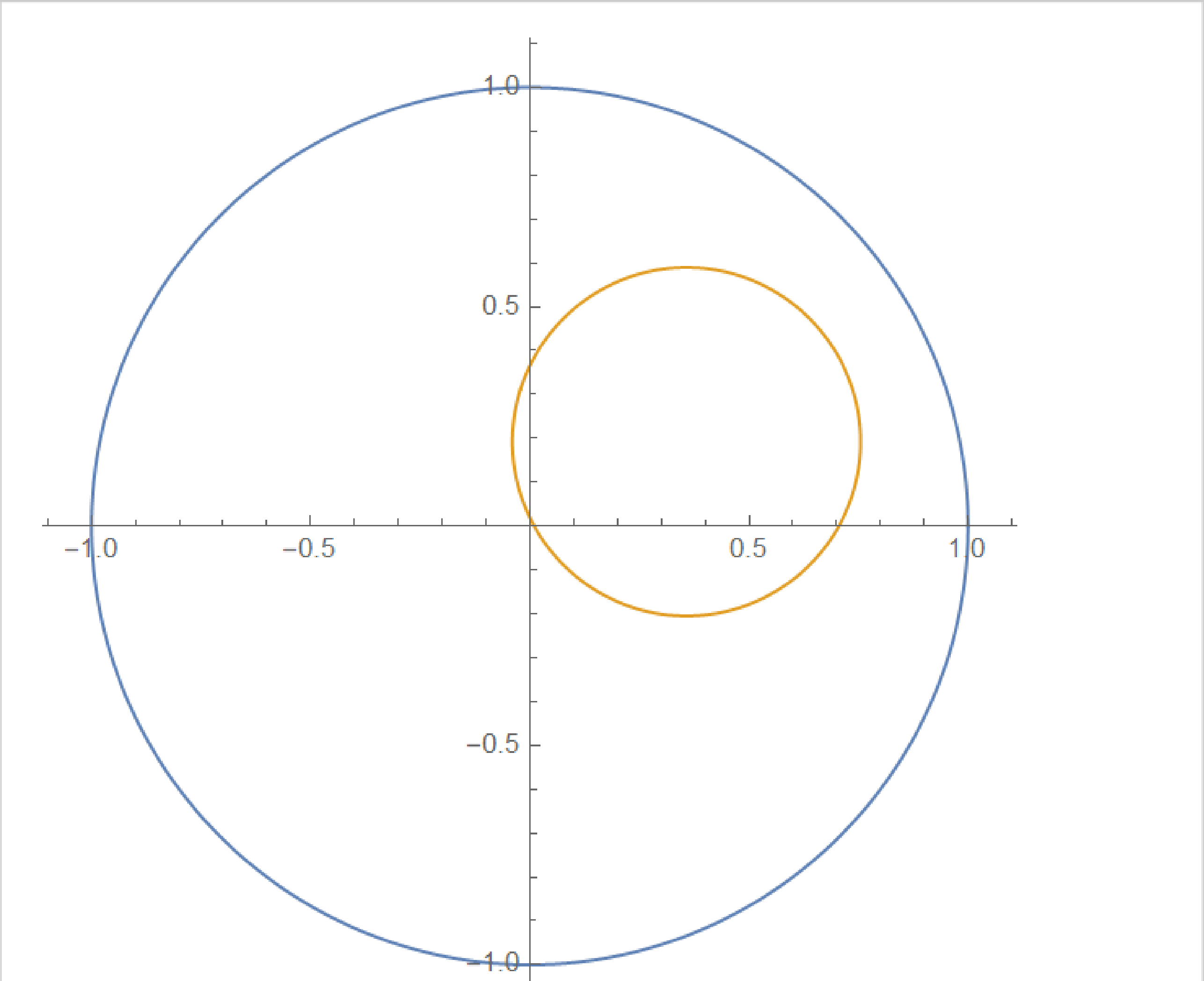
$$\xrightarrow{f(z) = \frac{1+z}{1-z}}$$

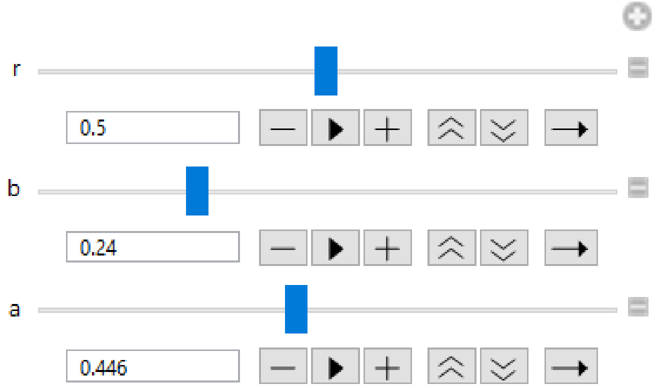


```
f[z_, alpha_] := (alpha - z) / (1 - Conjugate[alpha] z);
ref[r_, theta_, alpha_] := Re[f[r Exp[I theta], alpha]]
imf[r_, theta_, alpha_] := Im[f[r Exp[I theta], alpha]]
Manipulate[ParametricPlot[{{Cos[t], Sin[t]}, {ref[r, t, a + I b], imf[r, t, a + I b]}},
  {t, 0, 2 Pi}], {r, 0, 1}, {b, 0, Sqrt[1 - a^2]}, {a, 0, 1}]
```

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

$$\psi_\alpha(0) = \alpha \qquad \text{and} \qquad \psi_\alpha(\alpha) = 0.$$





$\psi_\alpha(0) = \alpha$

$\psi_\alpha(\alpha) = 0$

$\psi_\alpha(\psi_\alpha(0)) = 0$

$\psi_\alpha(\psi_\alpha(\alpha)) = \alpha$

$\Rightarrow \psi_\alpha = \psi_\alpha^{-1}$

$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$

$\frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$

$\frac{4\cancel{9}}{\cancel{9}8} = \frac{1}{2}$

$\frac{1\cancel{2}}{\cancel{2}4} = \frac{1}{4}$

$$\psi_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha}z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

$z = e^{i\theta}$

$$\psi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \overline{\alpha})} = e^{-i\theta}\frac{w}{\overline{w}}(-1)$$

$w = \alpha - e^{i\theta}$
$|w| \neq 0$

$$= \frac{w}{e^{i\theta}(-\overline{w})}$$

$$|\psi_\alpha(e^{i\theta})| = \left|\frac{w}{e^{i\theta}(-\overline{w})}\right| = 1$$

By Maximum Modulus know

$\psi_\alpha(z) \in \mathbb{D}$ if $|z| < 1$

as cant attain max interiorly

$$\psi_\alpha(0) = \alpha \quad \text{and} \quad \psi_\alpha(\alpha) = 0.$$

$$\begin{aligned}
(\psi_\alpha \circ \psi_\alpha)(z) &= \frac{\alpha - \frac{\alpha - z}{1-\overline{\alpha}z}}{1 - \overline{\alpha}\frac{\alpha - z}{1-\overline{\alpha}z}} \\
&= \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \overline{\alpha}z - |\alpha|^2 + \overline{\alpha}z} \\
&= \frac{(1-|\alpha|^2)z}{1-|\alpha|^2} \\
&= z,
\end{aligned}$$

Proof onto:

Given $w \in \mathbb{D}$ consider

$z = \psi_\alpha(w)$

Then $\psi_\alpha(z) = \psi_\alpha(\psi_\alpha(w))$

$= w$

Math 383: Complex Analysis: Fall '23 (Williams)

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Homepage:

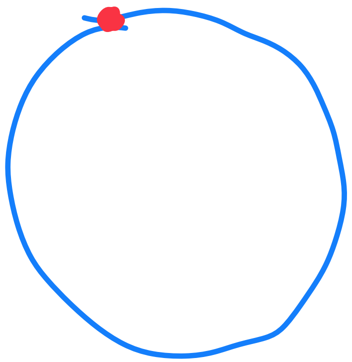
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

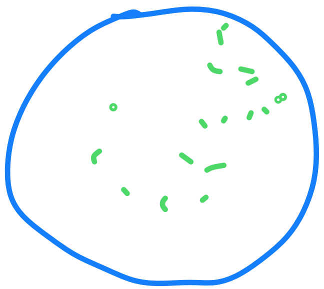
Lecture 22: 10-27-23: https://youtu.be/n43JVHQigBE

- Schwarz Lemma, Conformal Maps

Lecture from 2021: Lecture 17: 10/25/21: Schwarz Lemma, Automorphisms of the Disk: https://youtu.be/q4eZRrVPGA0 (slides)

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$





$f(\text{compact}) =$ compact set (f continuous)

Brute force: fix $w \in \partial\mathbb{D}$ find $z \in \partial\mathbb{D}$

st $\psi_\alpha(z) = w$

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

$\underbrace{\psi_b \circ \psi_a}_{\psi_c} : \mathbb{D} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{D}$

```
psi[z_, point_] := (point - z) / (1 - Conjugate[point] z)
Simplify[psi[psi[z, a], a]]
Simplify[psi[psi[z, a], b], Assumptions → {Abs[a] < 1 && Abs[b] < 1}]
psi[z, psi[b, a]]
(*Simplify[Numerator[-(1-a Conjugate[b]) psi[z,psi[a,b]]]] / Simplify[Denominator[-(1-a Conjugate[b]) psi[z,psi[a,b]]]]*)
g[z_, a_, b_] := Simplify[psi[psi[z, a], b] - psi[z, psi[b, a]], Assumptions → {Abs[a] < 1 && Abs[b] < 1}]
Abs[g[.2 - I / 11, .3 + I / 2, .4 + I / 5]]
```

z

$$\frac{a - b - z + b\, z \,\text{Conjugate}[a]}{-1 + z\, \text{Conjugate}[a] + (a - z)\, \text{Conjugate}[b]}$$

$$\frac{-z + \frac{a-b}{1-b\,\text{Conjugate}[a]}}{1 - \frac{z\,(\text{Conjugate}[a]-\text{Conjugate}[b])}{1-a\,\text{Conjugate}[b]}}$$

1.03072

$\psi_c : \mathbb{D} \to \mathbb{D}$

$0 \to c$

$c \to 0$

$(\psi_b \psi_a) : \mathbb{D} \to \mathbb{D}$

$a \to b$

$0 \to \psi_b(a)$

Only chance is $c = \psi_b(a)$

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

```
h[z_, a_, b_] := Abs[psi[psi[z, a], b] / psi[z, psi[b, a]]]
N[h[Exp[I Pi / 7] / 13, Exp[I Pi / 4] / 2, Exp[I Pi / 3] / 2]]
N[h[Exp[I Pi / 5] / 12, Exp[I Pi / 4] / 2, Exp[I Pi / 3] / 2]]
N[h[Exp[I Pi / Sqrt[2]] / 19, Exp[I Pi / 4] / 2, Exp[I Pi / 3] / 2]]
```

```
1.

1.

1.
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$$(\psi_b \circ \psi_a)(z) = e^{i\theta(a,b)} \psi_c(z)$$

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

Theorem 2.2 *If f is an automorphism of the disc, then there exist $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Sketch: $f: \mathbb{D} \to \mathbb{D}$ is an automorphism

$\exists\, a$ such that $f(a) = 0$.

$\psi_a(0) = a$

$f(\psi_a(0)) = f(a) = 0$

$(f \circ \psi_a)(0) = 0$

forces $f \circ \psi_a$ to be a rotation

The Schwarz lemma

Lemma 2.1 *Let* $f : \mathbb{D} \to \mathbb{D}$ *be holomorphic with* $f(0) = 0$. *Then*

(i) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.

(ii) *If for some* $z_0 \neq 0$ *we have* $|f(z_0)| = |z_0|$, *then* f *is a rotation.*

(iii) $|f'(0)| \leq 1$, *and if equality holds, then* f *is a rotation.*

(i) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.

Expand in a power series, study f(z)/z, look at in D(r)

$z \in D_0(r)$ so $|z| \leq r$ study $g(z) = \dfrac{f(z)}{z}$ ok at $z=0$ as $f(0)=0$, holomorphic!

$|g(z)| = \left|\dfrac{f(z)}{z}\right| \leq \dfrac{1}{|z|}$ as $f: \mathbb{D} \to \mathbb{D}$, $|f(z)| \leq 1$

$|g(z)| \leq \dfrac{1}{r}$ for all $|z| < r$ (Max Modulus)

take limit as $r \to 1$, get $|g(z)| \leq 1$

means $|f(z)| \leq |z|$

(ii) *If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then f is a rotation.*

For (ii), we see that $f(z)/z$ attains its maximum in the interior of $\mathbb{D}$ and must therefore be constant, say $f(z) = cz$. Evaluating this expression at z_0 and taking absolute values, we find that $|c| = 1$. Therefore, there exists $\theta \in \mathbb{R}$ such that $c = e^{i\theta}$, and that explains why f is a rotation.

$g(z) := \dfrac{f(z)}{z}$ know $|g(z)| \leq 1$

If $\exists\, z_0$ st $|f(z_0)| = |z_0| \Rightarrow |g(z_0)| = 1 \Rightarrow g$ constant by max modulus

$g(z) = c \Rightarrow f(z) = cz$

as $|f(z_0)| = |z_0| \Rightarrow |c| = 1$ or $c = e^{i\theta}$

(iii) $|f'(0)| \leq 1$, *and if equality holds, then f is a rotation.*

View g(z) = f(z)/z as a derivative at z=0….

Finally, observe that if $g(z) = f(z)/z$, then $|g(z)| \leq 1$ throughout $\mathbb{D}$, and moreover

$$g(0) = \lim_{z \to 0} \frac{f(z) - \underbrace{f(0)}_{=0}}{z - 0} = f'(0).$$

as $|g(z)| \leq 1$

know $|f'(0)| \leq 1$

Hence, if $|f'(0)| = 1$, then $|g(0)| = 1$, and by the maximum principle g is constant, which implies $f(z) = cz$ with $|c| = 1$.

The `Real' Schwarz lemma (w' David Thompson): American Mathematical Monthly. (**118** (October 2011), no. 8, page 725) pdf

https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/realschwarz10.pdf

Lemma 2.1 *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then*

(i) $|f(z)| \le |z|$ *for all* $z \in \mathbb{D}$.

(ii) *If for some* $z_0 \neq 0$ *we have* $|f(z_0)| = |z_0|$, *then* f *is a rotation.*

(iii) $|f'(0)| \le 1$, *and if equality holds, then* f *is a rotation.*

$f : (-1,1) \to (-1,1)$ real analytic fn, automorphism

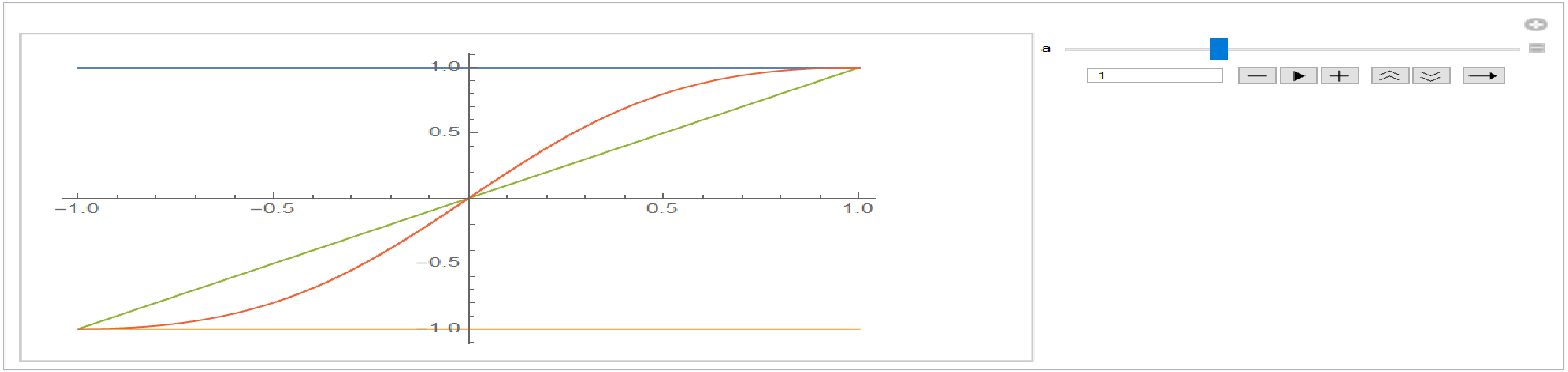
$f(0)=0$

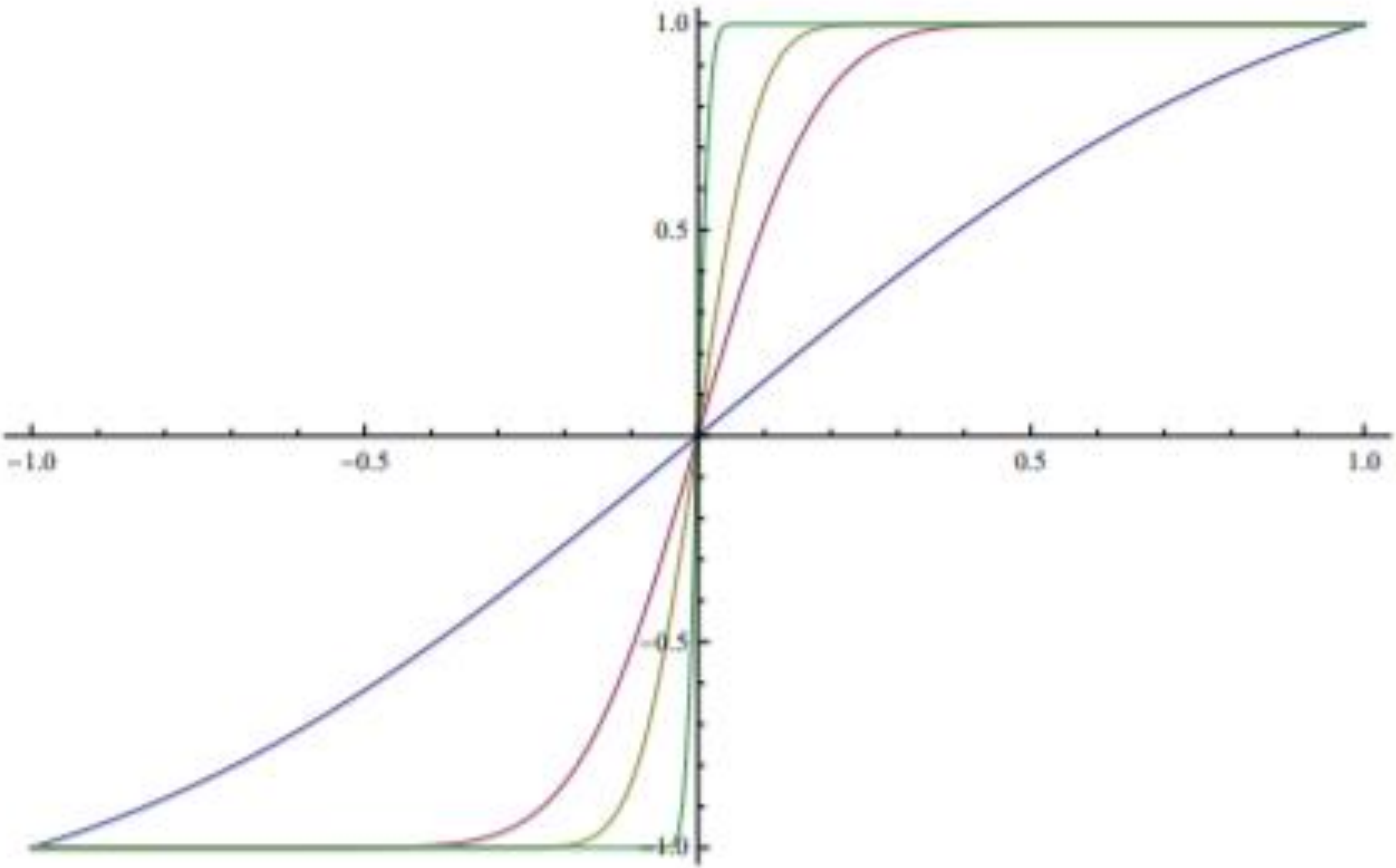
What is true about $|f'(0)|$?

It's interesting to consider the real analogue. In that situation, we're seeking a real analytic map g from $(-1, 1)$ to itself with $g(0) = 0$ and derivative $g'(0)$ as large as possible. After a little exploration, we quickly find two functions with derivative greater than 1 at the origin. The first is $g(x) = \sin(\pi x/2)$, which has $g'(0) = \pi/2 \in (1, 2)$. The second is actually an infinite family: letting $g_a(x) = (a + 1)x/(1 + ax^2)$ we see that g_a is real analytic on $(-1, 1)$ so long as $|a| \leq 1$, and $g_a'(0) = 1 + a$. Using this example, we see we can get the derivative as large as 2 at the origin. Unfortunately, if we take $|a| > 1$ then g_a is no longer a map from $(-1, 1)$ to itself; for example, $g_{1.01}(.995) > 1.00001$.

```
g[x_, a_] := (a + 1) x / (1 + a x^2)
Simplify[D[g[x, a], x]]
Manipulate[Plot[{1, -1, x, g[x, a]}, {x, -1, 1}], {a, 0, 3}]
```

$$-\frac{(1 + a)\ (-1 + a\ x^2)}{(1 + a\ x^2)^2}$$

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-0.5
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THE 'REAL' SCHWARZ LEMMA

STEVEN J. MILLER AND DAVID A. THOMPSON

ABSTRACT. The purpose of this note is to discuss the real analogue of the Schwarz lemma from complex analysis. We give two versions of a potential article; one is written to be a short note, while the other is written to be a box. We have tried to make the note and box versions as short as possible, but of course would be happy to add (or delete) details / images if that is desirable. We prefer the note version, as it gives us a chance to tell more of the story / set the stage.

1. NOTE VERSION

One of the most common themes in any complex analysis course is how different functions of a complex variable are from functions of a real variable. The differences can be striking, ranging from the fact that any function which is complex differentiable once must be complex differentiable infinitely often *and* further must equal its Taylor series, to the fact that any complex differentiable function which is bounded must be constant. Both statements fail in the real case; for the first consider $x^3\sin(1/x)$ while for the second just consider $\sin x$. In this note we explore the differences between the real and complex cases of the Schwarz lemma:

The Schwarz Lemma: *If f is a holomorphic map of the unit disk to itself that fixes the origin, then $|f'(0)| \le 1$; further, if $|f'(0)| = 1$ then f is an automorphism (in fact, a rotation).*

What this means is that we cannot have f locally expanding near the origin in the unit disk faster than the identity function, even if we were willing to pay for this by having f contracting a bit near the boundary. The largest possible value for the derivative at the origin of such an automorphism is 1. This result can be found in every good complex analysis book (see for example [Al, La, SS]), and serves as one of the key ingredients in the proof of the Riemann Mapping Theorem. For more information about the lemma and its applications, see the recent article in the Monthly by Harold Boas [Bo].

It's interesting to consider the real analogue. In that situation, we're seeking a real analytic map g from $(-1,1)$ to itself with $g(0) = 0$ and derivative $g'(0)$ as large as possible. After a little exploration, we quickly find two functions with derivative greater than 1 at the origin. The first is $g(x) = \sin(\pi x/2)$, which has $g'(0) = \pi/2 \in (1,2)$. The second is actually an infinite family: letting $g_a(x) = (a+1)x/(1+ax^2)$ we see that g_a is real analytic on $(-1,1)$ so long as $|a| \le 1$, and $g_a'(0) = 1 + a$. Using this example, we see we can get the derivative as large as 2 at the origin. Unfortunately, if we take $|a| > 1$ then g_a is no longer a map from $(-1,1)$ to itself; for example, $g_{1.01}(.995) > 1.00001$.

Notice both examples fail if we try to extend these automorphisms to maps on the unit disk. For example, when $z = 3i/5$ then already $\sin(\pi z/2)$ has absolute value exceeding 1, and thus we would not have an automorphism of the disk. For the family g_a, without loss of generality take $a > 0$. As $z \to i$ then $g_a(z) \to \frac{1+a}{1-a}i$, which is outside the unit disk if $a > 0$.

While it is easy to generalize our family $\{g_a\}$ to get a larger derivative at 0, unfortunately all the examples we tried were no longer real analytic on the entire interval $(-1,1)$. As every holomorphic function is also analytic (which means it equals its Taylor series expansion), it seems only fair to

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We thank our classmates from Math 302: Complex Analysis (Williams College, Fall 2010) for many enlightening conversations, especially David Gold and Liyang Zhang, as well as Jonathan Sondow for comments on an earlier draft. The first named author was partially supported by NSF grant DMS0970067.

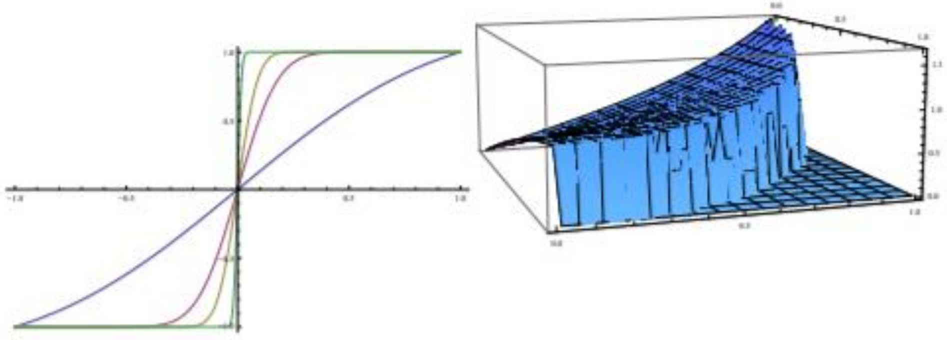

FIGURE 1. Plot of the scaled error functions. (1) Left: $\mathrm{Erf}(kx)/\mathrm{Erf}(x)$ for $k \in \{1, 5, 10, 50\}$ and $x \in (-1,1)$; (2) Right: Plot of $|\mathrm{Erf}(z)|$ for $|z| \le 1$.

require this property to hold in the real case as well. Interestingly, there is a family of real analytic automorphisms of the unit interval fixing the origin whose derivatives become arbitrarily large at 0. Consider $h_k(x) = \mathrm{erf}(kx)/\mathrm{erf}(k)$, where erf is the error function:

$$\mathrm{erf}(x) \;:=\; \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

We conclude with our main result, which is another example of the striking differences between functions of a real and functions of a complex variable.

The Real Analogue of the Schwarz Lemma: *Let $\mathcal{F}$ be the set of all real analytic automorphisms of $(-1,1)$ that fix the origin. Then $\sup_{f\in\mathcal{F}}|f'(0)| = \infty$; in other words, the first derivative at the origin can be made arbitrarily large by considering $f_k(x) = \mathrm{erf}(kx)/\mathrm{erf}(k)$.*

Proof: The error function has a series expansion converging for all complex z,

$$\mathrm{erf}(z) \;=\; \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)} \;=\; \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots\right)$$

(this follows by using the series expansion for the exponential function and interchanging the sum and the integral), and is simply twice the area under a normal distribution with mean 0 and variance 1/2 from 0 to x. From its definition, we see $\mathrm{erf}(-x) = -\mathrm{erf}(x)$, the error function is one-to-one, and for $x \in (-1,1)$ our function $\mathrm{erf}(kx)/\mathrm{erf}(k)$ is onto $(-1,1)$.

Using the Fundamental Theorem of Calculus, we see that $h_k'(x) = 2\exp(-k^2x^2)k)/\sqrt{\pi}\mathrm{erf}(k)$, and thus $h_k'(0) = 2k/\sqrt{\pi}\mathrm{erf}(k)$. As $\mathrm{erf}(k) \to 1$ as $k \to \infty$, we find $h_k'(0) \sim 2k/\sqrt{\pi} \to \infty$, which shows that, yet again, the real case behaves in a markedly different manner than the complex one. As h_k is an entire function with large derivative at 0, if we regard it as a map from the unit disk it must violate one of the conditions of the Schwarz lemma. From the series expansion of the error function, it's clear that if we take $z = iy$ then $h_k(iy)$ tends to infinity as $y \to 1$ and $k \to \infty$; thus h_k does not map the unit disk into itself, and cannot be a conformal automorphism (see Figure 1 for plots in the real and complex cases). □

2. Box Version

The Schwarz lemma states that if f is a holomorphic map of the unit disk to itself that fixes the origin, then $|f'(0)| \leq 1$; further, if $|f'(0)| = 1$ then f is an automorphism. It's interesting to consider the real analogue. In that situation, we're seeking a real analytic map g from $(-1,1)$ to itself that fixes the origin and has derivative $g'(0)$ as large as possible. After a little exploration, we find $h_k(x) = \operatorname{erf}(kx)/\operatorname{erf}(k)$, where erf is the error function:

$$\operatorname{erf}(x) \ := \ \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

The error function has a series expansion converging for all complex z,

$$\operatorname{erf}(z) \ = \ \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} \ = \ \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots \right),$$

and is simply twice the area under a normal distribution with mean 0 and variance 1/2 from 0 to x.

We have $h_k'(x) = 2\exp(-k^2x^2)k)/\sqrt{\pi}\operatorname{erf}(k)$, and thus $h_k'(0) = 2k/\sqrt{\pi}\operatorname{erf}(k)$. As $\operatorname{erf}(k) \to 1$ as $k \to \infty$, we see $h_k'(0) \sim 2k/\sqrt{\pi} \to \infty$, which shows that, yet again, the real case behaves in a markedly different manner than the complex one.

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Extra Credit: What other generalizations can we do?

$f : \mathbb{D} \to \mathbb{D}$ automorphism

$f(0) = 0$

$|f'(0)| \le 1$

Question: $|f''(0)|$?

Question: $\sup_{z \in \mathbb{D}} |f'(z)|$?

Question: max of $|f'(0)| + |f''(0)|$

$\frac{f'}{f}$

if $r < 1$

Question: $\frac{1}{2\pi i} \oint_{\partial D_{z_0}(r)} f(z)\, dz = 0$

$r > 1$ can blow up

Let Ω be an open subset of $\mathbb{C}$. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be **normal** if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω (the limit need not be in $\mathcal{F}$).

$f_n(z) = \frac{1}{n} z + \left(1 - \frac{1}{n}\right) z^2$

$f(z) = z$ as $n \to \infty$, $f_n(z) \to z^2$

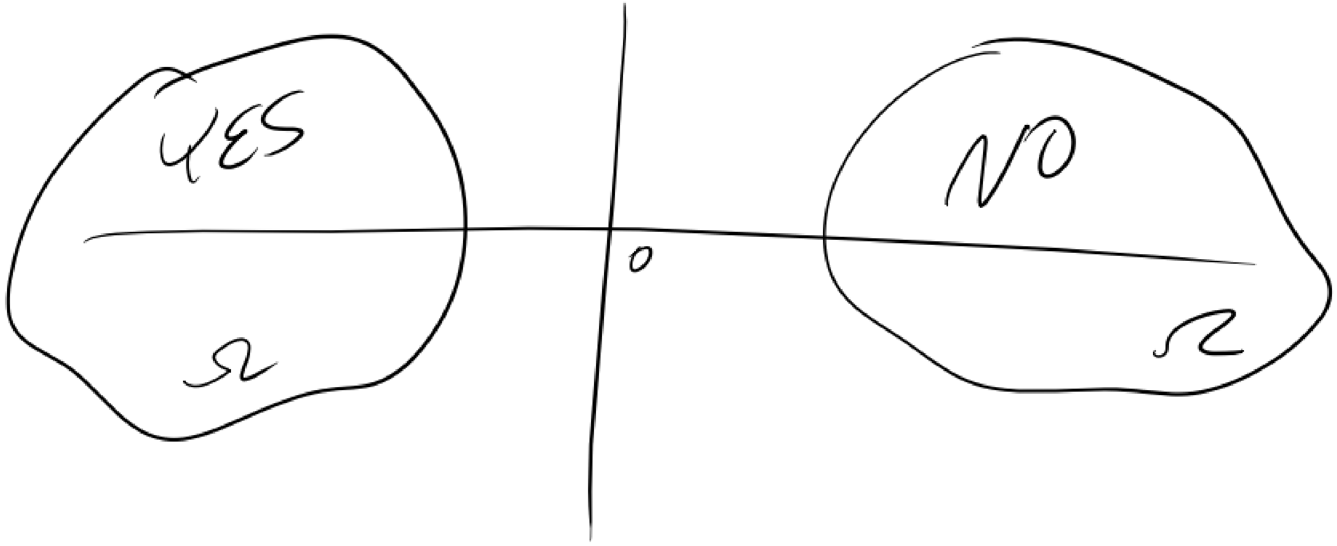
Uniform Continuity: $\forall \varepsilon > 0 \ \exists \delta$ st $\forall x, y$ st $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$ Note $\delta = \delta(\varepsilon)$

Continuity at x: $\forall \varepsilon > 0 \ \exists \delta = \delta(x, \varepsilon)$ st $\forall y$ with $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

The family $\mathcal{F}$ is said to be **uniformly bounded on compact subsets of** Ω if for each compact set $K \subset \Omega$ there exists $B_K > 0$, such that

$$|f(z)| \leq B_K \text{ for all } z \in K \text{ and } f \in \mathcal{F}.$$

$f_n(z) = e^{nz} = e^{nx} e^{iny}$ so $|f_n(z)| = e^{nx}$



Also, the family $\mathcal{F}$ is **equicontinuous** on a compact set K if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $z, w \in K$ and $|z - w| < \delta$, then

$$|f(z) - f(w)| < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

Uniform Continuity: $\forall \varepsilon > 0 \ \exists \delta$ st $\forall x, y$ st $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$ Note $\delta = \delta(\varepsilon)$

Difference is now for a family

Before: Given ε $\exists \delta = \delta(\varepsilon, f)$ st ...

Now: Given ε $\exists \delta = \delta(\varepsilon, \mathcal{F})$ st ...

Montel's theorem

Theorem 3.3 *Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then:*

(i) *$\mathcal{F}$ is equicontinuous on every compact subset of Ω.*

(ii) *$\mathcal{F}$ is a normal family.*

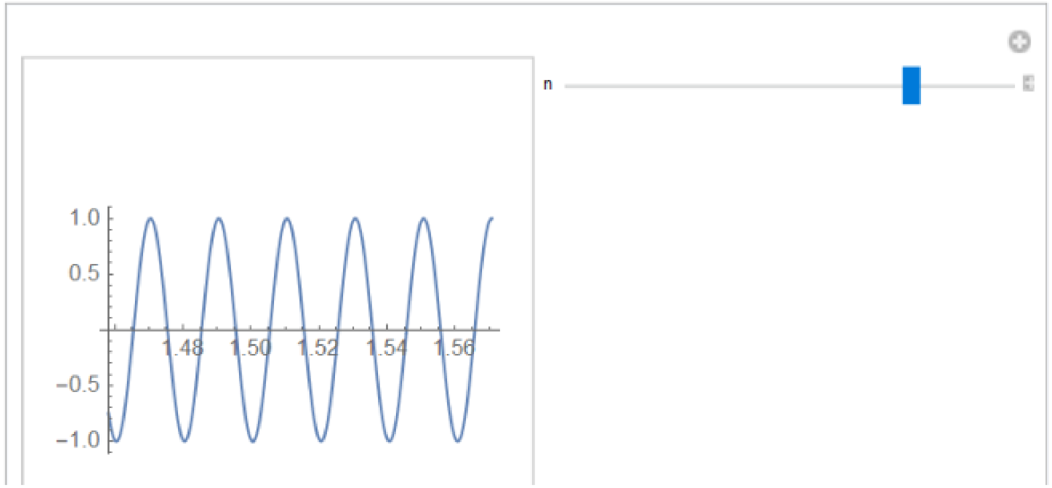
Ex: Real Example: $f_n(x) = \sin(nx)$ on $[-\pi, \pi]$ or $(-\pi, \pi)$

Since

uniformly bounded by 1

Not Equicontinuous!

```
f[x_, n_] := Sin[(4 Floor[n] + 1) x]
Manipulate[Plot[f[x, n], {x, Pi / 2 - 1 / Sqrt[n], Pi / 2}],
 {n, 1, 100}]
```



Math 383: Complex Analysis: Fall ‘23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 23: 10-30-23: https://youtu.be/uMyZ5mgy3sQ

•Bijections on the boundary, from the Geometric Series to the Riemann Zeta Function

No class Friday November 3rd; watch the following videos:

- Lecture 18: 10/27/21: Montel's Theorem and Results from Analysis: https://youtu.be/YAWP7TXRGJA (fix on error here: https://youtu.be/A2E5fVKyKXw) (slides) Already watched this….
- Lecture 20: 10/30/17: Riemann Mapping Theorem Overview): https://youtu.be/FhphhYFxIP0 (slides)
- Lecture 20: 11/01/21: Riemann Mapping Theorem (Proof), Differences between Real and Complex: https://youtu.be/yivEV2yhxgA (slides)
- Lecture 21: 11/03/21: Finishing Proof of the Riemann Mapping Theorem, Introduction to the Riemann Zeta Function, Partial Summation: https://youtu.be/-TpU7PdIEf0 (slides)

https://www.math.ucdavis.edu/~hunter/m201a_16/continuous.pdf

Theorem 9. The continuous image of a compact set is compact.

Proof. Suppose that $f : X \to Y$ is continuous and X is compact. If $\{G_\alpha : \alpha \in I\}$ is an open cover of $f(X)$, then $\{f^{-1}(G_\alpha) : \alpha \in I\}$ is an open cover of X, since the inverse image of an open set is open. Since X is compact, it has a finite subcover $\{f^{-1}(G_{\alpha_i}) : i = 1, 2, \dots, n\}$. Then $\{G_{\alpha_i} : i = 1, 2, \dots, n\}$ is a finite subcover of $f(X)$, which proves that $f(X)$ is compact. $\square$

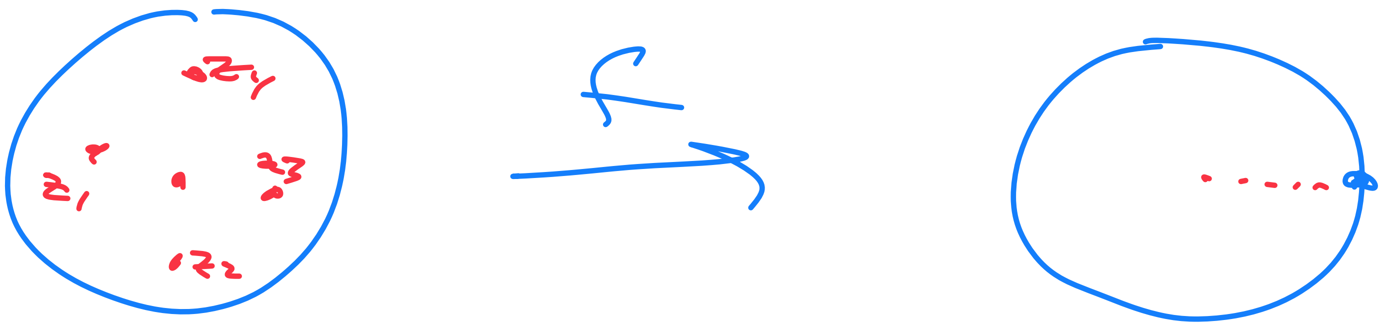
$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

Thm: $f: \mathbb{D} \to \mathbb{D}$ is 1-1, onto (bijection), holomorphic

Assume f extends to a holomorphic function beyond $\mathbb{D}$ on some open set Ω.

Then $f: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, all points on the boundary of the unit disk are hit.

Proof: wlog, assume miss a point in $\partial \mathbb{D}$: say 1.



$w_n = 1 - 1/n$, $f(z_n) = w_n$

$z_n = f^{-1}(w_n)$

$\lim_{n\to\infty} w_n = 1$

Seq of points z_n st $f(z_n) = w_n \to 1$

z_n seq of points with $f(z_n) = w_n \to 1$, $z_n \in \mathbb{D}$

Have subseq $z_{n_k} \in \mathbb{D}$ st $\lim_{k\to\infty} z_{n_k} = z^* \in \overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$

$|f(z^*)| \leq 1$ as each $|f(z_{n_k})| \leq 1$

Know $f(z_{n_k}) = w_{n_k} = 1 - 1/n_k$

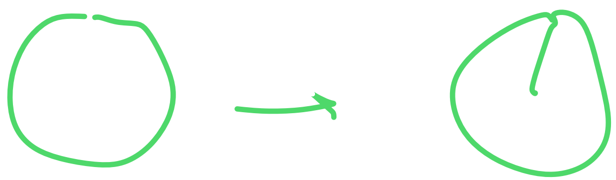
So $\forall k$ such that $n_k \geq N$, $|f(z_{n_k})| \geq 1 - 1/N$

So $|f(z^*)| = 1$ in fact $f(z^*) = 1$ (by continuity)

$\hookrightarrow$ if $f(z^*) \neq 1$ then $f(z_{n_k})$ is a small distance from 1 for all k large

If $z^* \in \mathbb{D}$ then max-modulus implies f constant, contradiction

Question: does $f : \partial\mathbb{D} \to \partial\mathbb{D}$?



From Shooting Hoops
to the Geometric Series Formula

The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

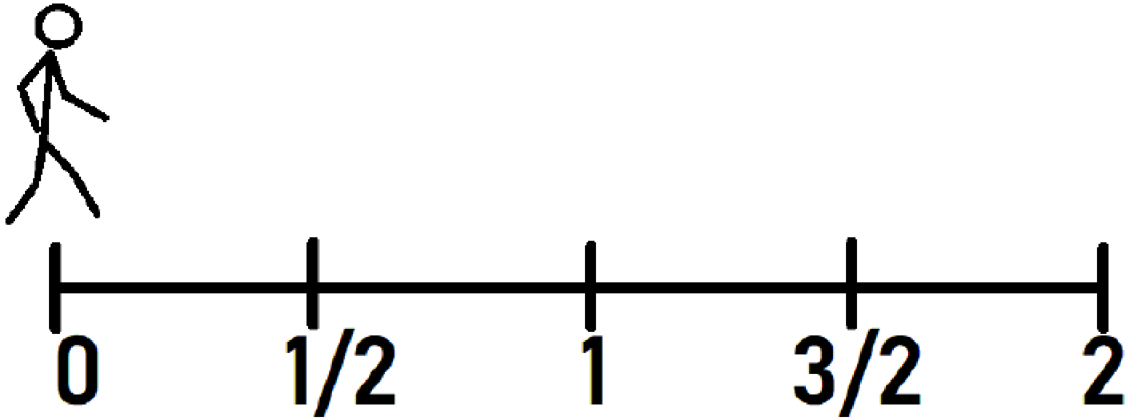
This is often proved by first computing the finite sum, up to r^n, and taking a limit. Note since $|r| < 1$ that each term r^n gets small fast…..

$(1 + r + r^2 + \ldots + r^{N-1}) + (r^N + r^{N+1} + \ldots)$ $\quad r^N(1 + r + r^2 + \ldots)$

The Geometric Series Converges if |r| < 1

$$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r}.$$

Why does this converge? Take r = ½. We then have 1 + ½ + ¼ + …. = $\frac{1}{1-\frac{1}{2}} = 2$, and we can view this as we start at 0, and each step covers half the distance to 2. We thus never reach it in finitely many steps, but we cover half the ground each time.



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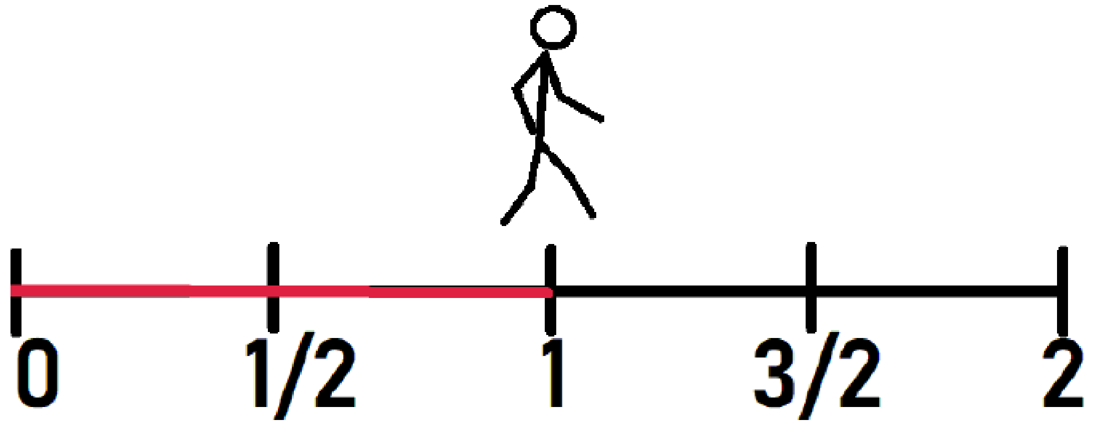

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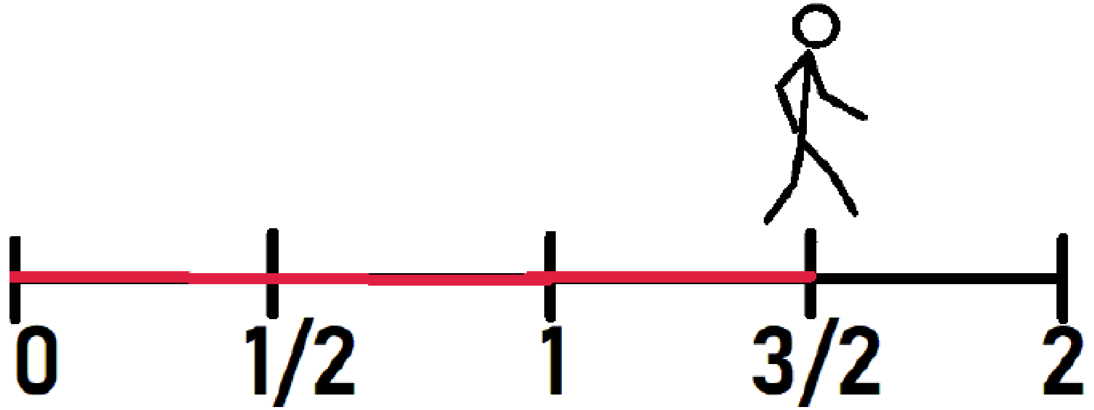

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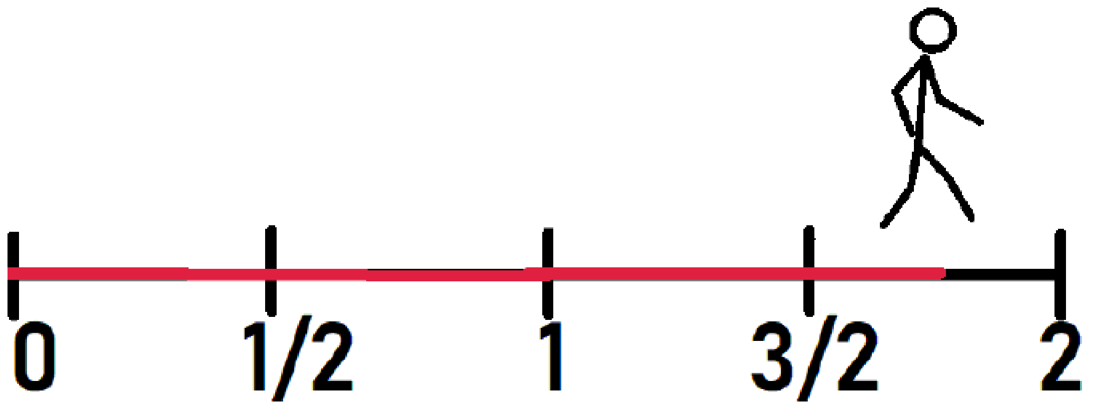

The Geometric Series Formula

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

Lemma: If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots + r^n = \frac{1 - r^{n+1}}{1-r}$.

Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \ldots + r^n$

Then $\quad r S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

What should we do now?

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Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \ldots + r^n$

Then $\quad r\,S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

Subtract: $S_n - r\,S_n = 1 - r^{n+1}$,

So $(1-r)\,S_n = 1 - r^{n+1}$, or S_n

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So $(1-r)\, S_n = 1 - r^{n+1}$, or $S_n = \frac{1 - r^{n+1}}{1-r}$.

If we let n go to infinity, we see r^{n+1} goes to

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Then $\quad r\,S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

Subtract: $S_n - r\,S_n = 1 - r^{n+1}$,

So $(1-r)\,S_n = 1 - r^{n+1}$, or $S_n = \frac{1 - r^{n+1}}{1 - r}$. $\quad = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}$

If we let n go to infinity, we see r^{n+1} goes to 0, so we get the infinite sum is $\frac{1}{1-r}$.

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

We will prove the Geometric Series Formula just by studying this basketball game!

Simpler Game: Hoops: Mathematical Formulation

Bird and Magic (I'm old!) alternate shooting; first basket wins.

- Bird always gets basket with probability p.
- Magic always gets basket with probability q.

Let x be the probability Bird wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1^{st} shot: p.
- **Bird** wins on 2^{nd} shot: $(1-p)(1-q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1 - p)(1 - q) \cdot p$.
- **Bird** wins on 3rd shot: $(1 - p)(1 - q) \cdot (1 - p)(1 - q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p+rp+r^2p+r^3p+\cdots \\ &= p\left(1+r+r^2+r^3+\cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p +$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p + (1 - p)(1 - q) * ???$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p + (1-p)(1-q)x$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \;=\; p \quad \text{or} \quad x \;=\; \frac{p}{1-r}.$$

Solving the Hoop Game: The Power of Perspective

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$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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Have

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \ = \ p \quad \mathrm{or} \quad x \ = \ \frac{p}{1-r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots \ = \ \frac{1}{1-r}.$$

Advanced Geometric Series Comments

Always carefully look at what you did, and be explicit on what you proved.

The geometric series formula is:

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

We proved this when $r = (1-p)(1-q)$, where p and q are the probabilities of making a basket for Bird and Magic. What are the ranges for p and q? We have what range of p and q?

Advanced Geometric Series Comments

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We proved this when $r = (1-p)(1-q)$, where p and q are the probabilities of making a basket for Bird and Magic. What are the ranges for p and q? We have $0 \leq p, q \leq 1$ BUT we cannot have $p=q=0$, or the game never ends. Thus we only proved the Geometric Series Formula for what range of r?

Advanced Geometric Series Comments

Always carefully look at what you did, and be explicit on what you proved.

The geometric series formula is:

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

We proved this when $r = (1-p)(1-q)$, where p and q are the probabilities of making a basket for Bird and Magic. What are the ranges for p and q? We have $0 \le p, q \le 1$ BUT we cannot have p=q=0, or the game never ends. Thus we only proved the Geometric Series Formula for $0 \le r < 1$. Is there a way to deduce the formula for $|r| < 1$ and r negative from what we have already done? (YES)

From the Geometric Series Formula to Primes

Application of the Geometric Series Formula: Infinitude of Primes!

One of the most important applications of the Geometric Series Formula is in Number Theory.

It is used in creating / understanding the Riemann Zeta Function, which gives us tremendous information about primes.

Remember primes are numbers with exactly two factors, 1 and themselves: 2, 3, 5, 7, 11, 13, 17, 19, 23, …. If you are divisible by two or more primes you are called composite, while 1 is called a unit. We will see it is convenient NOT to have 1 be a prime.

Euclid and the Infinitude of Primes

There are many proofs that there are infinitely many primes. This one goes back over 2000 years to Euclid….

Assume there are only finitely many primes, say $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, …, p_n.

Consider the new number $x = p_1 * p_2 * p_3 * \ldots * p_n + 1$.

$P_n\# = P_1 \cdot P_2 \cdots P_n$

Can this be divisible by p_1?

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Consider the new number $x = p_1 * p_2 * p_3 * \ldots * p_n + 1$.

Can this be divisible by p_1? No, the remainder is 1.

Can this be divisible by p_2?

Euclid and the Infinitude of Primes

There are many proofs that there are infinitely many primes. This one goes back over 2000 years to Euclid….

Assume there are only finitely many primes, say $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, …, p_n.

Consider the new number $x = p_1 * p_2 * p_3 * \ldots * p_n + 1$.
Can this be divisible by p_1? No, the remainder is 1.
Can this be divisible by p_2? No, the remainder is 1.

Continuing we see it cannot be divisible by ANY prime in our list. As we assumed our list was complete, we have found a new prime (either this number is prime, or it is divisible by a prime not on our list).

$\pi(x) = \#\{\text{primes } p \le x\}$ This gives $\pi(x) >> \text{Const} \log \log x$

Euclid and the Infinitude of Primes

Consider the numbers generated by Euclid’s method; it’s fun to try this process.

- We start with 2, then look at 2+1 and get 3 as the next number.
- Then $2 * 3 + 1 = 7$ for our next prime.
- Then $2 * 3 * 7 + 1 = 43$ which is also prime.

Do we always get a prime when we apply this? Do we get all the primes?

Euclid and the Infinitude of Primes

Consider the numbers generated by Euclid's method; It's fun to try this process.

- We start with 2, then look at 2+1 and get 3 as the next number.
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- Then 2 * 3 * 7 + 1 = 43 which is also prime.

Do we always get a prime when we apply this? Do we get all the primes?

We do not always get a prime – look at the next term!

- 2 * 3 * 7 * 43 + 1 = 1807 = 13 * 139.

https://www.youtube.com/watch?v=NOCsdhzo6Jg **(How They Fool Ya: Math parody of Hallelujah)**

The other questions are open….. We don't have to go far to find open questions about primes (others include are there infinitely many pairs of primes differing by 2, and can every even number at least 4 be written as the sum of two primes).

https://en.wikipedia.org/wiki/Euclid–Mullin_sequence

Euclid–Mullin sequence

From Wikipedia, the free encyclopedia

The **Euclid–Mullin sequence** is an infinite sequence of distinct prime numbers, in which each element is the least prime factor of one plus the product of all earlier elements. They are named after the ancient Greek mathematician Euclid, because their definition relies on an idea in Euclid's proof that there are infinitely many primes, and after Albert A. Mullin, who asked about the sequence in 1963.[1]

The first 51 elements of the sequence are

> 2, 3, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139, 2801, 11, 17, 5471, 52662739, 23003, 30693651606209, 37, 1741, 1313797957, 887, 71, 7127, 109, 23, 97, 159227, 643679794963466223081509857, 103, 1079990819, 9539, 3143065813, 29, 3847, 89, 19, 577, 223, 139703, 457, 9649, 61, 4357, 87991098722552272708281251793312351581099392851768893748012603709343, 107, 127, 3313, 227432689108589532754984915075774848386671439568260420754414940780761245893, 59, 31, 211... (sequence A000945 in the OEIS)

These are the only known elements as of September 2012. Finding the next one requires finding the least prime factor of a 335-digit number (which is known to be composite).

The Riemann Zeta Function ζ(s)

https://en.wikipedia.org/wiki/Greek_alphabet

Greek alphabet

From Wikipedia, the free encyclopedia

The **Greek alphabet** has been used to write the Greek language since the late ninth or early eighth century BC.[3][4] It is derived from the earlier Phoenician alphabet,[5] and was the first alphabetic script in history to have distinct letters for vowels as well as consonants. In Archaic and early Classical times, the Greek alphabet existed in many different local variants, but, by the end of the fourth century BC, the Euclidean alphabet, with twenty-four letters, ordered from alpha to omega, had become standard and it is this version that is still used to write Greek today. These twenty-four letters (each in uppercase and lowercase forms) are: Α α, Β β, Γ γ, Δ δ, Ε ε, Ζ ζ, Η η, Θ θ, Ι ι, Κ κ, Λ λ, Μ μ, Ν ν, Ξ ξ, Ο ο, Π π, Ρ ρ, Σ σ/ς, Τ τ, Υ υ, Φ φ, Χ χ, Ψ ψ, and Ω ω.

ζ(s)

There are many different ways of writing a Greek letter zeta; here is how Powerpoint displays it.

The Riemann Zeta Function ζ(s)

We define this function as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots$$

and for us we will take s > 1 which ensures the infinite sum converges (for those knowing more, s can be any complex number with real part at least 1).

Looking at this function, it is NOT clear why it is worth studying....

Integers and Primes

Most of us are familiar with the positive integers: 1, 2, 3, 4, 5, ….

What is the next integer after 2023?

Integers and Primes

Most of us are familiar with the positive integers: 1, 2, 3, 4, 5, ….

What is the next integer after 2023? 2024

What is the next integer after 2024?

Integers and Primes

Most of us are familiar with the positive integers: 1, 2, 3, 4, 5, ….

What is the next integer after 2023? 2024

What is the next integer after 2024? 2025

As you have hopefully noticed, there is not much mystery in the spacings between integers!

Integers and Primes

What about the primes: 2, 3, 5, 7, ….

What is the next prime after 2023?

Integers and Primes

What about the primes: 2, 3, 5, 7, ….

What is the next prime after 2023? 2027

What is the next prime after 2027?

Integers and Primes

What about the primes: 2, 3, 5, 7, ...

What is the next prime after 2023? 2027

What is the next prime after 2027? 2029

What is the next prime after 2029?

Integers and Primes

What about the primes: 2, 3, 5, 7, ….

What is the next prime after 2023? 2027

What is the next prime after 2027? 2029

What is the next prime after 2029? 2039

As you have hopefully noticed, it is a lot harder to find the next prime than to find the next integer!

The Riemann Zeta Function ζ(s) and Primes

We defined the Riemann Zeta Function (for s > 1) by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ ...$$

and now we note a remarkable property; we also have

$$\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ ....$$

Two questions: (1) Why is this true, and (2) Why do we care?

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

Why do we care?

The integers are completely understood. We even have a great formula for the n^{th} integer!

The Riemann zeta function connects the integers and the primes.

Perhaps we can pass from knowledge about the integers to knowledge about the primes....

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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If we take s=1 the sum becomes the Harmonic Series, which we showed diverges!

If there were only finitely many primes the product would ???.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

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If we take s=1 the sum becomes the Harmonic Series, which we showed diverges!

If there were only finitely many primes the product would converge!

Thus there are infinitely many primes! (Advanced: can prove more, can prove the sum of the reciprocals of the primes diverges.)

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} ,$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

The following is beyond the scope of this talk, but if we take s=2 then the sum is $\pi^2 / 6$, which is an irrational number (this means we cannot write it as a ratio of two integers).

If there were only finitely many primes then the product would be a finite product of rational numbers, and hence rational! For example, if only 2 and 3 are prime:

$$\prod_{p\ prime} \left(1 - \frac{1}{p^2}\right)^{-1} = \left(1 - \frac{1}{2^2}\right)^{-1} \left(1 - \frac{1}{3^2}\right)^{-1} = \left(\frac{3}{4}\right)^{-1} \left(\frac{8}{9}\right)^{-1} = \frac{4}{3}\frac{9}{8} = \frac{3}{2}$$

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

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Thus there are infinitely many primes!

The Riemann Zeta Function ζ(s) and Primes

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or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

We thus see the importance of the formula above, which connects sums over integers with products over primes.

It allows us to pass from knowledge of integers to knowledge of primes.

We now prove it, or at least sketch the proof.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

We need the Fundamental Theorem of Arithmetic: Every positive integer can be written uniquely as a product of prime powers, where we write the primes in increasing order, and we let the empty product be 1.

Thus 12 = $2^2 * 3$ and 90 = $2 * 3^2 * 5$, and there are no other ways to write these numbers. If 1 were prime, we would lose uniqueness: $2^2 * 3 = 1^{2020} * 2^2 * 3$.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1}\ldots.$

We will not give a fully rigorous argument.

What we do is consider a finite product, the product over the first P primes, and show that as P gets larger and larger we get more and more of the terms in the sum (once and only once), including all the terms up to P, and thus in the limit as we take all the primes we get the sum.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

We use the Geometric Series Formula to expand each factor.

$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{1}{\left(1 - \frac{1}{p^s}\right)}$ and this is a Geometric Series with $r = 1/p^s$.

Since $1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$, we have $\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots$

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

We use the Geometric Series Formula to expand each factor. If p = 2:

$\left(1 - \frac{1}{2^s}\right)^{-1} = \frac{1}{\left(1 - \frac{1}{2^s}\right)}$ and this is a Geometric Series with r = $1/p^s$.

Since $1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$, we have $\left(1 - \frac{1}{2^s}\right)^{-1} = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \ldots$

since $(2^2)^s = 4^s$, $(2^3)^s = 8^s$, ….

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

Let's look at multiplying the factors

$$\left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \ldots\right) * \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \ldots\right)$$

When we multiply out we get

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{12^s} + \frac{1}{16^s} + \frac{1}{18^s} + \frac{1}{24^s} + \frac{1}{27^s} + \frac{1}{32^s} + \frac{1}{36^s} + \cdots$$

We get exactly the numbers that have only 2 and 3 as prime factors....

My worse proof ever (this is the fixed version)

Irrationality measure and lower bounds for \(\pi(x)\) (with David Burt, Sam Donow, Matthew Schiffman and Ben Wieland), The Pi Mu Epsilon Journal **(14** (2017), no. 7, 421-429**)** pdf

Irrationality measure and lower bounds for $\pi(x)$

David Burt Sam Donow Steven J. Miller*
Matthew Schiffman Ben Wieland

August 10, 2017

Abstract

In this note we show how the irrationality measure of $\zeta(s) = \pi^2/6$ can be used to obtain explicit lower bounds for $\pi(x)$. We analyze the key ingredients of the proof of the finiteness of the irrationality measure, and show how to obtain good lower bounds for $\pi(x)$ from these arguments as well. While versions of some of the results here have been carried out by other authors, our arguments are more elementary and yield a lower bound of order $x/\log x$ as a natural boundary.

Math 383: Complex Analysis: Fall ‘23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 24: 11-01-23: https://youtu.be/0XjpgP9F5uw

- Stirling’s formula, evaluating and estimating sums

For $s > 0$ (or actually $\Re(s) > 0$), the **Gamma function** $\Gamma(s)$ is

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx = \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

Existence of $\Gamma(s)$

Study near 0 and near infinity

near $x=0$: $e^{-x} x^{s-1} \simeq (1 + x + x^2/2! + \cdots) x^{s-1} = x^{s-1} + \text{small}$

$$\int_{\varepsilon_1}^{\varepsilon_2} x^{s-1} dx = \begin{cases} \frac{x^s}{s}\Big|_{\varepsilon_1}^{\varepsilon_2} & \text{if } s \neq 0 \\ \ln(x)\Big|_{\varepsilon_1}^{\varepsilon_2} & \text{if } s = 0 \end{cases}$$

what happens as $\varepsilon_1 \to 0$? Need $s > 0$

near ∞: $|e^{-x} x^{s-1}| = \left|\frac{x^{s-1}}{e^x}\right| \le \left|\frac{x^s}{x^M/M!}\right| \le \frac{M!}{x^n}$

(x large)

$n \ge 2$ if M large relative to s

The Gamma function. The Gamma function $\Gamma(s)$ is

$\Gamma(n+1) = n!$
$n = 0, 1, 2, \ldots$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0. \quad \text{“} x^s \frac{dx}{x} \text{”}$$

$n=1$: $\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = e^{-x}\Big|_\infty^0 = 1$

$n=2$: $\Gamma(2) = \int_0^\infty x e^{-x} dx$ $= 1$

$u = x$ $\quad dv = e^{-x} dx$

$du = dx$ $\quad v = -e^{-x}$

$= uv\Big|_0^\infty - \int_0^\infty v\,du$

$\Gamma(3) = 2$

$\Gamma(4) = 6$

$\Gamma(s+1) = \int_0^\infty e^{-x} x^s dx$

$u = x^s$ $\quad dv = e^{-x} dx$

$du = s x^{s-1} dx$ $\quad v = -e^{-x}$

$= \int_0^\infty s x^{s-1} e^{-x} dx = s \int_0^\infty e^{-x} x^{s-1} dx = s\Gamma(s)$

Functional Eq: $\Gamma(s+1) = s\Gamma(s)$ $\Rightarrow$ $\Gamma(n+1) = n!$ n NON-NEG integer

Functional equation of $\Gamma(s)$: The Gamma function satisfies

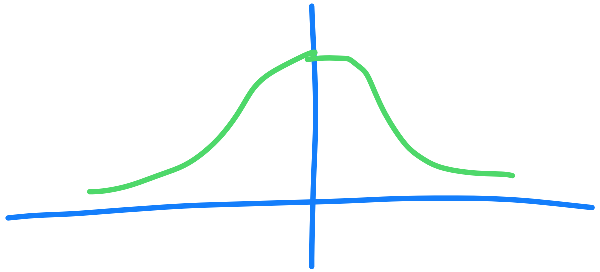
$$\Gamma(s+1) \;=\; s\Gamma(s).$$

This allows us to extend the Gamma function to all s. We call the extension the Gamma function as well, and it's well-defined and finite for all s save the negative integers and zero.

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\left(-\tfrac{1}{2}\right)! = \sqrt{\pi}$$

Moments of the standard normal



$$\tilde{\mu}_m = 2\int_0^\infty x^m \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$u = x^2/2$ so $x = 2^{1/2} u^{1/2}$

$du = x\,dx$ or $dx = 2^{1/2} \frac{1}{2} u^{-1/2} du = 2^{-1/2} u^{-1/2} du$

$x^m = 2^{m/2} u^{m/2}$

$$\tilde{\mu}_m = \frac{2}{\sqrt{2\pi}} \int_0^\infty 2^{m/2} u^{m/2} e^{-u}\, 2^{-1/2} u^{-1/2}\, du$$

$$= \pi^{-1/2} 2^{\frac{m}{2}} \int_0^\infty u^{\frac{m}{2}-\frac{1}{2}} e^{-u}\, du = \pi^{-\frac{1}{2}} 2^{\frac{m}{2}} \Gamma\left(\frac{m+1}{2}\right)$$

write as $\frac{m}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2} = \frac{m+1}{2}-1$

$m=0 \to \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

The cosecant identity. If s is not an integer, then

$$\Gamma(s)\Gamma(1-s) \;=\; \pi\csc(\pi s) \;=\; \frac{\pi}{\sin(\pi s)}.$$

$$\Gamma(1/2) = \sqrt{\pi}.$$

Gaussian

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Standard normal

$$\mu_{2n} = 2\int_0^\infty \underbrace{x^{2n} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx}_{} = (2n-1)!!$$

$u = x^2/2$

$$(*) \int_0^\infty e^{-u} u^{\text{power}}\, du$$

The Gamma function. The Gamma function $\Gamma(s)$ is

$\Gamma(n+1) = n!$
$n = 0, 1, 2, \ldots$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0.$$

"$x^s \frac{dx}{x}$"

Stirling's formula: As $n \to \infty$, we have

$$n! \approx n^n e^{-n} \sqrt{2\pi n};$$

by this we mean

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.$$

More precisely, we have the following series expansion:

$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots\right).$$

Crude upper/lower bounds.

$$n \leq n! \leq n^n$$

Note (n+1)!/n! = n+1; let's see what Stirling gives:

$$\frac{(n+1)^{n+1} e^{-(n+1)} \sqrt{2\pi(n+1)}}{n^n e^{-n} \sqrt{2\pi n}} = (n+1) \underbrace{\left(1+\frac{1}{n}\right)^n}_{\downarrow \text{ as } n\to\infty \atop e} e^{-1} \underbrace{\sqrt{\frac{n+1}{n}}}_{\downarrow \atop 1}$$

$$\approx n+1$$

Integral Test and the Poor Mathematician's Stirling

1 vs n

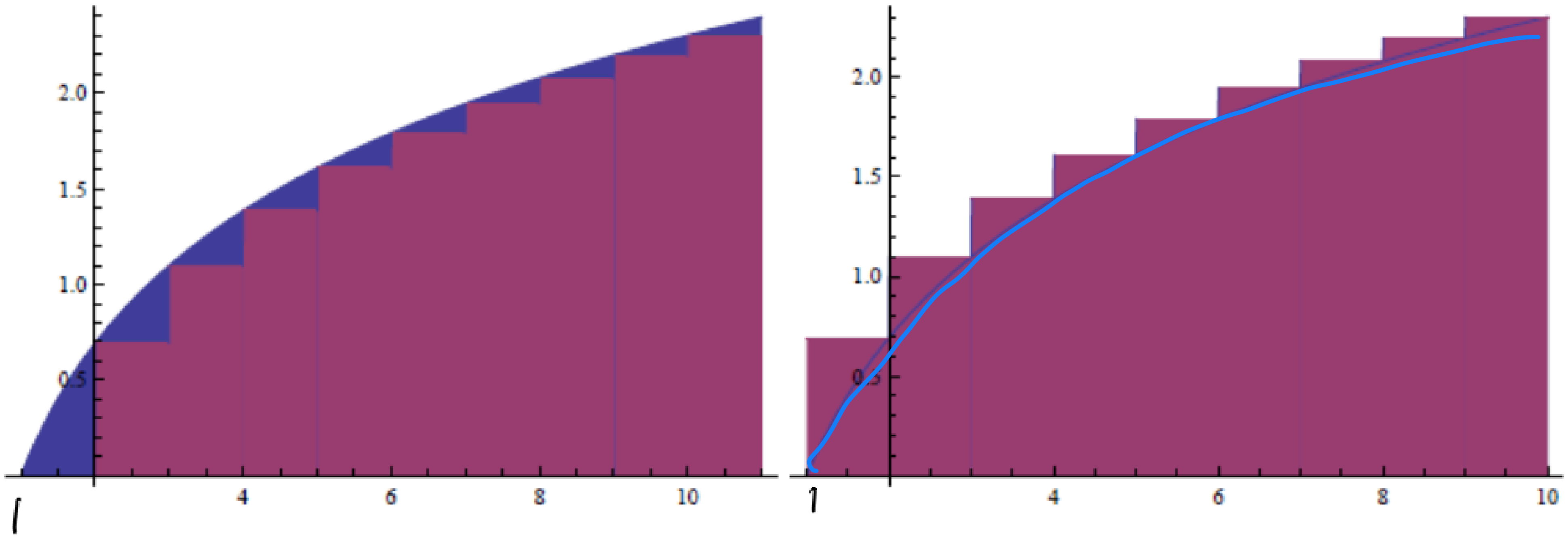
$GM(1,n) = \sqrt{1 \cdot n} = \sqrt{n}$

Poor man's Stirling. Let $n \geq 3$ be a positive integer. Then

$$n^n e^{-n} \cdot e \;\leq\; n! \;\leq\; n^n e^{-n} \cdot en.$$

$$\log P \;=\; \log n! \;=\; \log 1 + \log 2 + \cdots + \log n \;=\; \sum_{k=1}^{n} \log k.$$

$$\int_1^n \log t dt \;\leq\; \sum_{k=1}^{n} \log k \;\leq\; \int_2^{n+1} \log t dt.$$



Lower and upper bound for $\log n!$ when $n = 10$.

Stirling's Formula: Lower bound from Integral Test:

Guess $(t\log t)' = 1\cdot\log t - t\cdot\frac{1}{t} = \log t - 1$

$(t\log t + t)' = \log t$

$$(t\log t - t)\Big|_{t=1}^{n} \;\le\; \log n! \;\le\; (t\log t - t)\Big|_{t=2}^{n+1}$$

$$n\log n - n + 1 \;\le\; \log n! \;\le\; (n+1)\log(n+1) - (n+1) - (2\log 2 - 2).$$

We'll study the lower bound first. From

$$n\log n - n + 1 \;\le\; \log n!,$$

we find after exponentiating that

$$e^{n\log n - n + 1} \;=\; n^n e^{-n}\cdot e \;\le\; n!.$$

Dang 1st/last terms

$1 \qquad n$

average

$\sqrt{1\cdot n}$

$\frac{\log 1 + \log n}{2} = \log n^{1/2}$

Euler–Maclaurin formula

From Wikipedia, the free encyclopedia: https://en.wikipedia.org/wiki/Euler%E2%80%93Maclaurin_formula

If m and n are natural numbers and $f(x)$ is a real or complex valued continuous function for real numbers x in the interval $[m,n]$, then the integral

$$I = \int_m^n f(x)\,dx$$

can be approximated by the sum (or vice versa)

$$S = f(m+1) + \cdots + f(n-1) + f(n)$$

(see rectangle method). The Euler–Maclaurin formula provides expressions for the difference between the sum and the integral in terms of the higher derivatives $f^{(k)}(x)$ evaluated at the endpoints of the interval, that is to say $x = m$ and $x = n$.

Explicitly, for p a positive integer and a function $f(x)$ that is p times continuously differentiable on the interval $[m,n]$, we have

$$S - I = \sum_{k=1}^{p} \frac{B_k}{k!} \left(f^{(k-1)}(n) - f^{(k-1)}(m) \right) + R_p,$$

where B_k is the kth Bernoulli number (with $B_1 = \frac{1}{2}$) and R_p is an error term which depends on n, m, p, and f and is usually small for suitable values of p.

The formula is often written with the subscript taking only even values, since the odd Bernoulli numbers are zero except for B_1. In this case we have[1][2]

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)\,dx + \frac{f(n)+f(m)}{2} + \sum_{k=1}^{\left\lfloor \frac{p}{2} \right\rfloor} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m) \right) + R_p,$$

or alternatively

$$\sum_{i=m+1}^{n} f(i) = \int_m^n f(x)\,dx + \frac{f(n)-f(m)}{2} + \sum_{k=1}^{\left\lfloor \frac{p}{2} \right\rfloor} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m) \right) + R_p.$$

$$\underbrace{1+2+3+\cdots+n}_{S_1(n)} = \frac{n(n+1)}{2}$$

$1 \leq n \leq \frac{n}{2} \cdot \frac{n}{2} \leq S_1(n) \leq n^2$

$$1 + 2 + \cdots + \frac{n}{2} + \underbrace{\left(\frac{n}{2}+1\right) + \cdots + n}_{\frac{n}{2} \text{ terms, each at least } \frac{n}{2}}$$

Know $\frac{1}{4}n^2 \leq S_1(n) \leq n^2$ Correct growth rate

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Take n a power of 2…. (even/odd trick, zeta values)

$a_n = 1+2+\cdots+2^n$

$a_{n+1} = 1+2+\cdots+2^n+(2^n+1)+\cdots 2^{n+1}$

$= \text{evens} + \text{odds}$

$a_{n+1} = 2a_n + 2a_n - 2^n$

$a_{n+1} = 4a_n - 2^n$

Try $b_{n+1} = 4b_n \Rightarrow$ soln is $b_n = 4^n b_0$

Try $a_n = b_n + \alpha 2^n \quad a_{n+1} = b_{n+1} + \alpha 2^{n+1}$ Bring it over

$b_{n+1} + \alpha 2^{n+1} = 4b_n + 4\alpha 2^n - 2^n \quad 2\alpha \cdot 2^n = (4\alpha - 1)2^n$ So $2\alpha = 4\alpha - 1$

$1 = 2\alpha$

$\alpha = 1/2$

$a_n = 4^n b_0 + \frac{1}{2}2^n \quad a_0 = 1 \Rightarrow b_0 = \frac{1}{2} \Rightarrow a_n = \frac{1}{2}(2^n + 4^n)$

$$\zeta(2) = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \sum_{n \text{ even}} \frac{1}{n^2} + \sum_{n \text{ odd}} \frac{1}{n^2}$$

$$= \sum_{m=1}^{\infty} \frac{1}{(2m)^2} + \sum_{n \text{ odd}} \frac{1}{n^2}$$

$$= \frac{1}{4}\frac{\pi^2}{6} + \sum_{n \text{ odd}} \frac{1}{n^2}$$

evens $= 2+4+6+8+\cdots$

odds $= (2-1)+(4-1)+(6-1)+\cdots$

$= \text{evens} - 2^n$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Now $a_n = 1 + 2 + \dots + n$

$a_1 = 1 \rightarrow b_0 = 0$

$a_{n+1} = a_n + (n+1)$

Clean eq: $b_{n+1} = b_n$

Try $a_n = b_n + \alpha n^2 + \beta n$

$b_{n+1} + \alpha(n+1)^2 + \beta(n+1) = b_n + \alpha n^2 + \beta n + n + 1$

need $\alpha n^2 = \alpha n^2$

$2\alpha n + \beta n = \beta n + n \rightarrow \alpha = 1/2$

$\alpha + \beta = 1 \rightarrow \frac{1}{2} + \beta = 1 \rightarrow \beta = \frac{1}{2}$

Try $a_n = b_0 + \frac{1}{2} n^2 + \frac{1}{2} n$

$a_n = \frac{1}{2} n^2 + \frac{1}{2} n = \frac{1}{2} n(n+1)$

Stirling's Formula: Estimates from Dyadic Decompositions

$$n! \sim n^n e^{-n}\sqrt{2\pi n}$$

$$\mathcal{S}_0 = \{1,2,\ldots,n\} = \underbrace{\{1,2,\ldots,n/2\}}_{\text{product} \le (\frac{n}{2})^{n/2}} \cup \underbrace{\{n/2+1, n/2+2, \ldots, n\}}_{\text{product is} \le n^{n/2}} := \mathcal{S}_1 \cup \mathcal{S}_2.$$

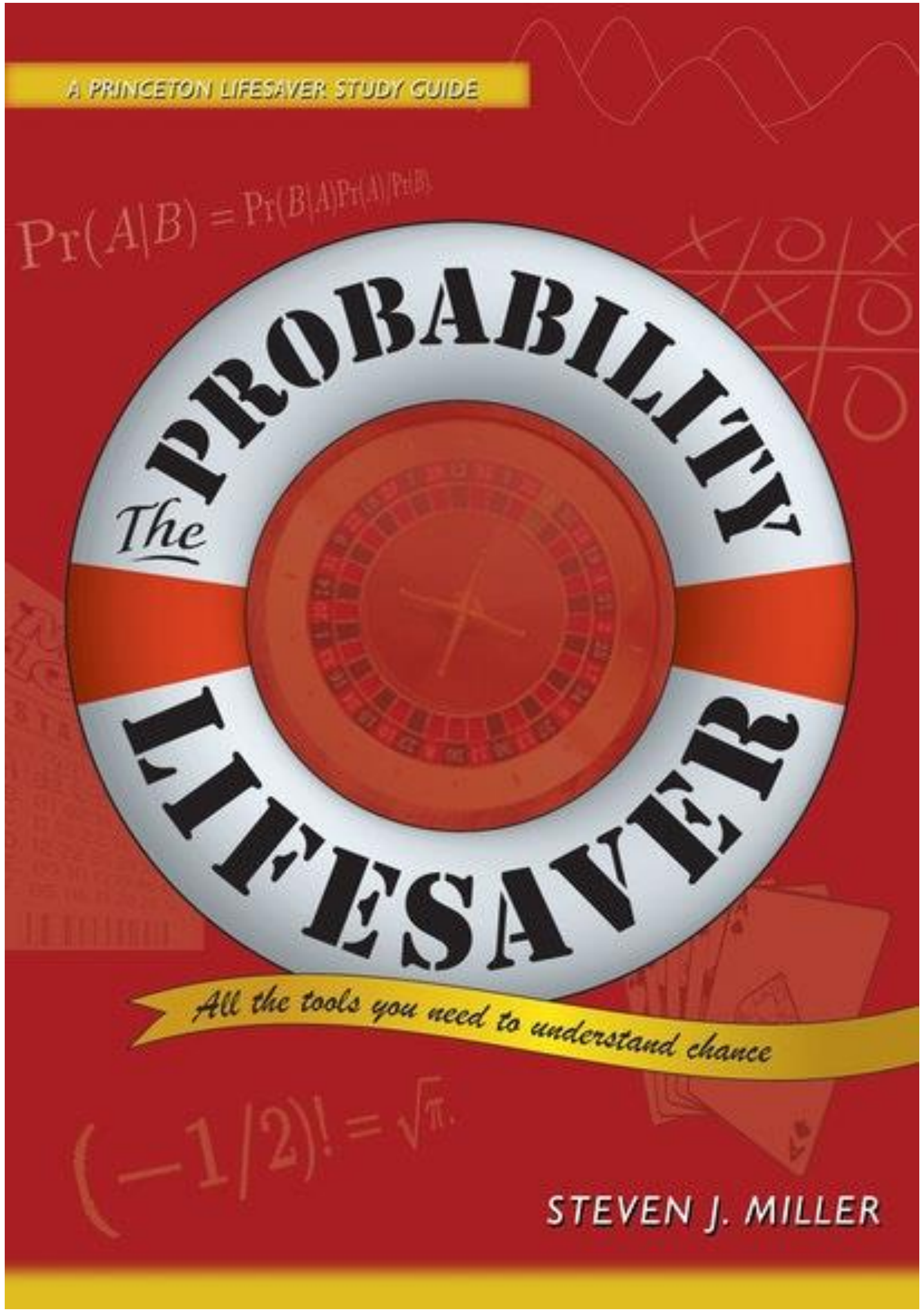
product is $\ge (n/2)^{n/2}$

So $n! \le \left(\frac{n}{2}\right)^{n/2} n^{n/2} = \left(\frac{1}{2}\right)^{n/2} n^n = \left(\frac{1}{\sqrt{2}}\right)^n n^n = (\sqrt{2})^{-n} n^n$

$1 - \frac{n}{4} \qquad \frac{n}{4} - \frac{2n}{4} \qquad \frac{2n}{4} - \frac{3}{4}n \qquad \frac{3}{4}n - n$

$$n! \le \left(\frac{n}{4}\right)^{\frac{n}{4}} \left(\frac{2n}{4}\right)^{\frac{n}{4}} \left(\frac{3n}{4}\right)^{\frac{n}{4}} n^{n/4}$$

$$= \left(\frac{1}{4}\cdot\frac{2}{4}\cdot\frac{3}{4}\right)^{\frac{n}{4}} n^n = \left(\sqrt[4]{\frac{32}{3}}\right)^{-n} n^n$$



Extra credit: Can you expand on the dyadic interval arguments / the Farmer Brown idea to get in the limit, at least for a sequence of n? In other words, can you show that it converges to n^n/e^{-n} times something small relative to the main term?

Additionally, can you prove the claims from class about the sums of powers? In particular, perturb and prove that the sum of the k-th powers is a polynomial of degree k+1 with constant term 0 and leading term n^{k+1} / (k+1)? Can you use the telescoping method and induction to show that the sum is a polynomial?

Looks like some of these results, with telescoping, are known: see https://www.jstor.org/stable/pdf/3026439.pdf

As this is such an important concept, let's work slowly and carefully through its application here. Our goal is to bound $n! = n(n-1)\cdots 2\cdot 1$. As each factor is at least 1 and at most n, we start with the trivial bound

$$1^n \leq n! \leq n^n.$$

Notice the *enormous* spread between our upper and lower bounds. The problem is our set $I_0 := \{1, 2, \dots, n\}$ is very large as $n \to \infty$, and thus it is horrible trying to find *one* upper bound for each factor, and *one* lower bound for each. The idea behind dyadic decompositions is to break this large interval into smaller ones, where the bounds are better, then put them together.

Explicitly, let's split our set in half:

$$\mathcal{S}_0 = \{1, 2, \dots, n\} = \{1, 2, \dots, n/2\} \cup \{n/2+1, n/2+2, \dots, n\} := \mathcal{S}_1 \cup \mathcal{S}_2.$$

In the first interval, each term is at least 1 and at most $n/2$, and thus we obtain

$$1^{n/2} \leq 1\cdot 2\cdots(n/2-1)(n/2) \leq (n/2)^{n/2}.$$

Similarly in the second interval each term is at least $n/2+1$, though we'll use $n/2$ as a lower bound as that makes the algebra cleaner, and at most n. Thus we find

$$(n/2)^{n/2} \leq (n/2+1)(n/2+2)\cdots(n-1)n \leq n^{n/2}.$$

Notice that we're still just using the trivial idea of bounding each term by the smallest or largest; the gain comes from the fact that the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are each half the size of the original set $\mathcal{S}_0$. Thus the upper and lower bounds are much better, as these sets have less variation. Multiplying the two lower (respectively, upper) bounds together gives a lower (respectively, upper) bound for $n!$:

$$1^{n/2}(n/2)^{n/2} \leq [1\cdot 2\cdots(n/2)]\,[(n/2+1)(n/2+2)\cdots n] \leq (n/2)^{n/2}n^{n/2},$$

which simplifies to

$$n^{n/2}\sqrt{2}^{-n} \leq n! \leq n^n\sqrt{2}^{-n}.$$

Notice how much better this is than our original trivial bound of $1 \leq n! \leq n^n$; the upper bound is very close (we have a $\sqrt{2}^{-n}$ instead of an $e^{-n}\sqrt{2\pi n}$), while the lower bound is significantly closer.

We now use the advice from shampoo: **lather, rinse, repeat**. We can break $\mathcal{S}_1$ and $\mathcal{S}_2$ into two smaller intervals, argue as above, and then break those new intervals further (though in practice we'll do something slightly different). We do all this in the next subsection; our purpose here was to introduce the method slowly and describe why it works so well. Briefly, the success is from a delicate balancing act. If we make things too small, there is no variation and no approximation – the numbers are what they are; if we have things too large, there is too much variation and the bounds are trivial. We need to find a happy medium between the two.

18.5.2 Lower bounds towards Stirling, I

We continue our elementary attack on $n!$, and build on the dyadic decomposition idea from the previous subsection. Instead of breaking each smaller set in half, what we will do is just break the earlier set (the one with smaller numbers). We thus end up with sets of different size, getting a chain of sets where each is half the size of the previous.

Explicitly, we study the factors of $n!$ in the intervals $I_1 = (n/2, n]$, $I_2 = (n/4, n/2]$, $I_3 = (n/8, n/4]$, ..., $I_N = (1, 2)$. Note on I_k that each of the $n/2^k$ factors is at least $n/2^k$. Thus

$$\begin{aligned} n! &= \prod_{k=1}^{N} \prod_{m \in I_k} m \\ &\geq \prod_{k=1}^{N} \left(\frac{n}{2^k}\right)^{n/2^k} \\ &= n^{n/2+n/4+n/8+\cdots+n/2^N} 2^{-n/2} 4^{-n/4} 8^{-n/8} \cdots (2^N)^{-n/2^N}. \end{aligned}$$

Let's look at each factor above slowly and carefully. Note the powers of n *almost* sum to n; they would if we just add $n/2^N = 1$ (since we're assuming $n = 2^N$). Remember, though, that $n = 2^N$; there is thus no harm in multiplying by $(n/2^N)^{n/2^N}$ as this is just 1^1 (**multiplying by one** is a powerful technique; see §A.12 for more applications of this method). We now have $n!$ is greater than

$$n^{n/2+n/4+n/8+\cdots+n/2^N+n/2^N} 2^{-n/2} 4^{-n/4} 8^{-n/8} \cdots (2^N)^{-n/2^N} (2)^{-n/2^N}.$$

Thus the n-terms gives n^n. What of the sum of the powers of 2? That's just

$$\begin{aligned} 2^{-n/2} 4^{-n/4} 8^{-n/8} \cdots (2^N)^{-n/2^N} \cdot 2^{-n/2^N} &= 2^{-n(1/2+2/4+3/8+\cdots N/2^N)} 2^{-2^N/2^N} \\ &> 2^{-n\left(\sum_{k=0}^{N} k/2^k\right)} 2^{-2^N/2^N} \\ &\geq 2^{-n\left(\sum_{k=0}^{\infty} k/2^k\right)} 2^{-1} \\ &= 2^{-2n-1} = \frac{1}{2} 4^{-n}. \end{aligned}$$

To see this, we use the following wonderful identity:

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2};$$

for a proof, see §11.1 (on differentiating identities involving the geometric series formula).

Putting everything together, we find

$$n! \geq \frac{1}{2} n^n 4^{-n},$$

which compares favorably to the truth, which is $n^n e^{-n}$. It's definitely much better than our first lower bound of $n^{n/2} 2^{-n/2}$.

As with many things in life, we can get a better result if we're willing
more work. For example, consider the interval $I_1 = (n/2, n]$. We can p
at the beginning and the end: n and $n/2+1$, $n-1$ and $n/2+2$, $n-2$ and
so on until $3n/4$ and $3n/4+1$; for example, if we have the interval (8
pairs are: (16,9), (15,10), (14,11), and (13,12). We now use one of the g
problems from calculus: if we want to maximize xy given that $x+y$
maximum occurs when $x = y = L/2$. This is frequently referred to a
Bob (or Brown) problem, and is given the riveting interpretation that if
to find the rectangular pen that encloses the maximum area for his c
given that the perimeter is L, then the answer is a square pen. Thus of
the one that has the largest product is $3n/4$ with $3n/4+1$, and the sma
$n/2+1$, which has a product exceeding $n^2/2$. We therefore decrease
of all elements in I_1 by replacing each product with $\sqrt{n^2/2} = n/\sqrt{2}$.
thought gives us that

$$n \cdot (n-1) \cdots \frac{3n}{4} \cdots \left(\frac{n}{2}+1\right) \cdot \frac{n}{2} \ \ge\ \left(\frac{n}{\sqrt{2}}\right)^{n/2} \ =\ \left(\frac{n\sqrt{2}}{2}\right.$$

a nice improvement over $(n/2)^{n/2}$, and this didn't require too much add

We now do a similar analysis on I_2; again the worst case is from the pair $n/2$
and $n/4+1$ which has a product exceeding $n^2/8$. Arguing as before, we find

$$\prod_{m \in I_2} m \ \ge\ \left(\frac{n}{\sqrt{8}}\right)^{n/4} \ =\ \left(\frac{n}{2\sqrt{2}}\right)^{n/4} \ =\ \left(\frac{n\sqrt{2}}{4}\right)^{n/4}.$$

At this point hopefully the pattern is becoming clear. We have almost exactly what we had before; the only difference is that we have a $n\sqrt{2}$ in the numerator each time instead of just an n. This leads to very minor changes in the algebra, and we find

$$n! \ \ge\ \frac{1}{2}(n\sqrt{2})^n 4^{-n} \ =\ \frac{1}{2} n^n (2\sqrt{2})^{-n}.$$

Notice how close we are to $n^n e^{-n}$, as $2\sqrt{2} \approx 2.82843$, which is just a shade larger than $e \approx 2.71828$. It's amazing how close our analysis has brought us to Stirling; we're within striking distance of it!

We end this section on elementary questions with a few things for you to try.

- Can you modify the above argument to get a reasonably good upper bound for $n!$?

- After reading the above argument, you should be wondering exactly how far can we push things. What if we didn't do a dyadic decomposition; what if instead we did say a triadic: $(2n/3, n]$, $(4n/9, 2n/3]$, ... Maybe powers of 2 are nice, so perhaps instead of thirds we should do fourths? Or perhaps fix an r and look at $(rn, n]$, $(r^2 n, rn]$, ... for some universal constant r. Using this and the pairing method described above, what is the largest lower bound attainable. In other words, what value of r maximizes the lower bound for the product.

Our proof in this section was *almost* entirely elementary. We used calculus in one step: we needed to know that $\sum_{k=0}^{\infty} kx^k$ equals $x/(1-x)^2$. Fortunately it's possible to prove this result *without* resorting to calculus. All we need is our work on memoryless processes from the basketball game of §1.2. I'll outline the argument in Exercise 18.8.19.

18.5.3 Lower bounds towards Stirling, II

We continue seeing just how far we can push elementary arguments. Of course, in some sense there is no need to do this; there are more powerful approaches that yield better results with less work. As this is true, we're left with the natural, nagging question: *why spend time reading this?*

There are several reasons for giving these arguments. Even though they're weaker than what we can prove, they need less machinery. To prove Stirling's formula, or good bounds towards it, requires results from calculus, real and complex analysis; it's nice to see what we can do just from basic properties of the integers. Second, there are numerous problems where we just need some simple bound. By carefully going through these pages, you'll get a sense of how to generate such elementary bounds, which we hope will help you in something later in life.

Again, the rest of the material in this subsection is advanced and not needed in the rest of the book. You may safely skip it, but I urge you to at least skim these arguments.

We now generalize our argument showing that $n! > (n/4)^n$ for $n = 2^N$ to any integer n; in other words, it was harmless assuming n had the special form $n = 2^N$. Suppose $2^k < n < 2^{k+1}$. Then we can write $n = 2^k + m$ for some positive $m < 2^k$, and use our previous result to conclude

$$n! \;=\; n \cdot (n-1) \cdots (2^k+1) \cdot (2^k)! \;>\; (2^k)^m \cdot (2^k)! \;>\; (2^k)^m \cdot (2^k/4)^{2^k}.$$

Our goal, then, is to prove that this quantity is greater than $(n/4)^n$. Here's one possible method: write

$$2^{km} \cdot (2^k/4)^{2^k} \;=\; (n/4)^{\alpha}.$$

If $\alpha > n$, then we're done. Taking logarithms, we find

$$k \cdot m \cdot \log 2 + 2^k \cdot \log(2)(k-2) \;=\; \alpha(\log(n) - 2\log 2).$$

Solving for α gives

$$\alpha = \frac{k \cdot m \cdot \log 2 + 2^k \cdot \log(2)(k-2)}{\log(n) - 2\log 2}.$$

Remember, we want to show that $\alpha > n$. Substituting in our prior expression $n = 2^k + m$, this is equivalent to showing

$$\frac{k \cdot m \cdot \log 2 + 2^k \cdot \log(2)(k-2)}{\log(2^k + m) - 2\log 2} > 2^k + m.$$

So long as $2^k + m > 4$, the denominator is positive, so we may multiply through without altering the inequality:

$$\log(2)(k(2^k + m) - 2^{k+1}) > (2^k + m)\log(2^k + m) - \log(2)2^{k+1} - 2m\log 2.$$

With a bit of algebra, we can turn this into a nicer expression:

$$\begin{aligned} \log(2^k)(2^k + m) &> (2^k + m)(\log(2^k + m) - 2m\log 2 \\ 2m\log 2 &> (2^k + m)\log(1 + m/2^k) \\ 2\log 2 &> (1 + 2^k/m)\log(1 + m/2^k). \end{aligned}$$

Let's write $t = m/2^k$. Then showing that $\alpha > n$ is equivalent to showing

$$2\log 2 > (1 + 1/t)\log(1 + t)$$

for $t \in (0,1)$. Why $(0,1)$? Since we know $0 < m < 2^k$, then $0 < m/2^k < 1$, so t is always between 0 and 1. While we're only really interested in whether this equation holds when t is of the form $m/2^k$, if we can prove it for all t in (0,1), then it automatically holds for the special values we care about. Letting $f(t) = (1 + 1/t)\log(1 + t)$, we see $f'(t) = (t - \log(1 + t))/t^2$, which is positive for all $t > 0$ (fun exercise: show that the limit as t approaches 0 of $f'(t)$ is $1/2$). Since $f(1) = 2\log 2$, we see that $f(t) < 2\log 2$ for all $t \in (0,1)$. Therefore $\alpha > n$, so $n! > (n/4)^n$ for all integer n.

18.5.4 Lower bounds towards Stirling, III

Again, this subsection may safely be skipped; it's the last in our chain of seeing just how far elementary arguments can be pushed. Reading this is a great way to see how to do such arguments, and if you continue in probability and mathematics there is a good chance you'll have to argue along these lines someday.

We've given a few proofs now showing that $n! > (n/4)^n$ for any integer n. However, we know that Stirling's formula tells us that $n! > (n/e)^n$. Why have we been messing around with 4, then, and where does e come into play? The following sketch doesn't *prove* that $n! > (n/e)^n$, but hints suggestively that e might come enter into our equations.

In our previous arguments we've taken n and then broken the number line up into the following intervals: $\{[n, n/2), [n/2, n/4), \dots\}$. The issue with this approach is that $[n, n/2)$ is a pretty big interval, so we lose a fair amount of information by approximating $n \cdot (n-1) \cdots \frac{n}{2}$ by $(n/2)^{n/2}$. It would be better if we could use a smaller interval. Therefore, let's think about using some ratio $r < 1$, and suppose $n = (1/r)^k$. We would like to divide the number line into $\{[n, rn), [rn, r^2n), \dots\}$, although the problem we run into is that $r^\ell n$ isn't always going to be an integer for every integer $\ell < k$. Putting that issue aside for now (*this is why this isn't a proof!*), let's proceed as we typically do: having broken up the number line, we want to say that $n!$ is greater than the product of the smallest numbers in each interval raised to the number of integers in that interval:

$$n! \;>\; (rn)^{(1-r)n}(r^2n)^{r\cdot(1-r)n}\cdot(r^3n)^{r^2\cdot(1-r)n}\cdots(r^k\cdot n)^{r^{k-1}\cdot(1-r)n}.$$

Since $r^{k+m}n < 1$ for all $m > 1$, we can extend this product to infinity:

$$n! \;>\; (rn)^{(1-r)n}(r^2n)^{r\cdot(1-r)n}\cdot(r^3n)^{r^2\cdot(1-r)n}\cdots(r^k\cdot n)^{r^{k-1}\cdot(1-r)n}\cdots.$$

While this lowers our value, it shouldn't change it too much. The reason is that $\lim_{x\to 0} x^x = 1$. Let's simplify this a bit. Looking at the n terms, we have

$$n^{(1-r+r-r^2+r^2-\cdots)n} \;=\; n^n$$

because the sum **telescopes**. Looking at the r terms we see

$$\begin{aligned} r^{n(1-r)(1+2r+3r^2+\cdots)} &= r^{n(1-r)/r(r+2r^2+3r^3+\cdots)} \\ &= r^{n(1-r)/r\cdot r/(1-r)^2} \\ &= r^{n/(1-r)}, \end{aligned}$$

where in the third step we use the identity

$$\sum_{k=1}^{\infty} kr^k \;=\; \frac{r}{(1-r)^2};$$

remember we used this identity earlier as well! Combining the two terms, we have

$$n! \;>\; (r^{1/(1-r)}n)^n.$$

To make this inequality as strong as possible, we want to find the largest possible value of $r^{1/(1-r)}$ for $r \in (0,1)$. Substituting $x = 1/(1-r)$, this becomes: what is the limit as $x \to \infty$ of $(1-1/x)^x$? Hopefully you've encountered this limit before; the first exposure to it is often from continuously compounded interest. It's just e^{-1} (see §B.3). There are two definitions of e^x, one as a series and one as this limit. Thus we see that this argument gives a heuristic proof (remember we only looked at special n that were a power of r) that $n! > (n/e)^n$.

Math 383: Complex Analysis: Fall ‘23 (Williams)

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Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 25: 11-03-23: No Class

Lecture 26: 11-06-23: https://youtu.be/mio4O-DpD2U

- Continuation of Zeta(s): multiple ways illustrating different insights

Plan for the day: Lecture 25: November 6, 2023:

- Continuation of Zeta(s)
- Theta Functions
- Gregory-Leibniz Formula
- Intro to Fourier Series:

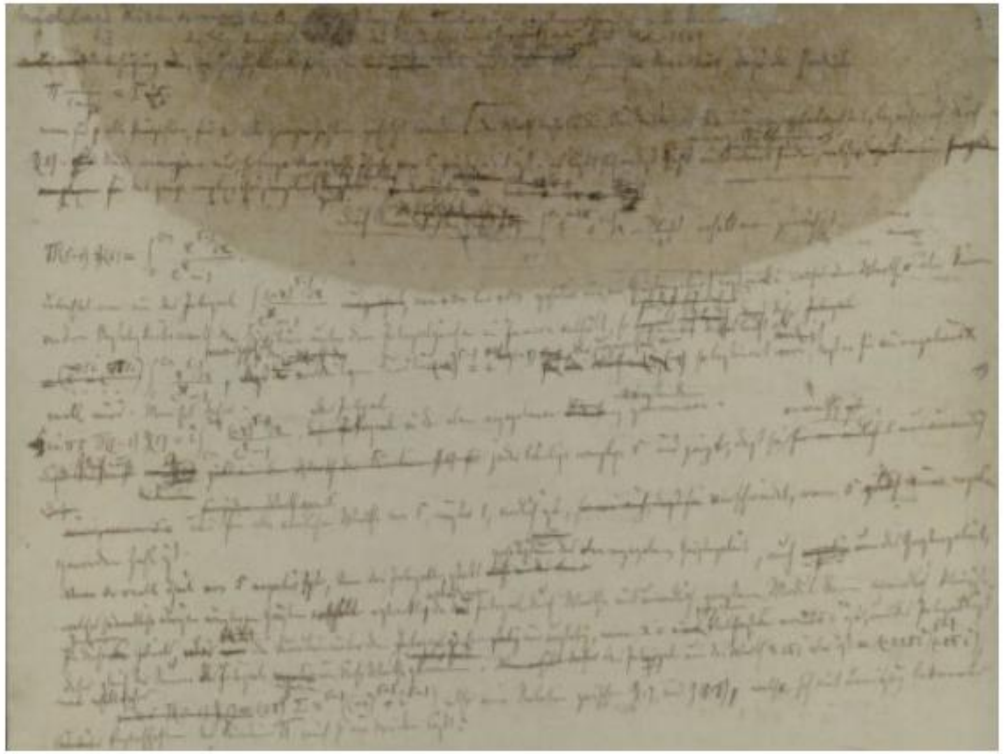
General items.

- Find the source of "miracles"

https://www.claymath.org/collections/riemanns-1859-manuscript/

German version:
https://www.claymath.org/sites/default/files/zeta.pdf

English version:
https://www.claymath.org/sites/default/files/ezeta.pdf



denn das Integral $\int d \log \xi(t)$ positiv um den Inbegriff der Werthe von t erstreckt, deren imaginärer Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $\frac{1}{T}$) gleich $\left(T \log \frac{T}{2\pi} - T\right) i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\xi(t) = 0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

because the integral $\int d\log\xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $\frac{1}{T}$) equal to $\left(T\log\frac{T}{2\pi}-T\right)i$; this integral however is equal to the number of roots of $\xi(t)=0$ lying within in this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.

Exercise 3.1.9. *Use the product expansion to prove $\zeta(s) \neq 0$ for $\Re s > 1$; this important property is not at all obvious from the series expansion. While it is clear from the series expansion that $\zeta(s) \neq 0$ for real $s > 1$, what happens for complex s is not apparent.*

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = \prod_p (1 - 1/p^s)^{-1} \quad Re(s) > 1$$

Prove sum converges without calculus! Dyadics!

$$[1, \infty) = [1, 2) \cup [2, 4) \cup [4, 8) \cup \cdots = \bigcup_{n=0}^{\infty} [2^n, 2^{n+1})$$

On $[2^k, 2^{k+1})$: $\frac{|2^{-s}|}{|2^{ks}|} = \left|\frac{1}{2^{(k+1)s}}\right| \leq \left|\frac{1}{n^s}\right| \leq \left|\frac{1}{2^{ks}}\right|$ all $n \in [2^k, 2^{k+1})$

For fixed s, upper and lower bounds differ by $|2^{-s}|$ (fixed!)

$\# [2^k, 2^{k+1}) = 2^k$

so $$\sum \left|\frac{1}{n^s}\right| \leq \sum_{k=0}^{\infty} 2^k \left|\frac{1}{2^{ks}}\right| = \sum_{k=0}^{\infty} \left(\frac{1}{|2^{s-1}|}\right)^k$$

as $Re(s) > 1$, $|2^{s-1}| = 2^{Re(s)} > 1$ so geometric series with ratio < 1, converges.

The following theorem is one of the most important theorems in mathematics:

Theorem 3.1.20 (Analytic Continuation of the Completed Zeta Function). *Define the completed zeta function by*

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s); \tag{3.17}$$

$\xi(s)$, *originally defined for* $\Re s > 1$, *has an analytic continuation to an entire function and satisfies the functional equation* $\xi(s) = \xi(1-s)$.

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$$

$$\lim_{s \to 1^+} \zeta(s) = \infty$$

subtract $\frac{1}{s-1}$

pole of order 1 at $s=1$
with residue 1

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty}\left[n^{-s} - \int_n^{n+1} x^{-s}\,dx\right] = \sum_{n=1}^{\infty}\int_n^{n+1}(n^{-s} - x^{-s})\,dx$$

ok if $Re(s) > 1$

$$n^{-s} = \int_{x=n}^{n+1} n^{-s}\,dx$$

$$|n^{-s} - x^{-s}| = \left|s\underbrace{\int_n^x y^{-1-s}\,dy}\right| \le |s|n^{-1-\sigma}$$

$\sigma = Re(s)$

$$\left.\frac{y^{-s}}{-s}\right|_n^x = \frac{1}{s}\left(n^{-s} - x^{-s}\right)$$

integrand is largest at $y = n$

length is at most 1, so $\left|\int_n^x y^{-1-s}\,dy\right| \le n^{-1-\sigma}\cdot 1$

Yields $\displaystyle\sum_{n=1}^{\infty} |s|\frac{1}{n^{1+\sigma}}$ converges as $\sigma = Re(s) > 0$

ok if $Re(s) > 0$

pulling out blowup at $s=1$

$$\zeta(s) = \sum_{n=1}^{\infty}\left[n^{-s} - \int_n^{n+1} x^{-s}\,dx\right] + \frac{1}{s-1}$$

(if $Re(s) > 0$)

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \left(1 - 2^{1-s}\right)\zeta(s)$$

Dirichlet eta function

$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \cdots\cdots$ $\quad Re(s) > 1$

$\eta(s) = 1 - 1/2^s + 1/3^s - 1/4^s + \cdots\cdots$ $\quad Re(s) > 1$

(maybe can do better)

$2^{1-s}\zeta(s) = \frac{2}{2^s}\zeta(s) = 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots\right)$

$\zeta(s) - 2^{1-s}\zeta(s) = \underbrace{1 - 1/2^s + 1/3^s - 1/4^s + 1/5^s - \cdots}_{\text{This is just } \eta(s)}$

$\zeta(s)\left(1 - 2^{1-s}\right) = \eta(s)$ or $\zeta(s) = \left(1 - 2^{1-s}\right)^{-1}\eta(s)$

(1) Prove $\eta(s)$ converges if $Re(s) > 0$

Consider $\lim_{s \to 1^+} \frac{s-1}{1-2^{1-s}}$

(2) $\eta(1) = 1 - 1/2 + 1/3 - 1/4 + \cdots = \ln(2)$

$\lim_{s\to 1^+} \frac{1}{1-2^{1-s}} = \lim_{s\to 1^+} \frac{1}{1-e^{(1-s)\log 2}} = \lim_{s\to 1^+} \frac{1}{1-[1+(1-s)\log 2 + \cdots]}$

$= \lim_{s\to 1^+} \frac{1}{(s-1)\log 2 + \cdots} \sim \frac{1}{s-1}\,\frac{1}{\log 2}$

$1 + 2 + 4 + 8 + 16 + 32 + \cdots = ?$

(1) ∞

(2) -1 geometric series with $r = 2$ so $\frac{1}{1-2} = -1$

2-adically true!

$$\underbrace{1 + x + x^2 + x^3 + \cdots}_{\text{ok if } |x| < 1} \longleftrightarrow \underbrace{\frac{1}{1-x}}_{\text{ok if } x \neq 1}$$

Continuation of the LHS same when $|x| < 1$, but defined for more x.

Analytic Continuation

Do you believe in miracles? (Or: Do you believe in unlikelihoods?)

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x) - 1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x > 0, \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

removes $n=0$, $-n$ and n counts as one term

$$= \sum_{n=1}^{\infty} e^{-\pi n^2 x}$$

will prove in Fourier Analysis

Is $\theta(x)$ well defined? or $\omega(x)$, with $x > 0$

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \sum_{n=1}^{\infty} \left(e^{-\pi n x}\right)^n$$

$$\le \sum_{n=1}^{\infty} \left(e^{-\pi x}\right)^n$$

Converges: geometric series:
ratio is $|e^{-\pi x}| < 1$

If $x = 0$: TROUBLE! $e^{-\pi \cdot 0} = 1$

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \text{implies} \qquad \int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2\pi x}\, dx = \frac{\Gamma\left(\frac{s}{2}\right)}{n^s \pi^{\frac{s}{2}}}$$

Why care? Sum over n: $\int_0^\infty x^{\frac{1}{2}s-1}\omega(x)\,dx$ $\qquad \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$

Connects $\zeta(s)$ to integral of $\omega(x)$

Prove Claim: Change of variables

$$e^{-t} \Rightarrow e^{-n^2\pi x} \quad \text{by} \quad \begin{array}{l} t = n^2\pi x \\ dt = n^2\pi\, dx \end{array}$$

$$n^{-s}\pi^{-s/2}\int_{x=0}^{\infty} e^{-n^2\pi x}(n^2\pi x)^{\frac{s}{2}-1} n^2\pi\, dx$$

$$\underbrace{n^{-s}\pi^{-s/2}\, n^{s}\pi^{s/2}}_{1}\int_{x=0}^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1}\, dx$$

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right)dx = \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)\,dx$$

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x)-1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x>0. \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

$\pi^{-\frac{1}{2}s}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)$ $\qquad$ $\omega(x) \longleftrightarrow \omega(1/x)$ related

$\int_0^\infty = \int_0^1 + \int_1^\infty$

$\omega(x)$ $\qquad$ $\omega(x)$ $\qquad$ change variables

$t = 1/x$ $\qquad$ $x: 1 \to \infty$

$dt = -1/x^2 \cdot dx$ $\qquad$ $t: 1 \to 0$

or $dx = -t^{-2}\,dt$ $\qquad$ $0 \to 1$

$\int_0^1 x^{\frac{1}{2}s-1}\omega(x) + \int_0^1 x^{\frac{1}{2}s-1}(\omega(1/x) + \text{other stuff})$

Bad: don't go from 0 to 1 as $\omega(x)$ blows up as $x \to 0$. Go from 1 to ∞

$\int_1^\infty$ keep, $\int_0^1 \to \int_1^\infty$

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \underbrace{\frac{1}{s(s-1)}}_{\text{Fixed piece, see pole at } s=1} + \underbrace{\int_1^\infty \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right)\omega(x)\,dx}_{\text{well defined for all } s}$$

rapid decay for $x \geq 1$

$s \to 1-s$ it is unchanged

Continued $\zeta(s)$ to all s, and values at s and $1-s$ are the same if we multiply by factors

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \left(1 - 2^{1-s}\right) \zeta(s)$$

The above is the Dirichlet eta function. It differs from the Riemann zeta function by the presence of (-1) raised to the n-th power, so it alternates in sign (the zeta function has all of its terms +1, while here we alternate +1, -1, …). It is amazing the consequences this has. The series for the zeta function converges if Re(s) > 1, but it does not converge if Re(s) > 0 for all such s; for example, if s = ½ we have the sum of the reciprocals of the square-roots of integers, which clearly diverges!

The eta function converges for all Re(s) > 0 – prove this! The following summation formulas may be useful. These are key tools of analytic number theorists, and are essentially discrete versions of integration by parts (in a shocker, called partial summation). In the argument below A_n = a_M + a_{m+1} + … + a_n.

Lemma 2.1 (Partial Summation: Discrete Version)

$$\sum_{M}^{N} a_n b_n = A_N b_N - A_{M-1} b_M + \sum_{M}^{N-1} A_n (b_n - b_{n+1})$$

Lemma 2.2 (Abel's Summation Formula - Integral Version) *Let $h(x)$ be a continuously differentiable function. Let $A(x) = \sum_{n \le x} a_n$. Then*

$$\sum_{n \le x} a_n h(n) = A(x)h(x) - \int_1^x A(u)h'(u)du$$

LECTURE 27

Class on Wednesday will be asynchronous. Please watch the following two videos (we may have covered a good amount of the first in class on Monday)

- 2021: Lecture 24: 11/12/21: Gregory-Leibniz Formula, Dirichlet L-functions, proof of RH (not!), Duality: https://youtu.be/K8RhtDyts7s (slides)
- 2021: Lecture 25: 11/15/21: Introduction to Fourier Analysis, Approximations to the Identity: https://youtu.be/YiFtCBbYe_I (slides)

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Lecture 28: 11-10-23: https://youtu.be/gfFtYYb4xPQ

- Number Theory and Complex Analysis

See also:

L-Functions and Random Matrix Theory:

- Introductory lectures on Random Matrix Theory and L-functions:
  - Part I (Classical RMT, Intro L-fns, Dirichlet): http://youtu.be/2PuUbk6gUMM (slides: part 1)
  - Part II (Convolving families, cusp forms: slides here): http://youtu.be/vJz6W24tDik (slides part 2)
- From the Manhattan Project to Elliptic Curves: MASON IV (3/7/20). pdf (video here: https://youtu.be/p15X3ERNvLs)
- Introduction to L-functions for SMALL students:

2021: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/intronumbertheory/

Today's lecture was a trimmed / adjusted version of the talk below. For more details and the slides, see below (as well as the links on the previous slide to more related talks). NOTE: The pages following this are the comments I made during class on the details of the contour integrals, finding the residues and contributions…. You are strongly urged to try to write things down with all the details, or read the book, and if you have questions reach out to me.

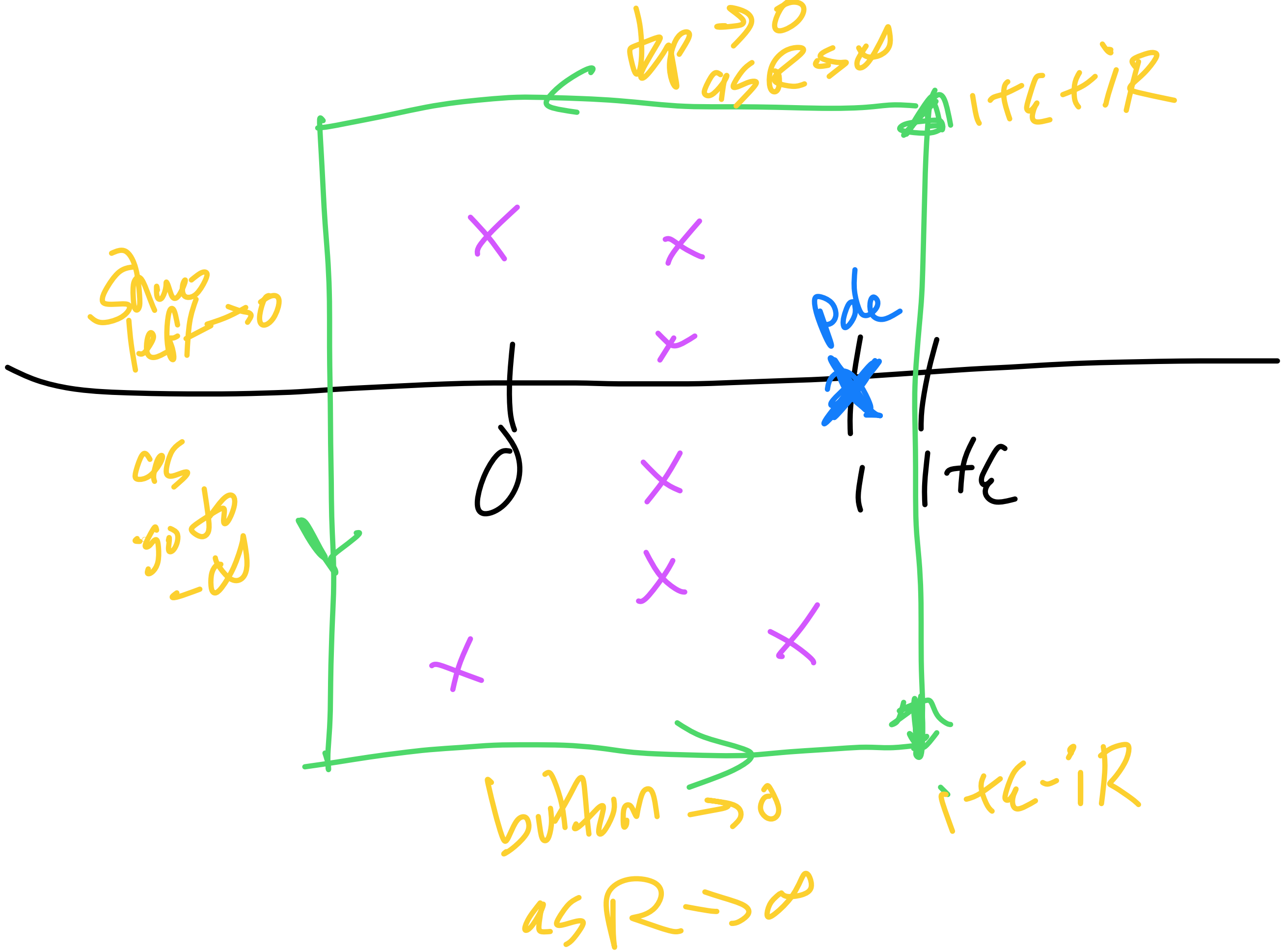
It is also a great exercise to show how sum_{p < x} log p ~ x implies sum_{p < x} 1 ~ x/log(x).

Part I (Classical RMT, Intro L-fns, Dirichlet): http://youtu.be/2PuUbk6gUMM (slides: part 1)

See the link to the slides above for most of the slides from today's lecture; there were a few comments on the final slides discussed, which follow.

Note: in case the hyperlink is not working, here is the link:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/Michigan2012Part1.pdf

$$\int -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}\, ds = \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{dx}{s} + \text{"Good"}$$



$$-\left(-\frac{x^1}{1}\right) = x$$

from pole of zeta at $s=1$

$$\zeta(s) = \frac{1}{s-1} + \cdots$$

zeros give $-\dfrac{x^\rho}{\rho}$

when $\zeta(\rho)=0$

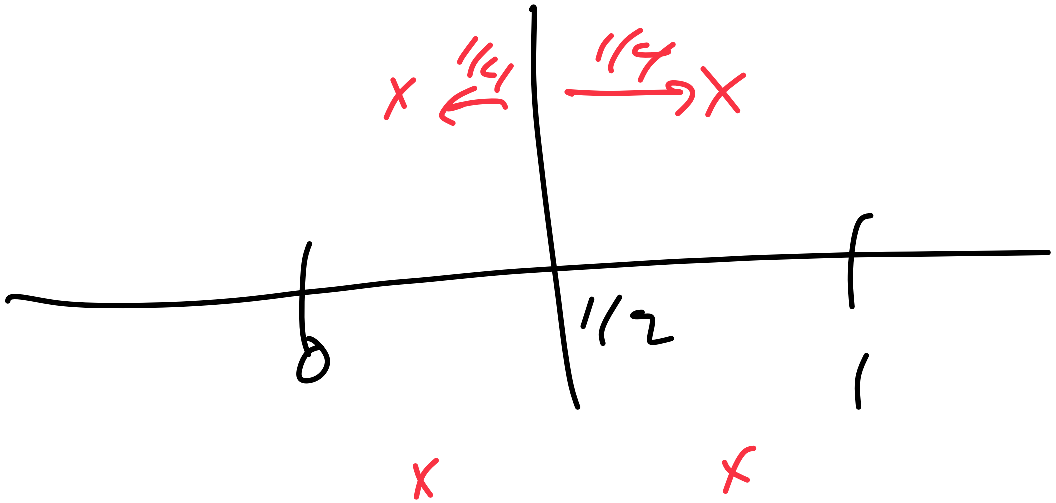
$$g(s) = (s-1)^{-n} + a(s-1)^{-1+1} + b(\ -\ -\ -\ )$$

$$= (s-1)^{-n}\underbrace{\left(1 + a(s-1) + b(s-1)^{2} + \cdots\right)}$$

$$\frac{g'(s)}{g(s)} = \frac{-n(s-1)^{-n-1}}{(s-1)^{-n}} = -\frac{n}{s-1}$$

$$x - \sum_{\substack{0 < \mathrm{Re}(\rho) < 1 \\ \zeta(\rho) = 0}} \frac{x^{\rho}}{\rho} + \cdots$$

zeros outside critical strip (- even #s)



$$\sum_{p \ldots} \log p \int_{Re(s)=1+\varepsilon} \underbrace{\left(\frac{x}{p}\right)^s \frac{dx}{s}}$$

if $p < x$ is 1

if $p > x$ is 0

$$= \sum_{p \leq x} \log p$$

$$x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \cdots = \sum_{p < x} \log p + \cdots$$

$$\sum_{p \le x} \log p \sim x$$

$$\sum_{p \le x} 1 \sim \frac{x}{\log x}$$

Prime Number Theorem

Partial Summation

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Lecture 29: 11-10-23: https://youtu.be/8TIW3bFr6XE

•Number Theory and Complex Analysis, Introduction to Fourier Analysis

See also:

L-Functions and Random Matrix Theory:

- •Introductory lectures on Random Matrix Theory and L-functions:
- •Part I (Classical RMT, Intro L-fns, Dirichlet): http://youtu.be/2PuUbk6gUMM (slides: part 1)
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•Introduction to L-functions for SMALL students:

2021: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/intronumbertheory/

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p\log\left(1-p^{-s}\right) \\
&= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \text{Good}(s).
\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.
$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
\end{aligned}
$$

$s = \sigma + it$

Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \left[\phi(s)e^{-\sigma\log p}\right] e^{-it\log p} ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Riemann Zeta Function: Let $\sum_\rho$ denote the sum over the zeros of $\zeta(s)$ in the critical strip, g an even Schwartz function of compact support and $\phi(r) = \int_{-\infty}^{\infty} g(u)e^{iru}du$. Then

$$\sum_\rho \phi(\gamma_\rho) \;=\; 2\phi\left(\frac{i}{2}\right) - \sum_p \sum_{k=1}^{\infty} \frac{2\log p}{p^{k/2}} g\left(k \log p\right)$$

$$+ \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{1}{iy - \frac{1}{2}} + \frac{\Gamma'(\frac{iy}{2} + \frac{5}{4})}{\Gamma(\frac{iy}{2} + \frac{5}{4}))} - \frac{1}{2}\log \pi\right) \phi(y)\, dy.$$

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma \in \mathbb{R}$. Then

$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

$$\sum_\rho h\left(\gamma_\rho \frac{\log(m/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} h(y)dy$$

$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}}$$

$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\left(\frac{1}{\log m}\right).$$

Theorem 3.1.20 (Analytic Continuation of the Completed Zeta Function). *Define the completed zeta function by*

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s);$$

$\xi(s)$, originally defined for $\Re s > 1$, has an analytic continuation to an entire function and satisfies the functional equation $\xi(s) = \xi(1-s)$.

Do you believe in miracles? (Or: Do you believe in unlikelihoods?)

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x)-1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x > 0, \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

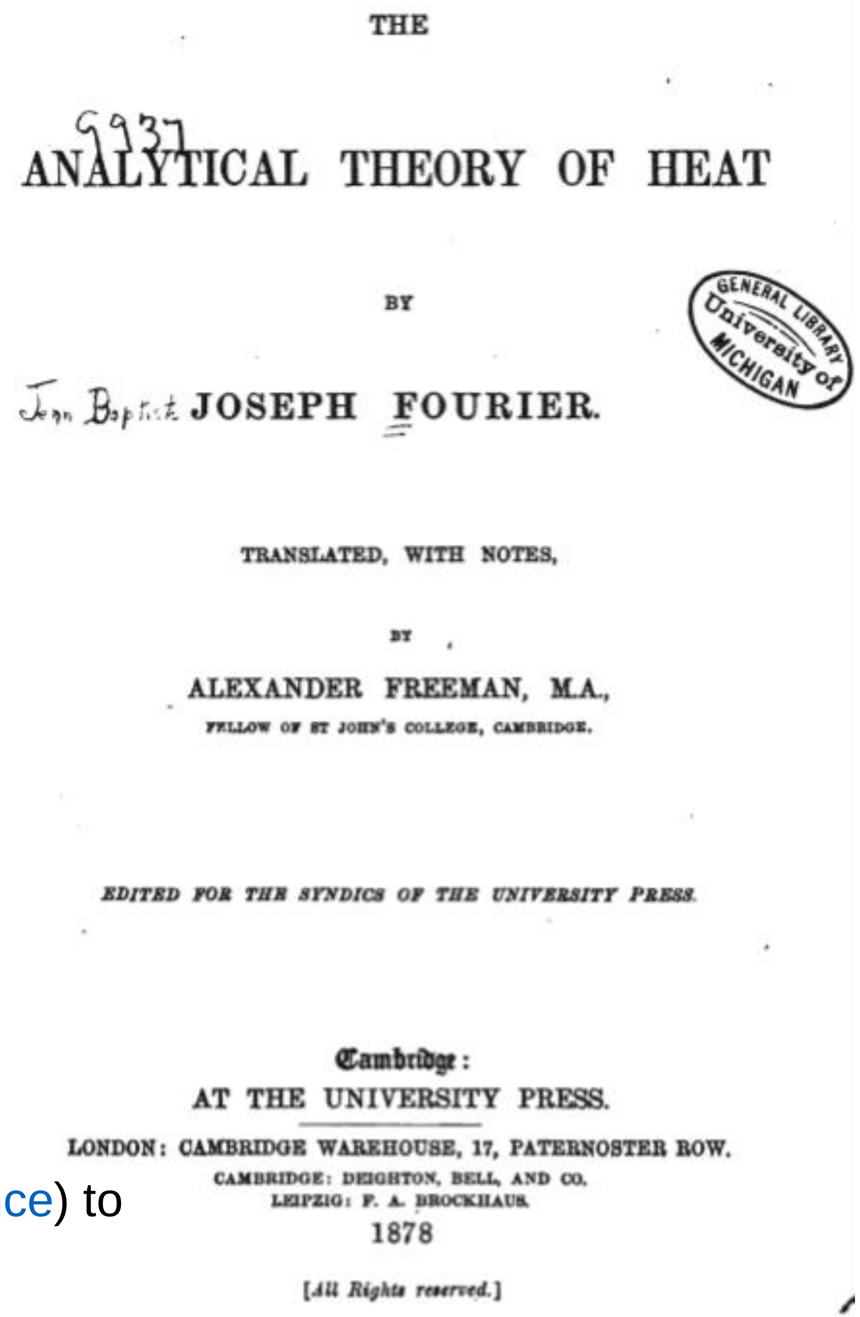
$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right)dx = \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)\,dx$$

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x)-1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x > 0, \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

Introduction to Fourier Series

https://www3.nd.edu/~powers/ame.20231/fourier1878.pdf

Jean-Baptiste Joseph Fourier



THE

9937

ANALYTICAL THEORY OF HEAT

BY

Jean Baptiste JOSEPH FOURIER.

TRANSLATED, WITH NOTES,

BY

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1878

21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France) to
16 May 1830 Paris, Kingdom of France

Inner or dot product:

$$\vec{v}\cdot\vec{w} = \langle \vec{v},\vec{w}\rangle = \sum_{i=1}^{n} v_i \overline{w}_i$$

else $\langle 1, i\rangle \cdot \langle 1, i\rangle = 1^2 + i^2 = 0$ DO NOT WANT

$f \leftrightarrow \langle f(\tfrac{0}{n}), f(\tfrac{1}{n}), \ldots, f(\tfrac{n-1}{n}), f(\tfrac{n}{n})\rangle$ $\quad g \leftrightarrow$ similar

$$f \cdot_n g = \sum_{k=0}^{n} f(\tfrac{k}{n})\,\overline{g(\tfrac{k}{n})}\,\frac{1}{n}$$

has a chance of a nice limit; if not, $f(x) = g(x) \equiv 1$ has $1 \cdot 1 \to \infty$

$$\lim_{n\to\infty} f \cdot_n g = \int_0^1 f(x)\,\overline{g(x)}\,dx =: f \cdot g \quad \text{for fns on } [0,1]$$

Check: $\int_0^1 \sin(2\pi x)\cos(2\pi x)\,dx = \frac{1}{2}\int_0^1 \sin(4\pi x)\,dx = 0$

What about $\sin(2\pi n x)$ and $\cos(2\pi n x)$? Or $\{e^{2\pi i n x}\}_{n=0}^{\infty}$

Exercise 11.1.3. *Let f, g and h be continuous functions on $[0,1]$, and $a, b \in \mathbb{C}$. Prove*

1. $\langle f, f\rangle \geq 0$, *and equals* 0 *if and only if* f *is identically zero;*
2. $\langle f, g\rangle = \overline{\langle g, f\rangle}$*;*
3. $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$.

Definition 11.1.5 (Orthogonal). *Two continuous functions on $[0,1]$ are orthogonal (or perpendicular) if their inner product equals zero.*

$$\oint_\gamma z^k \, dz = \begin{cases} 2\pi i & \text{if } k = -1 \\ 0 & \text{oth} \end{cases}$$

$$\int_{-1/2}^{1/2} e^{2\pi i(m-n)x} \, dx$$

For us, we use:

$$e_n(x) = e^{2\pi i n x} \qquad \langle e_m(x), e_n(x)\rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

$x \in [0,1]$ $\quad \in [-1/2, 1/2]$

$z = e^{i\theta}$

$dz = i e^{i\theta} \, d\theta$

$= i z \, d\theta$

Definition 11.1.8 (Periodic). *A function $f(x)$ is periodic with period a if for all $x \in \mathbb{R}$, $f(x+a) = f(x)$.*

Let f be continuous and periodic on $\mathbb{R}$ with period one. Define the n^{th} **Fourier coefficient** $\widehat{f}(n)$ of f to be

$e_n(x) = e^{2\pi i n x}$

$$\widehat{f}(n) = \langle f(x), e_n(x) \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx. \tag{11.13}$$

Returning to the intuition of $\mathbb{R}^m$, we can think of the $e_n(x)$'s as an infinite set of perpendicular unit directions. The above is simply the projection of f in the direction of $e_n(x)$. Often one writes a_n for $\widehat{f}(n)$.

Exercise 11.2.1. *Show*

$$\langle f(x) - \widehat{f}(n) e_n(x), e_n(x) \rangle = 0. \tag{11.14}$$

This agrees with our intuition: after removing the projection in a certain direction, what is left is perpendicular to that direction.

The N^{th} **partial Fourier series** of f is

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n) e_n(x).$$

l_p-norm: $\left(\int_0^1 |f(x)|^p dx\right)^{1/p}$

$$\widehat{f}(n) = \langle f(x), e_n(x) \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx$$

Exercise 11.2.2. *Prove* (assume f is periodic)

$|\widehat{f}(n)| \le \int_0^1 |f(x) e^{-2\pi i n x}| dx$
$= \int_0^1 |f(x)| dx$

1. $\langle f(x) - S_N(x), e_n(x) \rangle = 0$ *if* $|n| \le N$.

2. $|\widehat{f}(n)| \le \int_0^1 |f(x)| dx$. $= \|f\|_1$ L_1-norm on $[0,1]$

3. *Bessel's Inequality: if* $\langle f, f \rangle < \infty$ *then* $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \le \langle f, f \rangle = \|f\|_2^2$ $= \int_0^1 |f(x)|^2 dx$

4. *Riemann-Lebesgue Lemma: if* $\langle f, f \rangle < \infty$ *then* $\lim_{|n| \to \infty} \widehat{f}(n) = 0$ *(this holds for more general* f*; it suffices that* $\int_0^1 |f(x)| dx < \infty$*).*

there is an N and a C st for all $|n| > N$ have $|\widehat{f}(n)| \le \frac{C}{n^k}$

5. *Assume* f *is differentiable* k *times; integrating by parts, show* $|\widehat{f}(n)| \ll \frac{1}{|n|^k}$ *and the constant depends only on* f *and its first* k *derivatives.*

Give examples of L^1 and L^2: when is one contained in the other?

$[0,1]$

$$\int_0^1 \frac{1}{x^r}\,dx = \text{``}\frac{x^{1-r}}{1-r}\Big|_0^1\text{''}$$

infinite for $\infty > r \geq 1$

finite for $r < 1$

Consider $f(x) = 1/\sqrt{x}$

$f(x) \in L_1([0,1])$ as $\int_0^1 |f(x)|\,dx < \infty$

but $f(x) \notin L_2([0,1])$ as $\int_0^1 |f(x)|^2\,dx = \infty$

$L_1([0,1]) \not\subset L_2([0,1])$

Is $L_2([0,1]) \subset L_1([0,1])$?

Know $\int_0^1 |f(x)|^2\,dx < \infty$

Claim:

$|f(x)| \leq |f(x)|^2 + 1$

$\hookrightarrow$ if $|f| < 1$ ok

if $|f| \geq 1$ then $|f|^2 \geq |f|$

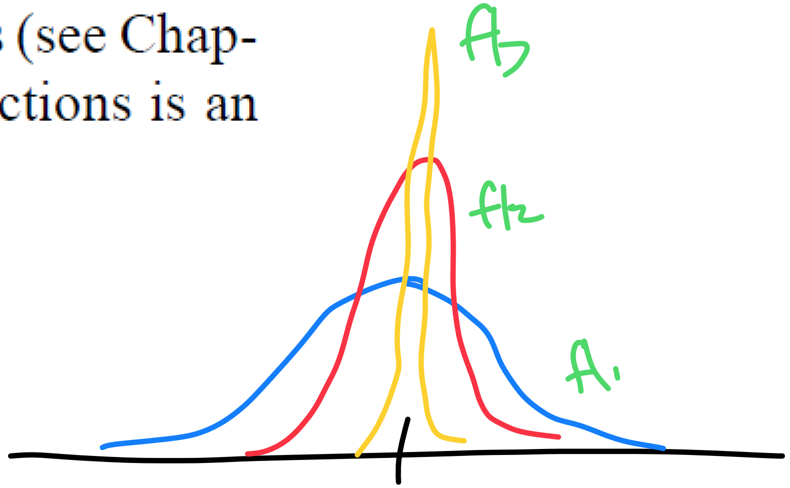
Have $\int_0^1 |f(x)|\,dx \leq \int_0^1 (|f|^2 + 1)\,dx$

$< \infty$

So yes, contained!

Note: Finite interval

We assume the reader is familiar with the basics of probability functions (see Chapter 8, especially §8.2.3). A sequence $A_1(x), A_2(x), A_3(x), \ldots$ of functions is an **approximation to the identity on** $[0, 1]$ if



1. for all x and N, $A_N(x) \geq 0$;
2. for all N, $\int_0^1 A_N(x)dx = 1$;
3. for all δ, $0 < \delta < \frac{1}{2}$, $\lim_{N\to\infty} \int_\delta^{1-\delta} A_N(x)dx = 0$.

Similar definitions hold with $[0, 1]$ replaced by other intervals; it is often more convenient to work on $[-\frac{1}{2}, \frac{1}{2}]$, replacing the third condition with

$$\lim_{N\to\infty} \int_{|x|>\delta} A_N(x)dx \;=\; 0 \quad \text{if} \quad 0 < \delta < \frac{1}{2}. \tag{11.17}$$

Assume $f(x)$ is a probability distribution.

Let $A_N(x) = \cancel{\frac{1}{N} f\left(\frac{x}{N}\right)} = N f(N x)$

often take f is the standard normal

Math 383: Complex Analysis: Fall '23 (Williams)

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Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 30: 11-15-23: https://youtu.be/sGUVKUYhEY0

•Fourier Analysis II: Convergence Theorems

Material taken from my book "Invitation to Modern Number Theory" (joint with Ramin Takloo-Bighash).

Exercise 11.2.8 (Important). *Let $A_N(x)$ be an approximation to the identity on $[-\frac{1}{2}, \frac{1}{2}]$. Let $f(x)$ be a* (periodic) *continuous function on $[-\frac{1}{2}, \frac{1}{2}]$. Prove*

$$\lim_{N\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)A_N(x)dx \;=\; f(0). \tag{11.21}$$

$A_N(-x)$ is also an approx to the identity

$$\int_{-1/2}^{1/2} f(x)A_N(x)dx \Leftrightarrow \int_{-1/2}^{1/2} f(x)A_N(0-x)dx$$ Same limit as $N\to\infty$

Study $\int_{-1/2}^{1/2} f(x)A_N(y-x)dx$ only x near y matters!

$$(f*A_N)(y) := \int_{-1/2}^{1/2} f(x)A_N(y-x)dx$$

$t = y-x$

$x = y-t$

$$= \int_{-1/2}^{1/2} f(y-t)A_N(t)dt$$

f is periodic, continuous, A_N is an approx to the identity

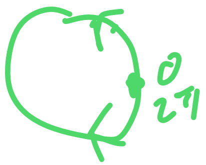
$$\int_{-1/2}^{1/2} f(y-x)\,A_N(x)\,dx - f(y)\cdot 1, \quad \text{with } 1 = \int_{-1/2}^{1/2} A_N(x)\,dx$$

$$= \int_{-1/2}^{1/2} \big[f(y-x) - f(y)\big] A_N(x)\,dx$$

$$= \int_{|x|\le\delta} \big[f(y-x) - f(y)\big] A_N(x)\,dx + \int_{\delta\le|x|\le\frac{1}{2}} \big[f(y-x) - f(y)\big] A_N(x)\,dx$$

By unif continuity, given $\varepsilon > 0$ $\exists\,\delta$ (indep of point) st if $|a-b| < \delta \Rightarrow |f(a) - f(b)| < \frac{\varepsilon}{2}$

$|f(y-x) - f(x)| \le 2\max|f|$



$$|\text{above}| \le \frac{\varepsilon}{2} \int_{|x|\le\delta} A_N(x)\,dx \le \varepsilon/2$$

$$|\text{above}| \le 2\max|f| \int_{\delta\le|x|\le\frac{1}{2}} A_N(x)\,dx$$

choose N so large that integral is at most $\dfrac{\varepsilon/2}{2\max|f|+1}$

11.2.3 Dirichlet and Fejér Kernels

We define two functions which will be useful in investigating convergence of Fourier series. Set

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

$$F_N(x) := \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{\sin^2(N\pi x)}{N \sin^2 \pi x}.$$

$e_n(x)$ is $e^{2\pi i n x}$

To compute use geometric series with $r = e^{2\pi i x}$

$$\sum_{n=-N}^{N} e^{2\pi i n x} = \sum_{n=-N}^{N} (e^{2\pi i x})^n$$

Taylor expand near origin, $F_n(x) \geq 0$ possible

$D_N(x)$ no chance

Here F stands for Fejér, D for Dirichlet. $F_N(x)$ and $D_N(x)$ are two important examples of (integral) **kernels**. By integrating a function against a kernel, we obtain a new function related to the original. We will study integrals of the form

$$g(x) = \int_0^1 f(y)K(x-y)dy. \tag{11.25}$$

Such an integral is called the **convolution** of f and K. The Fejér and Dirichlet kernels yield new functions related to the Fourier expansion of $f(x)$.

$$(f * D_N)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}$$

$$(f * F_N)(x) = \sum_{n=-N}^{N} \underbrace{\left(1 - \frac{|n|}{N}\right)}_{\text{weights}} \hat{f}(n) e^{2\pi i n x}$$

Theorem 11.2.11. *The Fejér kernels $F_1(x), F_2(x), F_3(x), \ldots$ are an approximation to the identity on $[0, 1]$.*

integrates to 1

integrate to 0

$$F_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{\overset{\leq 1}{\sin^2(N\pi x)}}{\underbrace{N\sin^2 \pi x}} \qquad F_N(x) = e_0(x) + \frac{N-1}{N}(e_{-1}(x) + e_1(x)) + \cdots$$

$\leq \frac{1}{N \sin^2 \pi x} \leq \frac{1}{N \sin^2 \pi \delta}$ if $|x| \geq \delta$

Shows the $\int_{|x|>\delta} F_N(x)dx$ goes to 0.

The Dirichlet kernels are not *an approximation to the identity.*

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

Theorem 11.3.1 (Fejér). *Let $f(x)$ be a continuous, periodic function on $[0,1]$. Given $\epsilon > 0$ there exists an N_0 such that for all $N > N_0$,*

$$|f(x) - T_N(x)| \leq \epsilon \tag{11.28}$$

for every $x \in [0,1]$. Hence as $N \to \infty$, $\underbrace{T_N f(x)}_{(f * F_N)(x)} \to f(x)$.

ALREADY DONE AS

F_N IS AN APPROX TO THE IDENTITY

$$\underbrace{(f * F_N)(x)}_{T_N(x)} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n)\, e^{2\pi i n x}$$

Definition 11.3.3 (Trigonometric Polynomials). *Any finite linear combination of the functions $e_n(x)$ is called a trigonometric polynomial.*

From Fejér's Theorem (Theorem 11.3.1) we immediately obtain the

Theorem 11.3.4 (Weierstrass Approximation Theorem). *Any continuous periodic function can be uniformly approximated by trigonometric polynomials.*

Remark 11.3.5. Weierstrass proved (many years before Fejér) that if f is continuous on $[a, b]$, then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$. This important theorem has been extended numerous times (see, for example, the Stone-Weierstrass Theorem in [Rud]).

Exercise 11.3.6. *Prove the Weierstrass Approximation Theorem implies the original version of Weierstrass' Theorem (see Remark 11.3.5).*

https://en.wikipedia.org/wiki/Stone%E2%80%93Weierstrass_theorem https://mast.queensu.ca/~speicher/Section14.pdf

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} \qquad S_N(x_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) D_N(x - x_0) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_0 - x) D_N(x) dx.$$

Theorem 11.3.8 (Dirichlet). *Suppose*

1. $f(x)$ *is real valued and periodic with period* 1*;*
2. $|f(x)|$ *is bounded;*
3. $f(x)$ *is differentiable at* x_0.

Then $\lim_{N\to\infty} S_N(x_0) = f(x_0)$.

$$S_N(x) = (f * D_N)(x)$$

Proof. Let $D_N(x)$ be the Dirichlet kernel. Previously we have shown that $D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$ and $\int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(x)dx = 1$. Thus

$$\begin{aligned} f(x_0) - S_N(x_0) &= f(x_0)\int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(x)dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_0 - x)D_N(x)dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x_0) - f(x_0 - x)]\, D_N(x)dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(x_0) - f(x_0 - x)}{\sin(\pi x)} \cdot \sin((2N+1)\pi x)dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)\sin((2N+1)\pi x)dx. \end{aligned} \tag{39}$$

We claim $g_{x_0}(x) = \frac{f(x_0)-f(x_0-x)}{\sin(\pi x)}$ is bounded. As f is bounded, the numerator is bounded. The denominator is only troublesome near $x = 0$; however, as f is differentiable at x_0,

$$\lim_{x\to 0} \frac{f(x_0 + x) - f(x_0)}{x} = f'(x_0). \tag{40}$$

Multiplying by 1 in a clever way (one of the most useful proof techniques) gives

$$\lim_{x\to 0} \frac{f(x_0 + x) - f(x_0)}{\sin(\pi x)} = \lim_{x\to 0} \frac{f(x_0 + x) - f(x_0)}{\pi x} \cdot \frac{\pi x}{\sin(\pi x)} = \frac{f'(x_0)}{\pi}, \tag{41}$$

where we used L'Hospital's rule to conclude that $\lim_{x\to 0} \frac{\pi x}{\sin(\pi x)} = 1$. Therefore $g_{x_0}(x)$ is bounded everywhere, say by B. As g_{x_0} is a bounded function, it is square-integrable, and thus the Riemann-Lebesgue Lemma (see Exercise 1.13) implies that its Fourier coefficients tend to zero. This completes the proof, as

$$i\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)\sin((2N+1)\pi x)dx = \Im\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)e^{2\pi i(2N+1)x}dx\right); \tag{42}$$

thus our integral is just the imaginary part of the $2N + 1^{\text{st}}$ Fourier coefficient, which tends to zero as $N \to \infty$. Hence as $N \to \infty$, $S_N(x_0)$ converges (pointwise) to $f(x_0)$.

Theorem 11.3.11 (Parseval's Identity). *Assume* $\int_0^1 |f(x)|^2 dx < \infty$. *Then*

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \;=\; \int_0^1 |f(x)|^2 dx.$$

One common application of pointwise convergence and Parseval's identity is to evaluate infinite sums. For example, if we know at some point x_0 that $S_N(x_0) \to f(x_0)$, we obtain

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x_0} \;=\; f(x_0).$$

Additionally, if $\int_0^1 |f(x)|^2 dx < \infty$ we obtain

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \;=\; \int_0^1 |f(x)|^2 dx.$$

Thus, if the terms in a series correspond to Fourier coefficients of a "nice" function, we can evaluate the series.

Exercise 11.3.15. *Let* $f(x) = \frac{1}{2} - |x|$ *on* $[-\frac{1}{2}, \frac{1}{2}]$. *Calculate* $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. *Use this to deduce the value of* $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *This is often denoted* $\zeta(2)$ *(see Exercise 3.1.7). See [BP] for connections with continued fractions, and [Kar] for connections with quadratic reciprocity.*

Exercise 11.3.16. *Let* $f(x) = x$ *on* $[0, 1]$. *Evaluate* $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Exercise 11.3.17. *Let* $f(x) = x$ *on* $[-\frac{1}{2}, \frac{1}{2}]$. *Prove* $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$. *See also Exercise 3.3.29; see Chapter 11 of [BB] or [Sc] for a history of calculations of* π.

Exercise 11.3.18. *Find a function to determine* $\sum_{n=1}^{\infty} \frac{1}{n^4}$*; compare your answer with Exercise 3.1.26.*

FOURIER TRANSFORM

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\, dx$$

The **Schwartz Space** $\mathcal{S}(\mathbb{R})$ is the space of all infinitely differentiable functions whose derivatives are rapidly decreasing. Explicitly,

$$\forall j, k \geq 0, \quad \sup_{x \in \mathbb{R}} (|x| + 1)^j |f^{(k)}(x)| < \infty.$$

Ex: e^{-x^2}

Ex: $\begin{cases} e^{-\frac{1}{(x-a)(b-x)}} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

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https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 31: 11-17-23: https://youtu.be/d4QZPmR_LXA

- Fourier Analysis III: Poisson Summation, Probability

Notes from my book with Ramin Takloo-Bighash: An Invitation to Modern Number Theory (see also my book page here)

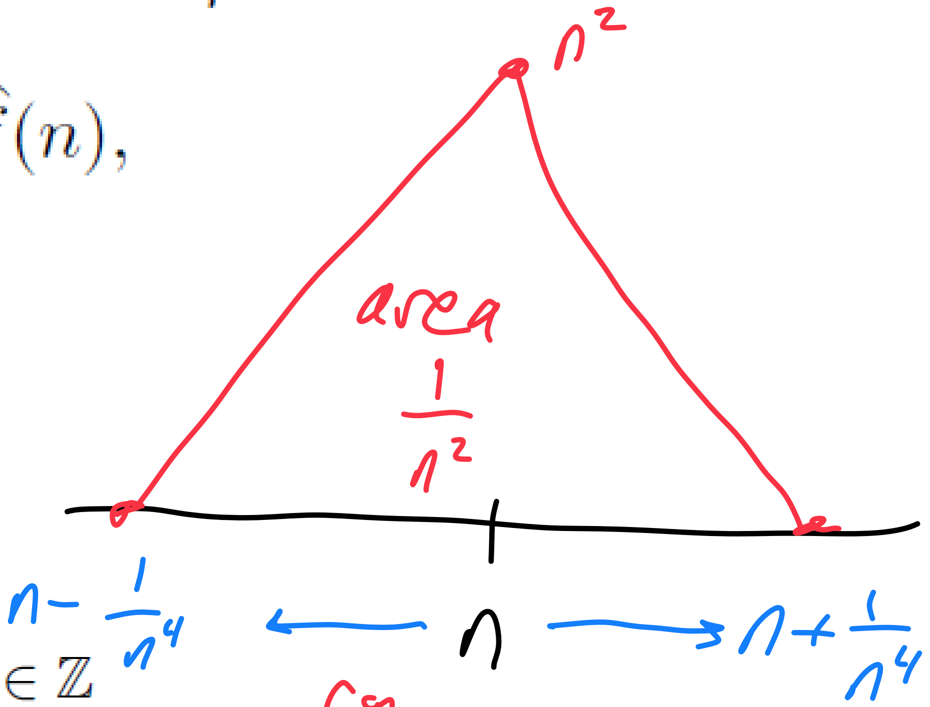
We say a function $f(x)$ decays like x^{-a} if there are constants x_0 and C such that for all $|x| > x_0$, $|f(x)| \leq C/|x|^a$.

Theorem 11.4.6 (Poisson Summation). *Assume f is twice continuously differentiable and that f, f' and f'' decay like $x^{-(1+\eta)}$ for some $\eta > 0$. Then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where $\hat{f}$ is the Fourier transform of f.

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$



$$\int_{-\infty}^{\infty} f(x)\,dx < \infty$$

Exercise 11.4.7. *Consider*

$$f(x) = \begin{cases} n^6\left(\frac{1}{n^4} - |n-x|\right) & \text{if } |x-n| \leq \frac{1}{n^4} \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Show $f(x)$ is continuous but $F(0)$ is undefined. Show $F(x)$ converges and is well defined for any $x \notin \mathbb{Z}$.

Sketch of proof of Poisson Summation:

- $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ exists and is periodic

  our decay condition + comparison test with a p-series

  ↳ here $p = 1+h > 0$ so converges

- $F'(x) = \sum_{n=-\infty}^{\infty} f'(x+n)$

  ↳ show $\sum_{n=-N}^{N} f'(x+n)$ converges to $F'(x)$ as $N \to \infty$

  ok to truncate as tail is small by our decay condition

F is continuous, decays, periodic, differentiable, bounded

USE DIRICHLET'S THEOREM!

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{F}(n)\, e^{2\pi i n x}$$ Fourier Series converges pointwise

Here $\hat{F}(n) = \int_0^1 F(x)\, e^{-2\pi i n x}\, dx$ Fourier coeff of $F:[0,1] \to \mathbb{R}$

Also have Fourier transform of $f: \mathbb{R} \to \mathbb{R}$

$\hookrightarrow$ $$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x y}\, dx$$

In Dirichlet's Theorem, take $x = 0$

$\hookrightarrow$ $$F(0) = \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{F}(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

and done $\blacksquare$

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

$$e^{-2\pi i n x} = e^{-2\pi i n (x+m)}$$

$$\hat{F}(n) = \int_0^1 F(x)\, e^{-2\pi i n x}\, dx$$

$$= \int_0^1 \sum_{m=-\infty}^{\infty} f(x+n)\, e^{-2\pi i n x}\, dx$$

Fubini

Decay

$$= \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m)\, e^{-2\pi i n x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i n x}\, dx$$

$$= \hat{f}(n)$$

Convolutions and Probability:

X_k with density f_k consider $X_1 + X_2$

Claim $f_{X_1+X_2}(x) = \int_{-\infty}^{\infty} f_{X_1}(t)\, f_{X_2}(x-t)\, dt$

CDF: $F_{X_1+X_2}(x) = \mathrm{Prob}(X_1 + X_2 \le x) = \int_{t=-\infty}^{\infty} f_{X_1}(t) \int_{x_2=-\infty}^{x-t} f_{X_2}(x_2)\, dx_2\, dt$

$$= \int_{t=-\infty}^{\infty} f_{X_1}(t)\, F_{X_2}(x-t)\, dt$$

take d/dx

Call this convolution: $(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = (g * f)(x)$

Definition 19.6.2 (Moment generating function) *Let X be a random variable with density f. The moment generating function of X, denoted $M_X(t)$, is given by $M_X(t) = \mathbb{E}[e^{tX}]$. Explicitly, if X is discrete then*

$$M_X(t) \;=\; \sum_{m=-\infty}^{\infty} e^{tx_m} f(x_m),$$

$$\mu_k' = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

k^{th} moment

while if X is continuous then

$$M_X(t) \;=\; \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

Note $M_X(t) = G_X(e^t)$, or equivalently $G_X(s) = M_X(\log s)$.

Theorem 19.6.3 *Let X be a random variable with moments μ'_k.*

1. *We have*

$$M_X(t) \;=\; 1+\mu'_1 t+\frac{\mu'_2 t^2}{2!}+\frac{\mu'_3 t^3}{3!}+\cdots;$$

in particular, $\mu'_k = d^k M_X(t)/dt^k \Big|_{t=0}$.

2. *Let α and β be constants. Then*

$$M_{\alpha X+\beta}(t) \;=\; e^{\beta t} M_X(\alpha t).$$

Useful special cases are $M_{X+\beta}(t) = e^{\beta t}M_X(t)$ and $M_{\alpha X}(t) = M_X(\alpha t)$; when proving the central limit theorem, it's also useful to have $M_{(X+\beta)/\alpha}(t) = e^{\beta t/\alpha}M_X(t/\alpha)$.

3. *Let X_1 and X_2 be independent random variables with moment generating functions $M_{X_1}(t)$ and $M_{X_2}(t)$ which converge for $|t| < \delta$. Then*

$$M_{X_1+X_2}(t) \;=\; M_{X_1}(t)M_{X_2}(t).$$

More generally, if $X_1, \ldots, X_N$ are independent random variables with moment generating functions $M_{X_i}(t)$ which converge for $|t| < \delta$, then

$$M_{X_1+\cdots+X_N}(t) = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_N}(t).$$

If the random variables all have the same moment generating function $M_X(t)$, then the right hand side becomes $M_X(t)^N$.

<u>Proof of (2)</u>

$$G_X(t) = E[e^{tX}]$$

$$Y = \alpha X + \beta$$

$$G_Y(t) = E[e^{tY}]$$

$$= E[e^{t(\alpha X+\beta)}]$$

$$= E[e^{t\alpha X} e^{t\beta}]$$

$$= e^{\beta t} E[e^{(\alpha t) X}]$$

$$= e^{\beta t} M_X(\alpha t) \quad \blacksquare$$

Rescale to have mean 0, st dev 1

Proof of (3) Recall $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

$M_{X_1+X_2}(t) = E\left[e^{t(X_1+X_2)}\right]$ Assume X_1, X_2 are indep

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t(x_1+x_2)} f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1\, dx_2$$

$$= \underbrace{\int_{x_1=-\infty}^{\infty} e^{tx_1} f_{X_1}(x_1)\,dx_2}_{M_{X_1}(t)} \underbrace{\int_{x_2=-\infty}^{\infty} e^{tx_2} f_{X_2}(x_2)\,dx_2}_{M_{X_2}(t)}$$

There exist distinct probability distributions which have the same moments. In other words, knowing all the moments doesn't always uniquely determine the probability distribution.

Example 19.6.6 *The standard examples given are the following two densities, defined for $x \geq 0$ by*

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2\pi x^2}}\, e^{-(\log^2 x)/2} \\ f_2(x) &= f_1(x)\left[1 + \sin(2\pi \log x)\right]. \end{aligned} \tag{19.2}$$

It's a nice calculation to show that these two densities have the same moments; they're clearly different (see Figure 19.1).

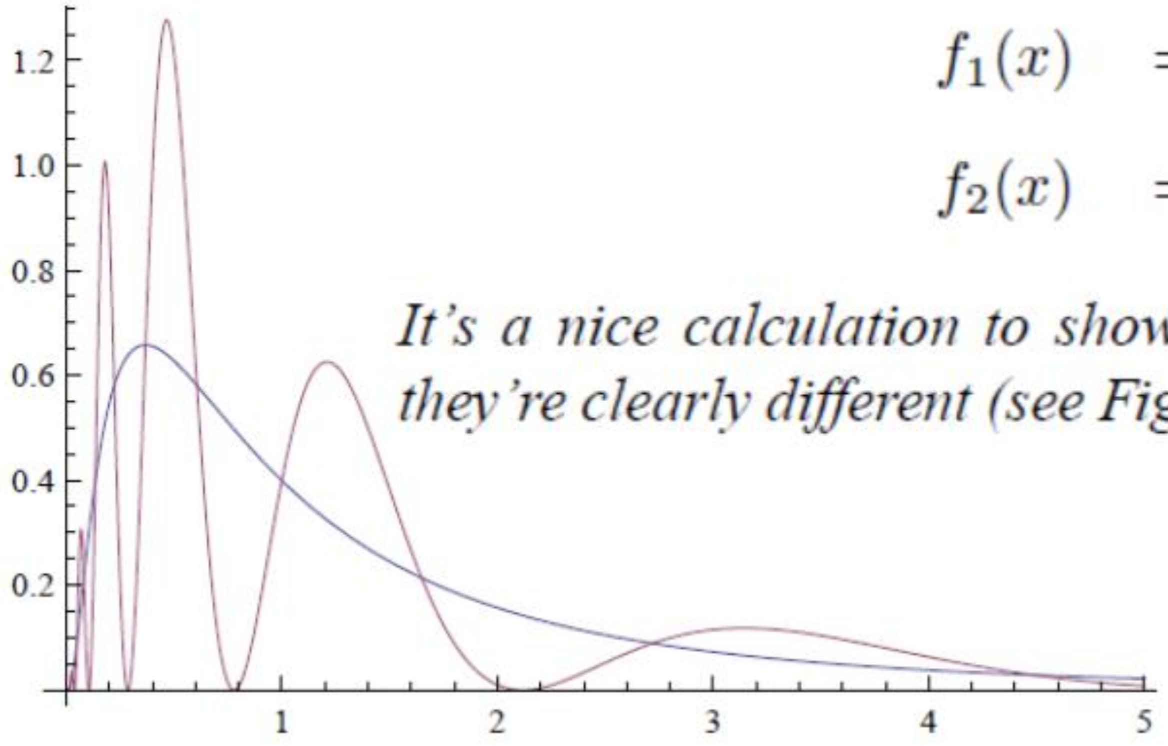


Figure 19.1: Plot of $f_1(x)$ and $f_2(x)$ from (19.2).

$$g(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (19.3)$$

Taylor Series is identically zero as

$g^{(n)}(0) = 0$

by L'Hopital

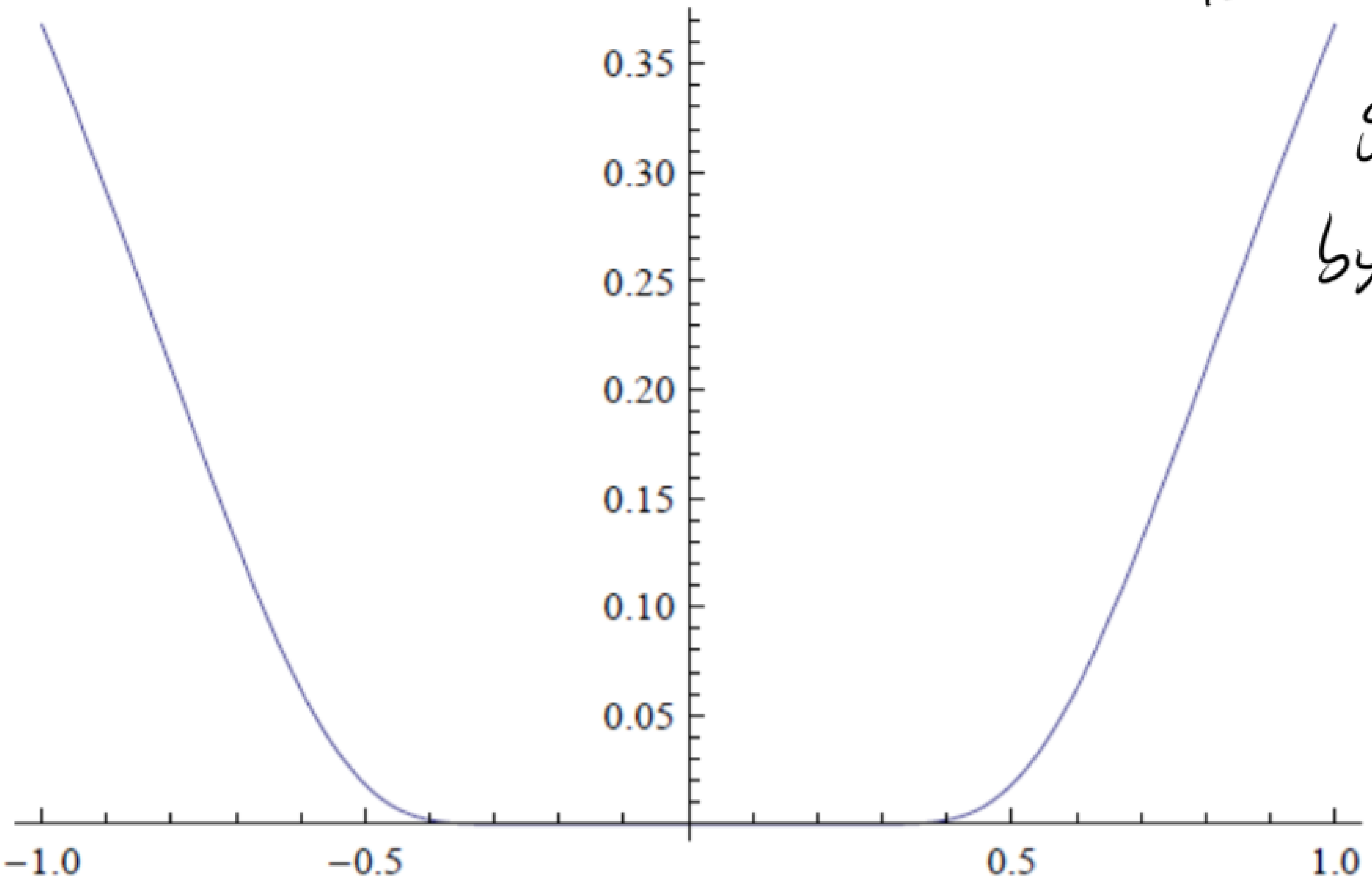


Figure 19.2: Plot of $g(x)$ from (19.3).

Characteristic functions, Convolutions and Random Variables:

$$M_X(t) = \int_{-\infty}^{\infty} f(x)\, e^{tx}\, dx$$

$$\Phi(t) = M_X(it) = \int_{-\infty}^{\infty} f(x)\, e^{itx}\, dx$$

characteristic function

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \left(\frac{-t}{2\pi}\right) x}\, dx$$

$$= \hat{f}\left(\frac{-t}{2\pi}\right)$$

Characteristic Function of the Standard Normal: An old friend returns….

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \; e^{itx} \, dx$$

Math 383: Complex Analysis: Fall ‘23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 32: 11-20-23:

- Fourier Analysis IV: Probability, Differential Equations, Fun Problems

Notes from my book with Ramin Takloo-Bighash: An Invitation to Modern Number Theory (see also my book page here)

Characteristic Function of the Standard Normal: An old friend returns….

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{itx}\,dx$$

$t=0$: $$I = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

Then $$I^2 = \frac{1}{2\pi}\int_{x=-\infty}^{\infty} e^{-x^2/2}\,dx \int_{y=-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2}\, r\,dr\,d\theta = 1$$

$$\Rightarrow I = \pm 1 \Rightarrow 1$$

Try $t = iy,\ y \in \mathbb{R}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-yx}\, dx \quad \text{from } t = iy,\ y^2 = -t^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(x^2 - 2yx + y^2 - y^2\right)\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+y)^2/2} e^{y^2/2}\, dx$$

$$= e^{y^2/2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\tilde{x}^2/2}\, d\tilde{x}}_{} \qquad \tilde{x} = x + y$$

$$= e^{y^2/2} \quad \text{or} \quad e^{-t^2/2}$$

Sketch of the Proof of the Central Limit Theorem:

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

$$\hat{h}(y) = \int_{-\infty}^{\infty} h(x)\, e^{-2\pi i x y}\, dx$$

$$= \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t)\, g(x-t)\, e^{-2\pi i (x - t + t) y}\, dt\, dx$$

$$= \underbrace{\int_{t=-\infty}^{\infty} f(t)\, e^{-2\pi i t y}}_{\hat{f}(y)} \underbrace{\left[\int_{x=-\infty}^{\infty} g(x-t)\, e^{-2\pi i (x-t) y}\, dx \right]}_{\hat{g}(y)} dt$$

$$u = x - t$$

$$\int_{u=-\infty}^{\infty} g(u)\, e^{-2\pi i u y}\, du$$

$$\hat{f}(y) \cdot \hat{g}(y)$$

X_k iid r.v. mean 0, st dev 1, finite higher moments ...

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{CLT} \text{Normal}(0,1)$$

Var(Sum of indep R.V.) = Sum of the variances

$$M_{\frac{X_1+\cdots+X_n}{\sqrt{n}}}(t) = \prod_{k=1}^{n} M_{\frac{X_k}{\sqrt{n}}}(t) = \prod_{k=1}^{n} M_{X_k}\left(\frac{t}{\sqrt{n}}\right)$$

$$= M_X\left(t/\sqrt{n}\right)^n$$

One can also study problems on $\mathbb{R}$ by using the Fourier Transform. Its use stems from the fact that it converts multiplication to differentiation, and vice versa: if $g(x) = f'(x)$ and $h(x) = xf(x)$, prove that $\widehat{g}(y) = 2\pi iy\widehat{f}(y)$ and $\frac{d\widehat{f}(y)}{dy} = -2\pi i\widehat{h}(y)$. This and Fourier Inversion allow us to solve problems such as the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0 \tag{11.95}$$

Proof: $g(x) = f'(x)$

Then $\hat{g}(y) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi ixy}\,dx$ assume f, f' are 'nice' decay at ∞ at a decent rate

$u = e^{-2\pi ixy}$ $dv = f'(x)dx$

$du = -2\pi iy e^{-2\pi ixy}$ $v = f(x)$

$\hat{g}(y) = \cancel{uv\Big|_{-\infty}^{\infty}}^{0} - \int_{-\infty}^{\infty} v\,du = 2\pi iy\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx = 2\pi iy\hat{f}(y)$

assuming decay

Laplace transform

60 languages

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From Wikipedia, the free encyclopedia

In mathematics, the **Laplace transform**, named after its discoverer Pierre-Simon Laplace (/ləˈplɑːs/), is an integral transform that converts a function of a real variable (usually t, in the *time domain*) to a function of a complex variable s (in the complex frequency domain, also known as **s-domain**, or **s-plane**). The transform has many applications in science and engineering, mostly as a tool for solving linear differential equations.[1] In particular, it transforms ordinary differential equations into algebraic equations and convolution into multiplication.[2][3] For suitable functions *f*, the Laplace transform is defined by the integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

https://en.wikipedia.org/wiki/Laplace_transform

Integral kernel

$K(s,t) = e^{-st}$

$\int_0^\infty f(t)\,K(s,t)\,dt$

Laplace transform

60 languages

Article Talk Read Edit View history Tools

From Wikipedia, the free encyclopedia

In mathematics, the **Laplace transform**, named after its discoverer Pierre-Simon Laplace (/ləˈplɑːs/), is an integral transform that converts a function of a real variable (usually t, in the *time domain*) to a function of a complex variable s (in the complex frequency domain, also known as **s-domain**, or **s-plane**). The transform has many applications in science and engineering, mostly as a tool for solving linear differential equations.[1] In particular, it transforms ordinary differential equations into algebraic equations and convolution into multiplication.[2][3] For suitable functions *f*, the Laplace transform is defined by the integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

https://en.wikipedia.org/wiki/Laplace_transform

$$\begin{aligned}\hat{f}(\omega) &= \mathcal{F}\{f(t)\} \\ &= \mathcal{L}\{f(t)\}|_{s=i\omega} = F(s)|_{s=i\omega} \\ &= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt\,.\end{aligned}$$

Properties of the unilateral Laplace transform

Property	Time domain	s domain	Comment
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$	Can be proved using basic rules of integration.
Frequency-domain derivative	$tf(t)$	$-F'(s)$	F' is the first derivative of F with respect to s.
Frequency-domain general derivative	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, nth derivative of $F(s)$.
Derivative	$f'(t)$	$sF(s) - f(0^-)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second derivative	$f''(t)$	$s^2 F(s) - sf(0^-) - f'(0^-)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General derivative	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^-)$	f is assumed to be n-times differentiable, with nth derivative of exponential type. Follows by mathematical induction.
Frequency-domain integration	$\frac{1}{t} f(t)$	$\int_s^\infty F(\sigma)\, d\sigma$	This is deduced using the nature of frequency differentiation and conditional convergence.
Time-domain integration	$\int_0^t f(\tau)\, d\tau = (u * f)(t)$	$\frac{1}{s} F(s)$	$u(t)$ is the Heaviside step function and $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Frequency shifting	$e^{at} f(t)$	$F(s - a)$	
Time shifting	$f(t - a)u(t - a)$	$e^{-as} F(s)$	$a > 0$, $u(t)$ is the Heaviside step function

Time scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$a > 0$
Multiplication	$f(t)g(t)$	$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} F(\sigma)G(s-\sigma)\,d\sigma$	The integration is done along the vertical line $\operatorname{Re}(\sigma) = c$ that lies entirely within the region of convergence of F.[24]
Convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$	
Circular convolution	$(f * g)(t) = \int_0^T f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$	For periodic functions with period T.
Complex conjugation	$f^*(t)$	$F^*(s^*)$	
Cross-correlation	$(f \star g)(t) = \int_0^\infty f(\tau)^*\, g(t+\tau)\,d\tau$	$F^*(-s^*)\cdot G(s)$	
Periodic function	$f(t)$	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$	$f(t)$ is a periodic function of period T so that $f(t) = f(t+T)$, for all $t \geq 0$. This is the result of the time shifting property and the geometric series.
Periodic summation	$f_P(t) = \sum_{n=0}^{\infty} f(t-Tn)$ $f_P(t) = \sum_{n=0}^{\infty}(-1)^n f(t-Tn)$	$F_P(s) = \frac{1}{1-e^{-Ts}}F(s)$ $F_P(s) = \frac{1}{1+e^{-Ts}}F(s)$	

Fourier transform [edit]

Further information: Fourier transform § Laplace transform

The Fourier transform is a special case (under certain conditions) of the bilateral Laplace transform. While the Fourier transform of a function is a complex function of a *real* variable (frequency), the Laplace transform of a function is a complex function of a *complex* variable. The Laplace transform is usually restricted to transformation of functions of t with $t \geq 0$. A consequence of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. Unlike the Fourier transform, the Laplace transform of a distribution is generally a well-behaved function. Techniques of complex variables can also be used to directly study Laplace transforms. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function. This perspective has applications in probability theory.

The Fourier transform is equivalent to evaluating the bilateral Laplace transform with imaginary argument $s = i\omega$ or $s = 2\pi i\xi$[27] when the condition explained below is fulfilled,

$$\begin{aligned}\hat{f}(\omega) &= \mathcal{F}\{f(t)\} \\ &= \mathcal{L}\{f(t)\}|_{s=i\omega} = F(s)|_{s=i\omega} \\ &= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt\,.\end{aligned}$$

This convention of the Fourier transform ($\hat{f}_3(\omega)$ in Fourier transform § Other conventions) requires a factor of $\frac{1}{2\pi}$ on the inverse Fourier transform. This relationship between the Laplace and Fourier transforms is often used to determine the frequency spectrum of a signal or dynamical system.

The above relation is valid as stated if and only if the region of convergence (ROC) of $F(s)$ contains the imaginary axis, $\sigma = 0$.

Selected Laplace transforms

Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence	Reference
unit impulse	$\delta(t)$	1	all s	inspection
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$		time shift of unit impulse
unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$	integrate unit impulse
delayed unit step	$u(t-\tau)$	$\frac{1}{s}e^{-\tau s}$	$\mathrm{Re}(s) > 0$	time shift of unit step
rectangular impulse	$u(t) - u(t-\tau)$	$\frac{1}{s}(1-e^{-\tau s})$	$\mathrm{Re}(s) > 0$	
ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$	integrate unit impulse twice
nth power (for integer n)	$t^n \cdot u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$ $(n > -1)$	integrate unit step n times
qth power (for complex q)	$t^q \cdot u(t)$	$\frac{\Gamma(q+1)}{s^{q+1}}$	$\mathrm{Re}(s) > 0$ $\mathrm{Re}(q) > -1$	[30][31]
nth root	$\sqrt[n]{t} \cdot u(t)$	$\frac{1}{s^{\frac{1}{n}+1}}\Gamma\left(\frac{1}{n}+1\right)$	$\mathrm{Re}(s) > 0$	Set $q = 1/n$ above.
nth power with frequency shift	$t^n e^{-\alpha t} \cdot u(t)$	$\frac{n!}{(s+\alpha)^{n+1}}$	$\mathrm{Re}(s) > -\alpha$	Integrate unit step, apply frequency shift
delayed nth power with frequency shift	$(t-\tau)^n e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{n! \cdot e^{-\tau s}}{(s+\alpha)^{n+1}}$	$\mathrm{Re}(s) > -\alpha$	integrate unit step, apply frequency shift, apply time shift

exponential decay	$e^{-\alpha t}\cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}(s) > -\alpha$	Frequency shift of unit step
two-sided exponential decay (only for bilateral transform)	$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 - s^2}$	$-\alpha < \mathrm{Re}(s) < \alpha$	Frequency shift of unit step
exponential approach	$(1 - e^{-\alpha t})\cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\mathrm{Re}(s) > 0$	unit step minus exponential decay
sine	$\sin(\omega t)\cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$	[32]
cosine	$\cos(\omega t)\cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$	[32]
hyperbolic sine	$\sinh(\alpha t)\cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\mathrm{Re}(s) > \|\alpha\|$	[33]
hyperbolic cosine	$\cosh(\alpha t)\cdot u(t)$	$\frac{s}{s^2-\alpha^2}$	$\mathrm{Re}(s) > \|\alpha\|$	[33]
exponentially decaying sine wave	$e^{-\alpha t}\sin(\omega t)\cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}(s) > -\alpha$	[32]
exponentially decaying cosine wave	$e^{-\alpha t}\cos(\omega t)\cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}(s) > -\alpha$	[32]
natural logarithm	$\ln(t)\cdot u(t)$	$-\frac{1}{s}\left[\ln(s)+\gamma\right]$	$\mathrm{Re}(s) > 0$	[33]
Bessel function of the first kind, of order n	$J_n(\omega t)\cdot u(t)$	$\frac{\left(\sqrt{s^2+\omega^2}-s\right)^n}{\omega^n\sqrt{s^2+\omega^2}}$	$\mathrm{Re}(s) > 0$ $(n > -1)$	[34]
Error function	$\mathrm{erf}(t)\cdot u(t)$	$\frac{1}{s}e^{(1/4)s^2}\left(1-\mathrm{erf}\,\frac{s}{2}\right)$	$\mathrm{Re}(s) > 0$	[34]

https://web.williams.edu/Mathematics/sjmiller/public_html/209/HW/209HWmay12.pdf

Question 1 (40 points) : Find the Laplace Transforms of: (1) $\cos(2t)$; (2) $4t^7 - 11t^3 + 1$; (3) t^2e^{3t}; (4) $\cosh(t) = \frac{e^t+e^{-t}}{2}$.

Question 2 (30 points) : Find the Inverse Laplace Transform of the following (the table in the book or online at http://en.wikipedia.org/wiki/Laplace_transforms#Table_of_selected_Laplace_transforms might be useful): (1) $F(s) = \frac{3}{s^2+4}$; (2) $F(s) = \frac{2}{s^2+3s-4}$; (3) $F(s) = \frac{8s^2-4s+12}{s(s^2+4)}$.

Question 3 (10 points) : Use the Laplace transform to solve $y'' - y' - 6y = 0$ with $y(0) = 1$, $y'(0) = -1$.

Question 4 (10 points) : Use the Laplace transform to solve $y'''' - 4y = 0$ with $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$ and $y'''(0) = 0$. (NOTE: for those looking for additional problems, #17 from Section 6.2 is a good one.)

Question 5 (10 points) : Solve $y'' + y = f(t)$, where $f(t) = 1$ for $0 \le t < 3\pi$ and 0 if $3\pi \le t < \infty$ and subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

CHAPTER 6: THE LAPLACE TRANSFORM

SECTION 6.1: DEFN OF THE LAPLACE TRANSFORM

Danger with improper integrals: $\lim_{\substack{A\to\infty \\ B\to\infty}} \int_{-A}^{B} f(x)\,dx$ could depend on how go to infinity

↳ ex: $\int_{-A}^{2A} f(t)\,dt$ vs $\int_{-A}^{A} f(t)\,dt$ for $f(t) = \frac{t}{t^2+1}$

Piecewise cont fn: finite # points $\alpha = t_0 < t_1 < t_2 < \cdots < t_n = \beta$ st f is cont on (t_{i-1}, t_i) and ~~limit exists~~ left and right hand limits exist for each t_i

THM 6.1.1 (integral test): f piecewise cont for $t \geq a$, $|f(t)| \leq g(t)$ $\forall t \geq M$ then

- $\int_M^\infty g(t)\,dt$ conv $\Rightarrow$ $\int_a^\infty f(t)\,dt$ conv
- If instead $f(t) \geq g(t) \geq 0$ $\forall t \geq M$ then
- $\int_M^\infty g(t)\,dt$ diverges $\to$ $\int_a^\infty f(t)\,dt$ diverges

Integral Transforms: $F(s) = \int_\alpha^\beta K(s,t) f(t)\,dt$, K the kernel.

Laplace Transform: $\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

↳ Why Use it?

① convert DiffEq for f to algebraic problem for $\mathcal{L}[f(t)] = F(s)$

② solve for $F(s)$

③ invert and find f from F (can be hard!)

THM 6.1.2: f piecewise cont on $[0, A]$ for ~~some~~ all A, $|f(t)| \le K e^{at}$ $\forall t \ge M$
for constants a, $K \ge 0$, $M \ge 0$
Then $\mathcal{L}[f(t)]$ exists for $s > a$

Proof: Use $\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \int_M^\infty e^{-st} f(t)\,dt$

Note: "Big" table of Laplace Transforms in Section 6.2

↳ ex: $\mathcal{L}[1] = 1/s \quad s > 0$

$\mathcal{L}[e^{at}] = 1/(s-a) \quad s > a$

$\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2} \quad s > 0$

Linear Operator: $\mathcal{L}[c_1 f_1(t) + \dots + c_k f_k(t)] = c_1 \mathcal{L}[f_1(t)] + \dots + c_k \mathcal{L}[f_k(t)]$

SECTION 6.2: SOLN OF INITIAL VALUE PROBLEMS

THM 6.2.1: f cont, f' piecewise cont on any $[0, A]$, $|f(t)| \le Ke^{at}$ $t \ge M$.

Then $\mathcal{L}[f'(t)]$ exists $\forall\, t \ge a$ and $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$

Proof: assume for simplicity f, f' cont

integrate by parts: $\int_0^A e^{-st} f'(t)\,dt$ and let $A \to \infty$

CORR: $f, f', \dots, f^{(n-1)}$ cont, $f^{(n)}$ piecewise cont on any $[0, A]$

and $|f^{(i)}(t)| \le Ke^{at}$. Then $\mathcal{L}[f^{(n)}(t)]$ exists and equals

$$s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Ex: $y'' - y' - 2y = 0$ with $y(0) = 1$, $y'(0) = 0$

↳ know sol'n comes from e^{rt}, get $\frac{2}{3} e^{-t} + \frac{1}{3} e^{2t}$

↳ Now use Laplace Transform

$\mathcal{L}[y''] - \mathcal{L}[y'] - \mathcal{L}[2y] = 0$, set $Y(s) = \mathcal{L}[y(t)]$

$\Rightarrow \left(s^2 Y(s) - s\,y(0) - y'(0)\right) - \left(s\,Y(s) - y(0)\right) - 2\,Y(s) = 0$

$(s^2 - s - 2)Y(s) + (1-s)\,y(0) - y'(0) = 0$ but $y(0) = 1$, $y'(0) = 0$

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$$

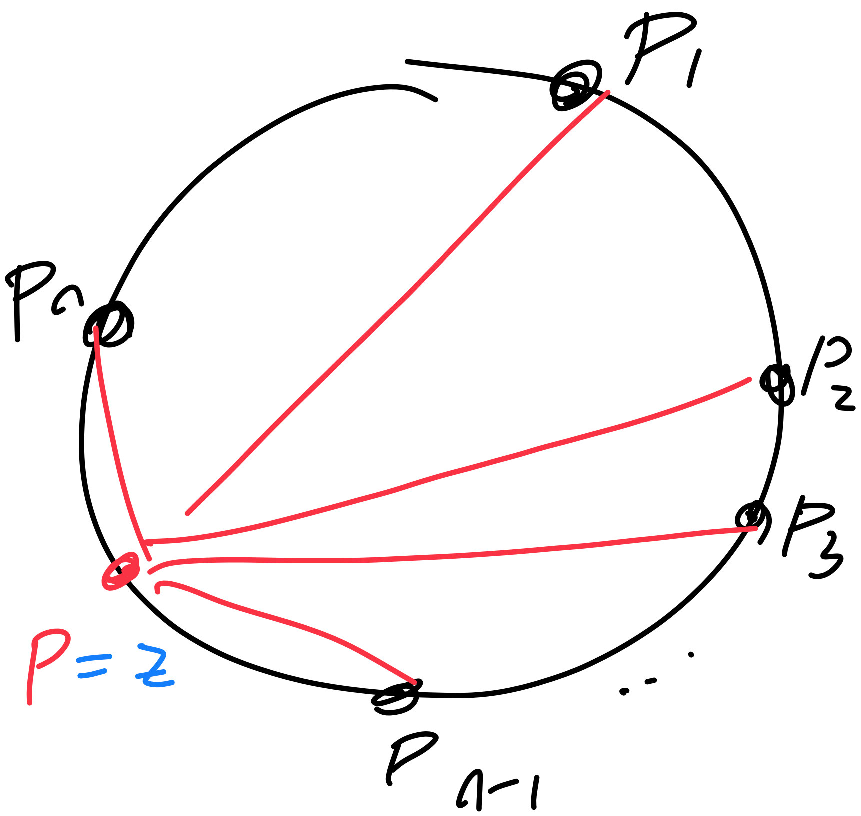
↳ partial fractions! $\frac{a}{s-2} + \frac{b}{s+1} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}$

know $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

Thus $y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$

Question! $\mathcal{L}[y_1] = \mathcal{L}[y_2]$ iff $y_1 = y_2$?

↳ For us want $\mathcal{L}^{-1}$ unique: can appeal to sol'ns to diff eq unique....



n points

find (x,y) on unit circle

st product of distances

to points is at least 1

or prove a point exist.

$$f(z) = \prod_{k=1}^{n} (z - z_k) \quad \text{study } |f|$$

$$f(c) \to |f(c)| = 1$$

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Lecture 32b: 11-20-23:

- Bonus Lecture: Fourier Analysis V: Complex to CLT: https://youtu.be/YwDJh6V8C3Y

Notes from my book with Ramin Takloo-Bighash: An Invitation to Modern Number Theory (see also my book page here)

$X_1, X_2, \ldots, X_n$ indep, identically distr rand vars

wlog mean is 0, variance is 1, assume higher moments are finite

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \underbrace{f_X(x)}_{\text{density of } X} dx$$

$g(x) = x$ mean, $\mu = E[X]$

$g(x) = x^k$ k^{th} moment

$$\mathrm{Var}(X) = E[(X-\mu)^2] = E[X^2] - \underbrace{E[X]^2}_{=\sigma^2}$$

Translation game

X has mean μ, $\quad Y = X - \mu$ has 0

X has stdev σ. $\quad Y = X/\sigma$ has stdev 1

Fact: $E[X_1 + \cdots + X_n] = \sum_{k=1}^{n} E[X_k]$

$Var(X_1 + \cdots + X_n) = \sum_{k=1}^{n} Var(X_k)$ if indep

Consider $X_1 + X_1 \quad (X_2 = X_1)$

$X_1 - X_1 \quad (X_2 = -X_1)$

$$\mathrm{Var}(a_1 X_1 + \cdots + a_n X_n) = \sum_{k=1}^{n} a_k^2 \, \mathrm{Var}(X_k)$$

$$\mathbb{E}\left[\left(aX - \mathbb{E}[aX]\right)^2\right]$$

$$= \mathbb{E}\left[(aX)^2\right] - \mathbb{E}[aX]^2$$

$$= a^2 \mathbb{E}[X^2] - a^2 \mathbb{E}[X]^2$$

$$= a^2 \mathrm{Var}(X^2)$$

So $\mathrm{StDev}(aX) = |a| \, \mathrm{StDev}(X)$

Why assume all means are 0, all stdevs are 1

$X_1 + \cdots + X_n$

mean is zero

Variance is $n \cdot 1$

Study $\dfrac{X_1 + \cdots + X_n}{\sqrt{n}}$

Mean 0

stdev 1

Then $\operatorname{Var}\left(\dfrac{X_1 + \cdots + X_n}{\sqrt{n}}\right) = \sum_{k=1}^{n} \operatorname{Var}\left(\dfrac{X_k}{\sqrt{n}}\right)$

$= \sum_{k=1}^{n} \dfrac{1}{n} \operatorname{Var}(X_k)$

$= \dfrac{1}{n} \cdot n \cdot 1 = 1$

$$\lim_{n \to \infty} Z_n \quad \text{where } Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

CLT: $Z_n \to \text{Normal}(0,1)$

Miracle is b/c wlog mean $= 0$, stdev $= 1$

$$f_{\frac{X_1}{\sqrt{n}}+\cdots+\frac{X_n}{\sqrt{n}}}(t) = \left(f_{\frac{X_1}{\sqrt{n}}} * \cdots * f_{\frac{X_n}{\sqrt{n}}}\right)(t)$$

Take Fourier Transform

$$\hat{f}_{Z_n}(t) = \prod_{k=1}^{n} \hat{f}_{\frac{X_k}{\sqrt{n}}}(t) = \left(\hat{f}_{\frac{X}{\sqrt{n}}}(t)\right)^n$$

$f_X(x) \qquad Y = X/a \qquad f_Y(y)$

$$\hat{f}_Y(y) = \int_{t=-\infty}^{\infty} f_Y(t)\, e^{-2\pi i y t}\, dt$$

$$= \int_{t=-\infty}^{\infty} f_{\frac{X}{a}}(t)\, e^{-2\pi i y t}\, dt = \int_{t=-\infty}^{\infty} a\, f_X(at)\, e^{-2\pi i y t}\, dt$$

CDF: $\text{Prob}(Y \le y) = \int_{-\infty}^{y} f_Y(t)\, dt$

$$\frac{d}{dy} \text{Prob}(Y \le y) = \frac{d}{dy}\left[F_Y(y) - F_Y(-\infty)\right] = f_Y(y)$$

$$\text{Prob}(Y \le y) = \text{Prob}(X \le ay) \to \frac{d}{dy}\text{Prob}(Y \le y) = \frac{d}{dy}\int_{-\infty}^{ay} f_X(t)\, dt$$

$$f_Y(y) = \frac{d}{dy}\left[F_X(ay) - F_X(-\infty)\right] = f_X(ay) \cdot a$$

$$\hat{f}_Y(y) = \int_{t=-\infty}^{\infty} a\, f_X(at)\, e^{-2\pi i y t}\, dt$$

$u = at$

$du = a\,dt$

$$= \int_{t=-\infty}^{\infty} f_X(at)\, e^{-2\pi i \frac{y}{a}\cdot at} \cdot a\,dt$$

$$= \int_{u=-\infty}^{\infty} f_X(u)\, e^{-2\pi i \frac{y}{a} u}\, du$$

$$\hat{f}_Y(y) = \hat{f}_X\left(\frac{y}{a}\right) \quad \text{if} \quad Y = X/a$$

Ex: $\hat{f}_{\frac{X}{\sqrt{n}}}(t) = \hat{f}_X\left(\frac{t}{\sqrt{n}}\right)$

Had $\hat{f}_{Z}(t) = \left(\hat{f}_{\frac{X}{\sqrt{n}}}(t)\right)^n$

$= \left(\hat{f}_{X}\left(\frac{t}{\sqrt{n}}\right)\right)^n$

$$\hat{f}_X(y) = \int_{-\infty}^{\infty} f_X(x)\, e^{-2\pi i x y}\, dx$$

If $y=0$, $\hat{f}_X(0) = \int_{-\infty}^{\infty} f_X(x)\,dx = 1$ as f is a density

$$\hat{f}_X(y) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x y}\, dx \qquad y \to z \text{ holomorphic (analytic)}$$

$$= \underbrace{\hat{f}_X(0)}_{1} + \underbrace{\hat{f}_X'(0)\, y}_{0} + \underbrace{\hat{f}_X''(0)}_{-4\pi^2} \frac{y^2}{2!} + \cdots$$

$$\hat{f}_X'(y) = \int_{-\infty}^{\infty} f_X(x)\,(-2\pi i x)\, e^{-2\pi i x y}\, dx$$

$$= -2\pi i \int_{-\infty}^{\infty} x f_X(x)\, e^{-2\pi i x y}\, dx$$

$$\hat{f}_X'(0) = -2\pi i \int_{-\infty}^{\infty} x\, f_X(x)\, dx = -2\pi i\, \mathbb{E}[X] = 0$$

$$f_X''(0) = -4\pi^2 \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = -4\pi^2 \operatorname{Var}(X)$$

$$= -4\pi^2$$

Have $\left(\hat{f}_X(t/\sqrt{n})\right)^n$

$$= \left(1 + 0 - 4\pi^2 \frac{t^2}{2n} + \cdots\right)^n$$

$t^3/n^{3/2}$ t^4/n^2

If (see product) then (take log)
unless (with young kids)

$$\log \hat{f}_Z(t) = n \log\Big(1 - \underbrace{4\pi^2 \frac{t^2}{2n} + \text{terms of size } \frac{t^3}{n^{3/2}}}_{u} \Big)$$

$$\log(1-u) = -u - u^2/2 - u^3/3 \cdots$$

we have $n\left[\left(-\frac{4\pi^2 t^2}{2n} + \text{terms of size } \frac{t^3}{n^{3/2}}\right) + \left(\text{stuff of size } \frac{1}{n^2}\right) + \cdots\right]$

$$= -2\pi^2 t^2 + \left(\text{size } \frac{t^3}{\sqrt{n}} + \text{size } \frac{1}{n} + \cdots\right)$$

So $\log \hat{f}_Z(t) \to -2\pi^2 t^2$ Then

$$\hat{f}_Z(t) \to e^{-2\pi^2 t^2}$$

$$e^{-2\pi^2 t^2} = e^{-(2\pi)^2 t^2/2}$$

Get rid of 2π in the Fourier Transform to use the characteristic function yields $e^{-t^2/2}$

$$\lim_{n\to\infty}\left(1 \pm \frac{x}{n}\right)^n = e^{\pm x}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n^2}\left(1-\frac{x}{n}\right)^{n^2} = \lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^{n^2} = e^{-x^2}$$

$$= \lim_{n\to\infty}\left(\underbrace{\left(1+\frac{x}{n}\right)^n}_{}\right)^n\left(\underbrace{\left(1-\frac{x}{n}\right)^n}_{}\right)^n$$

$$= \lim_{n\to\infty}\left(e^{x}\right)^n\left(e^{-x}\right)^n$$

$$= 1$$

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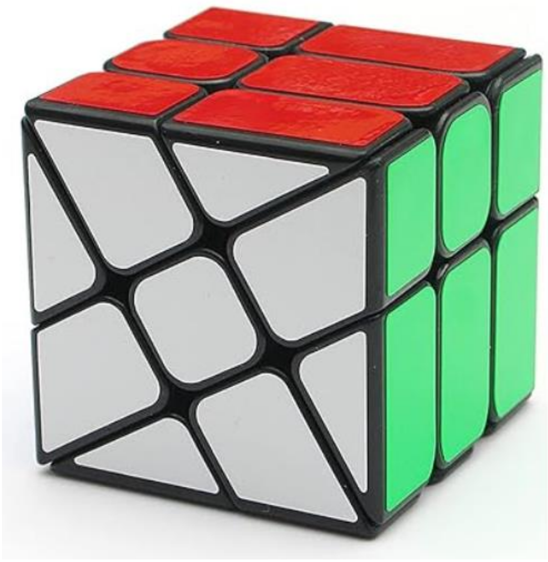
Lecture 33: 11-27-23: https://youtu.be/65b0Jh1DIns

- Method of Stationary Phase
- Previous year's iteration: Lecture 27: 11/15/17: Laplace's Method, Stirling's Formula: https://youtu.be/AycMlf4Mbyo (2015 lecture: https://youtu.be/GvKI5I_cfDQ)



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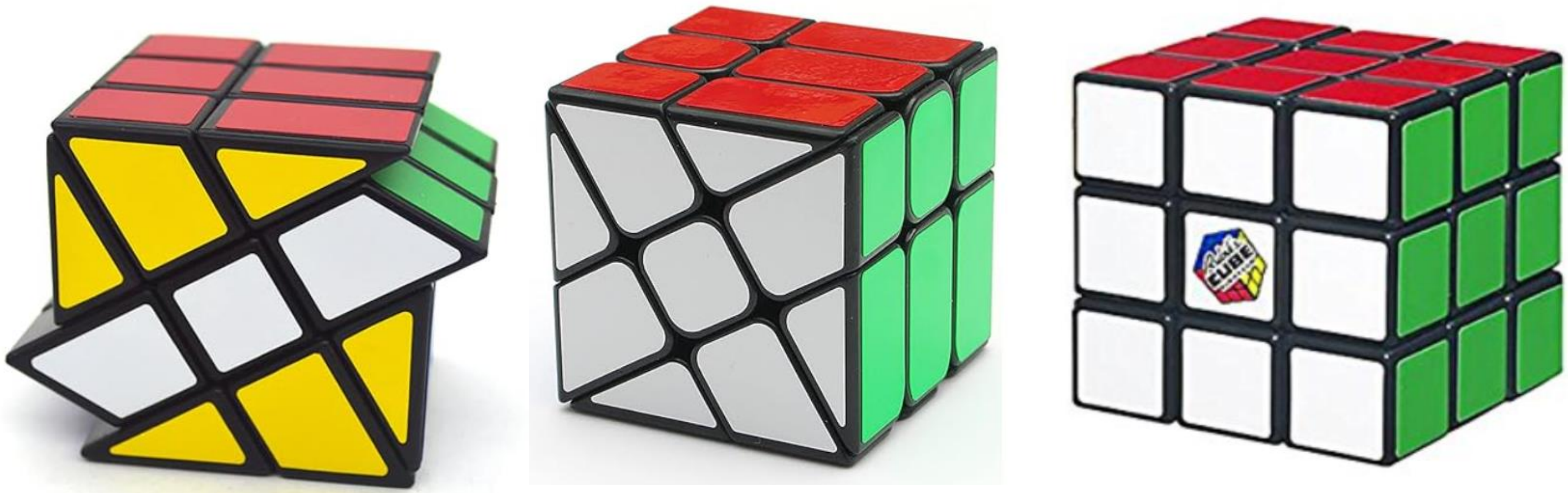


Plan for the day:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Stirling’s formula: intuition
- Stirling Approximation (Integral Test Review)
- Stirling from Central Limit Theorem
- Taylor Series Review
- Stationary Phase / Critical Points

General items.

- Often easier to pass to a continuous analogue to study discrete problem
- Intuition from Taylor series: “higher” terms eventually negligible….

The Gamma function. The Gamma function $\Gamma(s)$ is

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0.$$

$$x^{s-1} dx = x^s \frac{dx}{x}$$

Stirling's formula: As $n \to \infty$, we have

$$n! \approx n^n e^{-n} \sqrt{2\pi n};$$

by this we mean

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.$$

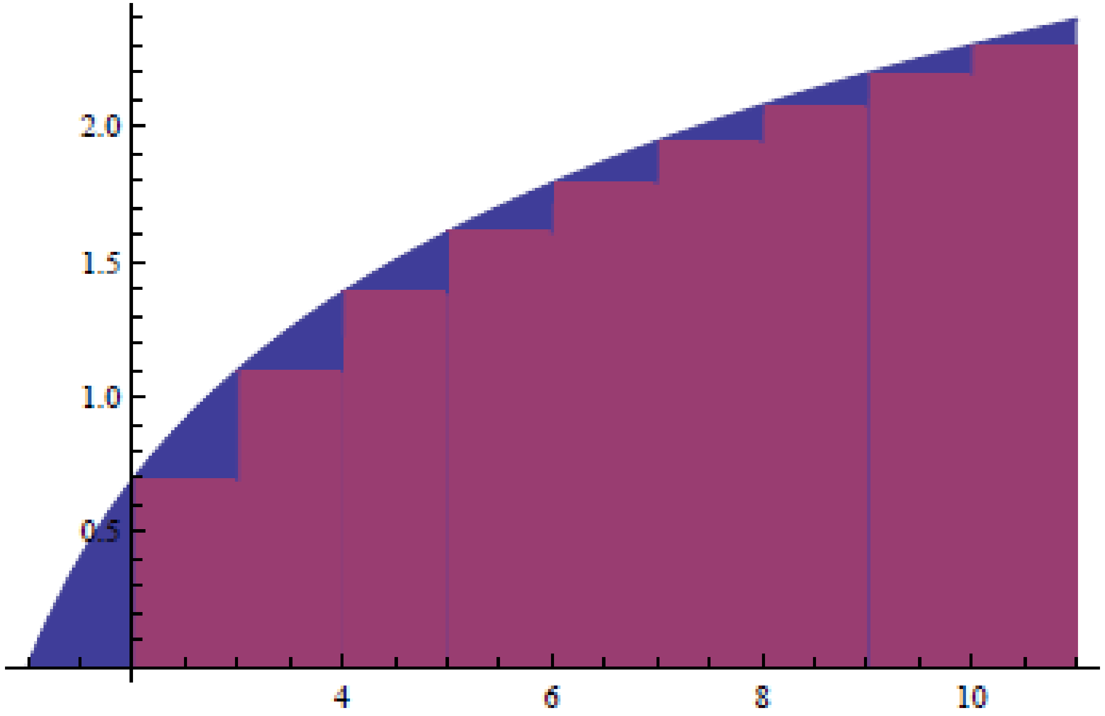
More precisely, we have the following series expansion:

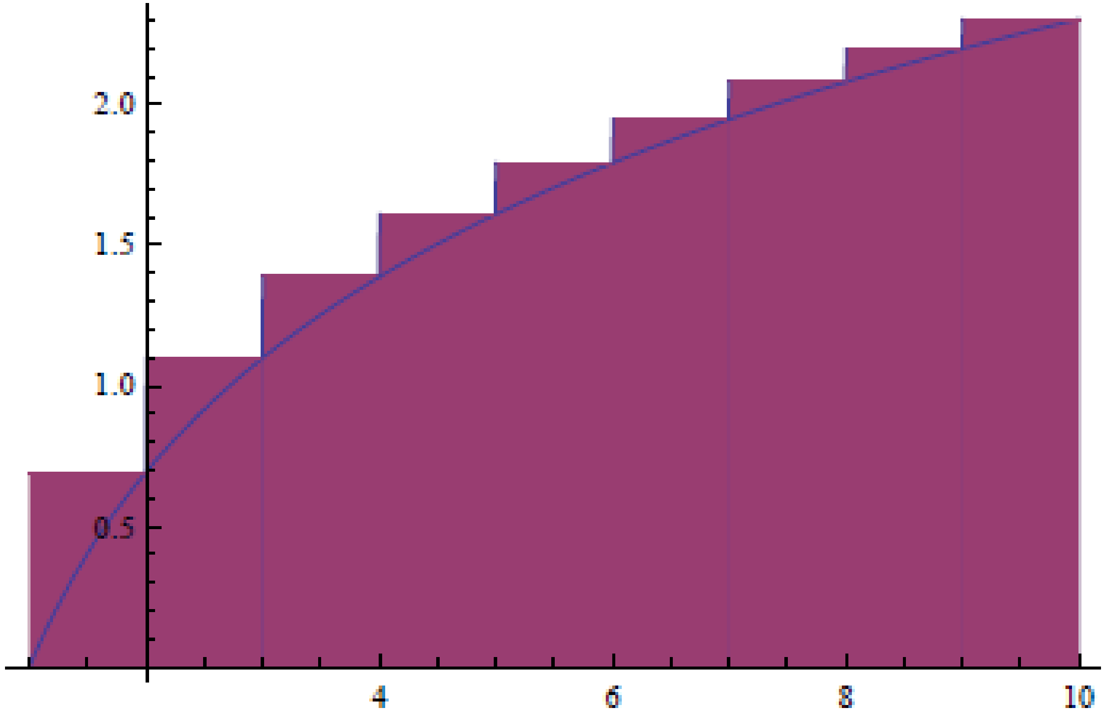
$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots\right).$$

Poor man's Stirling. Let $n \geq 3$ be a positive integer. Then

$$n^n e^{-n} \cdot e \;\leq\; n! \;\leq\; n^n e^{-n} \cdot en.$$

$\log n! = \sum_{k=1}^{n} \log k \approx \int_1^n \log t \, dt$ $\qquad GM(e, en) = e\sqrt{n}$





https://en.wikipedia.org/wiki/Integral_test_for_convergence
and my Math 150 (Calc III) lecture on the integral test: https://youtu.be/ujJbUpCab6M

Crude upper/lower bounds.

$$1 \le n \le \quad n! \quad \le n^n$$

Note (n+1)!/n! = n+1; let's see what Stirling gives:

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

$$(n+1)! \approx (n+1)^{n+1} e^{-(n+1)} \sqrt{2\pi(n+1)}$$

$$\frac{(n+1)!}{n!} \approx \frac{(n+1)^{n+1}}{n^n} \frac{e^{-n-1}}{e^{-n}} \frac{\sqrt{2\pi(n+1)}}{\sqrt{2\pi n}}$$

$$\approx (n+1) \underbrace{\left(1+\frac{1}{n}\right)^n e^{-1}}_{e \cdot 1/e} \sqrt{1+\frac{1}{n}} \simeq (n+1)$$

$\Gamma(s+1) = s\Gamma(s)$ so $\Gamma(n+1) = n!$ n non-neg integer

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx$$

Integrand largest when $(e^{-x} x^n)' = 0$

$\hookrightarrow$ $-e^{-x}x^n + e^{-x} n x^{n-1} = 0 \Rightarrow e^{-x} x^{n-1}(-x+n) = 0$ so $x=n$

$\hookrightarrow$ largest value is $e^{-n} n^n$

let $x = n+t$ could do $x = n + g(n)$

Get $e^{-(n+t)} (n+t)^n = e^{-n} e^{-t} n^n (1+\frac{t}{n})^n$

$$= e^{-n} n^n \boxed{e^{-t}(1+t/n)^n}$$

$\log(e^{-t}(1+\frac{t}{n})^n) = -t + n\log(1+\frac{t}{n}) = -t + n\left(\frac{t}{n} - \frac{t^2}{2n^2} + \frac{t^3}{3n^3} \cdots\right)$

$= 0 - t^2/2n + t^3/3n^2 - \cdots$, ok if $\frac{t^2}{n} \to 0$

The Central Limit Theorem and Stirling

$\mu_X = \lambda \quad Var_X = \lambda$

X has a Poisson distribution with parameter λ means

$$\text{Prob}(X = n) = \begin{cases} \frac{\lambda^n e^{-\lambda}}{n!} & \text{if } n \geq 0 \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases} \quad (\lambda > 0)$$

If $X_1, \ldots, X_N$ are independent, identically distributed random variables with mean μ, variance σ^2 and a little more (such as the third moment is finite, or the moment generating function exists), then $X_1 + \cdots + X_N$ converges to being normally distributed with mean $n\mu$ and variance $n\sigma^2$.

$X_k \sim$ Poiss (λ_k) Then $X_1 + \cdots + X_n \sim$ Poiss $(\lambda_1 + \cdots + \lambda_n)$

if all $\lambda_k = 1$ Then it's Poiss(n)

Proof Poisson + Poisson = Poisson (if independent: stable distribution!): Algebra!

$$\text{Prob}(X = n) = \begin{cases} \frac{\lambda^n e^{-\lambda}}{n!} & \text{if } n \geq 0 \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

$X_1 \sim \text{Poiss}(\lambda_1)$ $X_2 = \text{Poiss}(\lambda_2)$ Show $X = X_1 + X_2 \sim \text{Poiss}(\lambda_1 + \lambda_2)$

$$\text{Prob}(X = n) = \sum_{m=0}^{n} \text{Prob}(X_1 = m)\,\text{Prob}(X_2 = n-m)$$

$$= \sum_{m=0}^{n} \frac{\lambda_1^m e^{-\lambda_1}}{m!} \frac{\lambda_2^{n-m} e^{-\lambda_2}}{(n-m)!} \frac{n!}{n!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{m=0}^{n} \binom{n}{m} \lambda_1^m \lambda_2^{n-m}$$

$$= \frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}$$

Definition 19.6.2 (Moment generating function) *Let X be a random variable with density f. The moment generating function of X, denoted $M_X(t)$, is given by $M_X(t) = \mathbb{E}[e^{tX}]$. Explicitly, if X is discrete then*

$$M_X(t) = \sum_{m=-\infty}^{\infty} e^{tx_m} f(x_m),$$

$$\mu_k' = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

k^{th} moment

while if X is continuous then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

Note $M_X(t) = G_X(e^t)$, or equivalently $G_X(s) = M_X(\log s)$.

Proof Poisson + Poisson = Poisson (if independent: stable distribution!): MGF!

$$\text{Prob}(X = n) = \begin{cases} \frac{\lambda^n e^{-\lambda}}{n!} & \text{if } n \geq 0 \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases} \qquad M_X(t) = \sum_{m=-\infty}^{\infty} e^{tx_m} f(x_m)$$

$X \sim \text{Poiss}(\lambda)$

$$M_X(t) = E\left[e^{tX}\right] = \sum_{n=0}^{\infty} e^{tn} \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} e^{tn} \lambda^n$$

expansion of e^u with $u = e^t \lambda$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} (e^t \lambda)^n = e^{-\lambda} e^{e^t \lambda}$$

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) \quad \text{if } X_k \sim \text{Poiss}(\lambda_k) \text{ (indep)}$$

$$= e^{-\lambda_1} e^{e^t \lambda_1} e^{-\lambda_2} e^{e^t \lambda_2} = e^{-(\lambda_1+\lambda_2)} e^{e^t(\lambda_1+\lambda_2)}$$

MGF of $\text{Poiss}(\lambda_1+\lambda_2)$

Show if $X \sim \text{Poiss}(\lambda)$

Find mean $\mathbb{E}[X]$

Find Var $Var(X) = \mathbb{E}[(X-\mu)^2]$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Do for $\lambda = 1$ and $\lambda = n$

Definition 20.2.1 (Normal distribution) *A random variable X is normally distributed (or has the normal distribution, or is a Gaussian random variable) with mean μ and variance σ^2 if the density of X is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We often write $X \sim N(\mu, \sigma^2)$ to denote this. If $\mu = 0$ and $\sigma^2 = 1$, we say X has the standard normal distribution.

Theorem 20.2.2 (Central Limit Theorem (CLT)) *Let $X_1, \dots, X_N$ be independent, identically distributed random variables whose moment generating functions converge for $|t| < \delta$ for some $\delta > 0$ (this implies all the moments exist and are finite). Denote the mean by μ and the variance by σ^2, let*

$$\overline{X}_N = \frac{X_1 + \cdots + X_N}{N}$$

and set

$$Z_N = \frac{\overline{X}_N - \mu}{\sigma/\sqrt{N}}.$$

Then as $N \to \infty$, the distribution of Z_N converges to the standard normal (see Definition 20.2.1 for a statement).

(Stirling by CLT)

$X_k \sim Poiss(1)$, $X = X_1 + \cdots + X_n \sim Poiss(n)$: $Prob(X = n) = \frac{n^n e^{-n}}{n!}$

$Var(X) = n$

$Mean(X) = n$

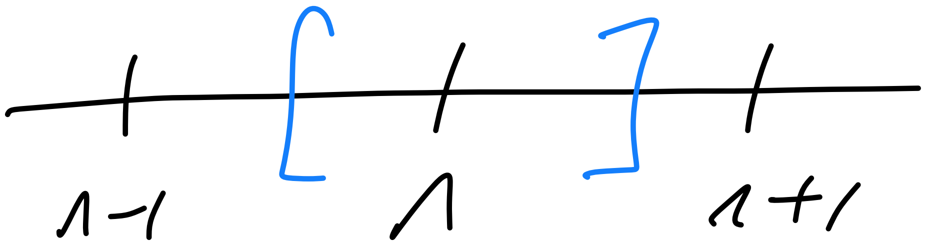
$StDev(X) = \sqrt{n}$

$X = X_1 + \cdots + X_n$

$X \sim Poiss(n)$

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{\sqrt{2\pi n}} \exp\left(-(x-n)^2/2n\right) dx = \frac{1}{\sqrt{2\pi n}} \int_{-1/2}^{1/2} e^{-t^2/2n} dt.$$

$\mu = n \qquad \sigma^2 = n$



CLT:

$$Prob(X = n) \approx \int_{n-1/2}^{n+1/2} \frac{1}{\sqrt{2\pi n}} e^{-\frac{(x-n)^2}{2n}} dx$$

integrand is 1 when $x = n$, when $x = n \pm \frac{1}{2}$ get $e^{-1/8n} \approx 1 - \frac{1}{8n} \approx 1$

So $\frac{n^n e^{-n}}{n!} \approx \frac{1}{\sqrt{2\pi n}} \cdot 1 \Rightarrow n! \approx n^n e^{-n} \sqrt{2\pi n}$

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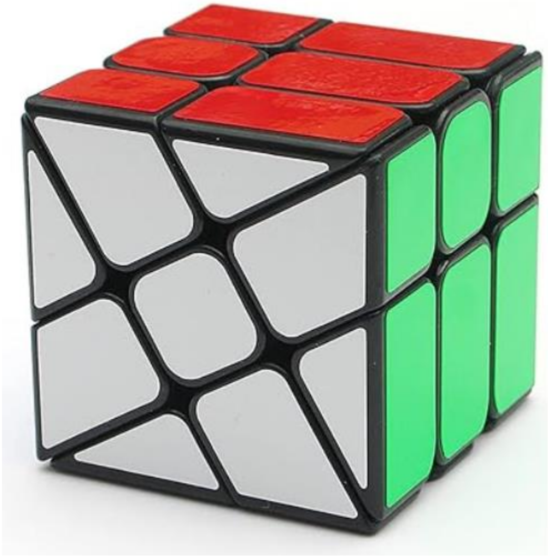
Lecture 34: 11-29-23:

•Method of Stationary Phase II: https://youtu.be/NCalCkFs2a0

•Previous year's iteration: Lecture 27: 11/15/17: Laplace's Method, Stirling's Formula: https://youtu.be/AycMlf4Mbyo (2015 lecture: https://youtu.be/GvKI5I_cfDQ)



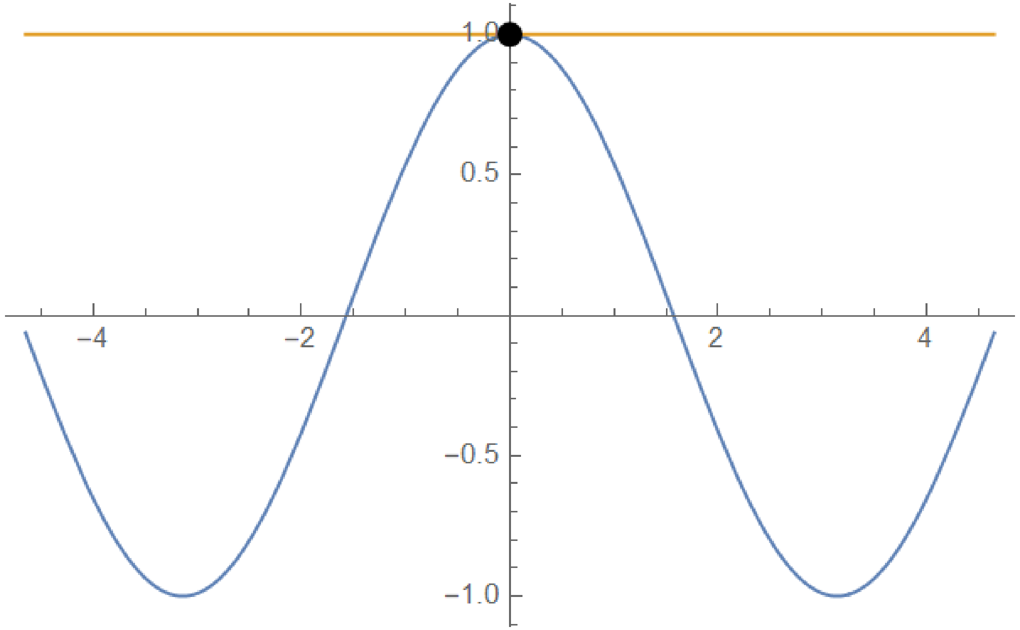
Taylor Series

Goal is to see how well Taylor Series approximate functions,

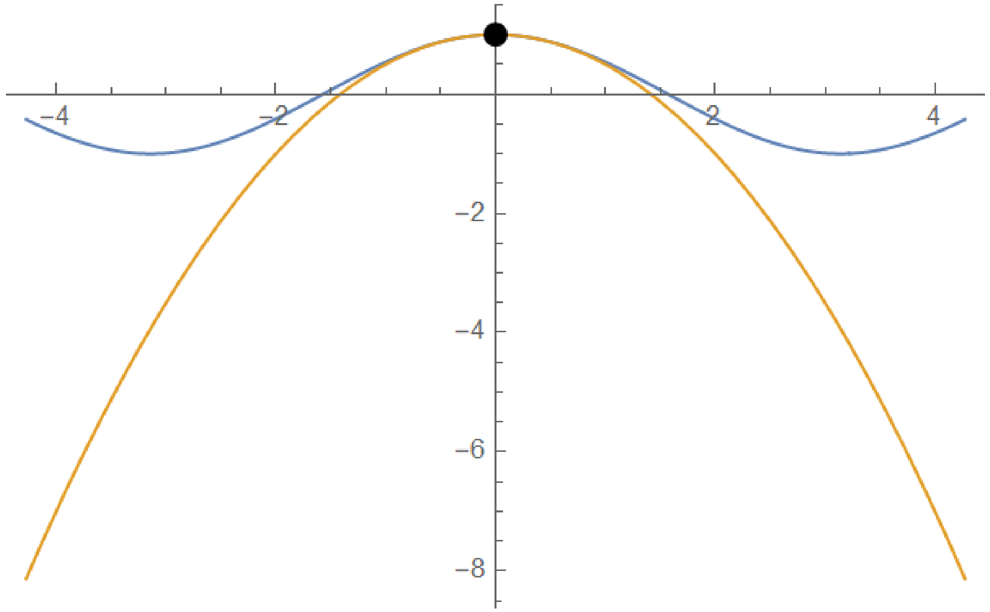
how listtle later terms change approximation ....

For definiteness, will do Cos[x]

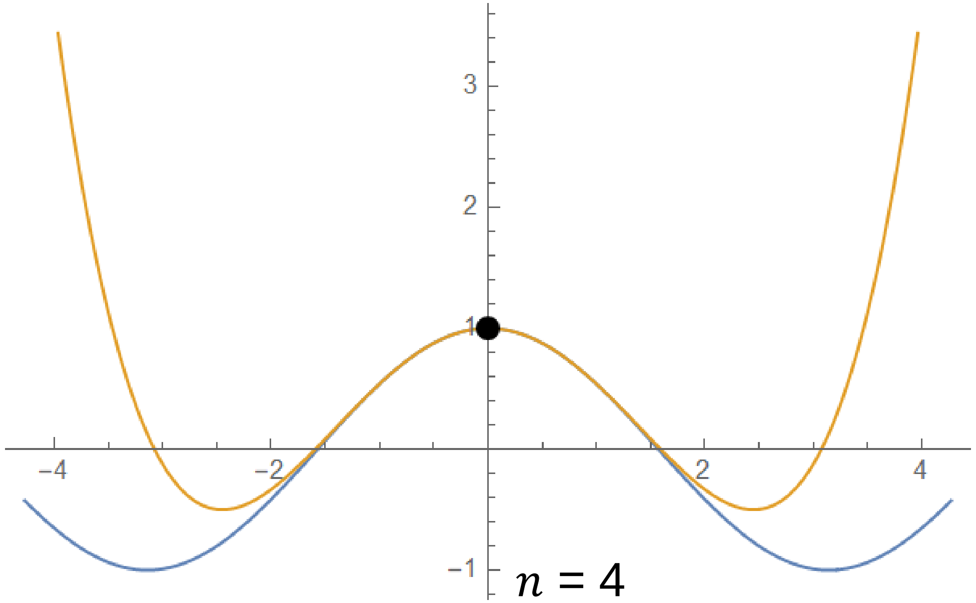
```
coeff[x0_, n_] := If[Mod[n, 4] == 0, Cos[x0],
   If[Mod[n, 4] == 1, - Sin[x0],
    If[Mod[n, 4] == 2, - Cos[x0], Sin[x0]]
   ]];
approx[x_, x0_, n_] := Sum[coeff[x0, nn] (x - x0) ^ nn / nn!, {nn, 0, n}]

Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},

     Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi / 2}, {n, 1, 40},

  {c, 4, .01}]
```



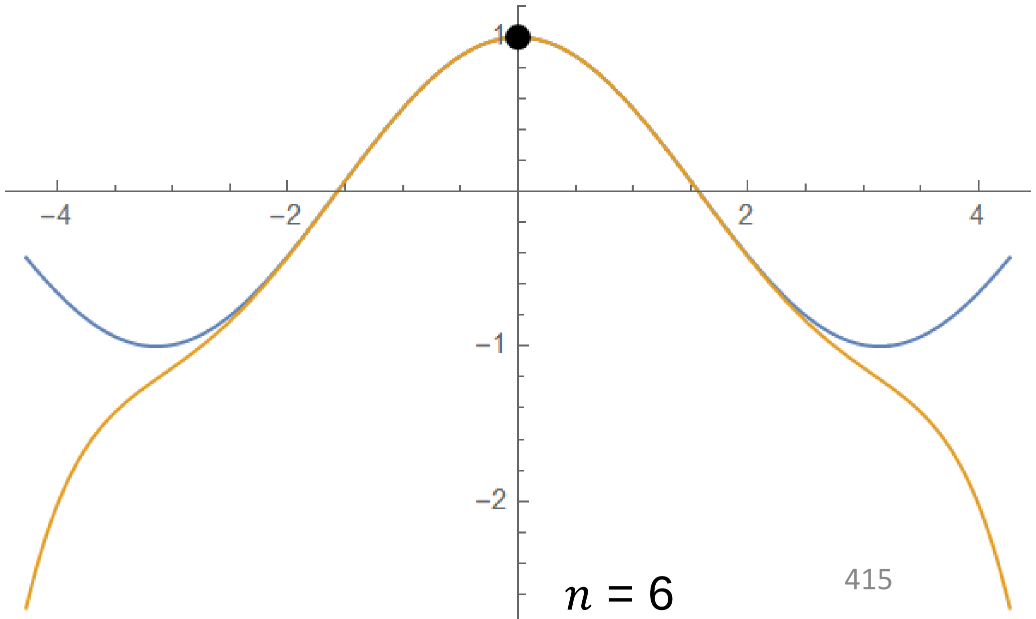
$n = 0$



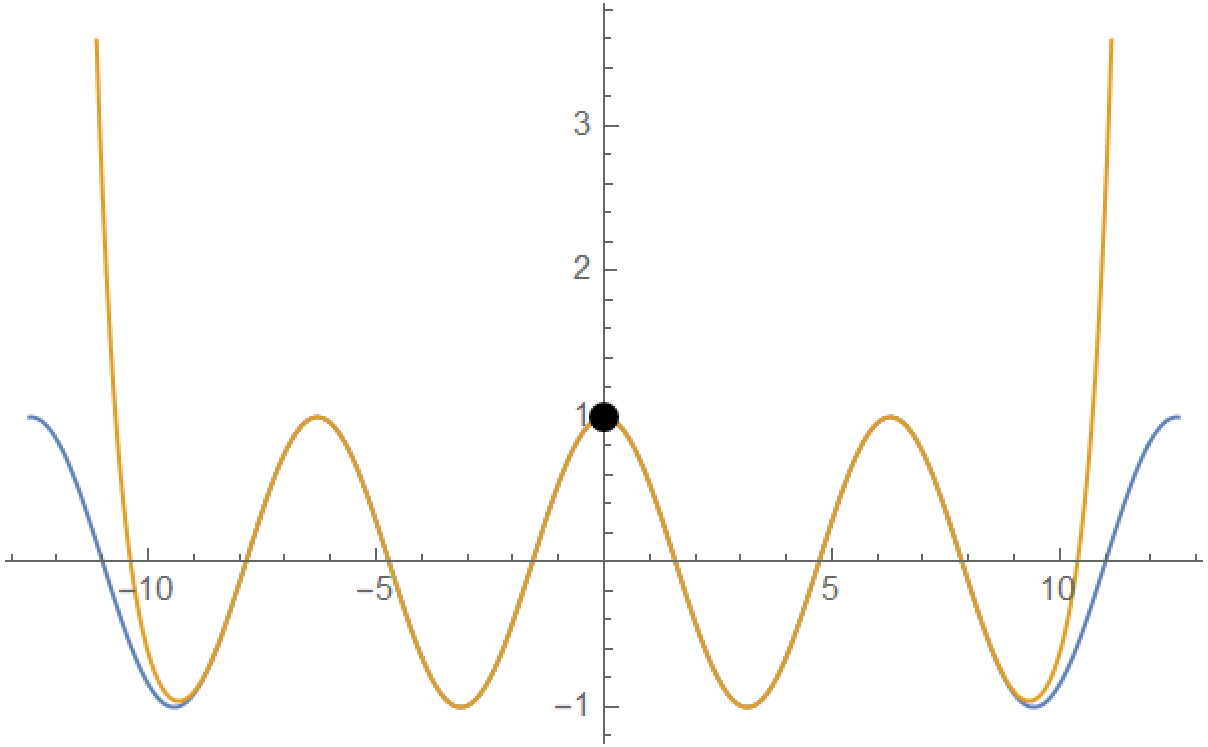
$n = 2$



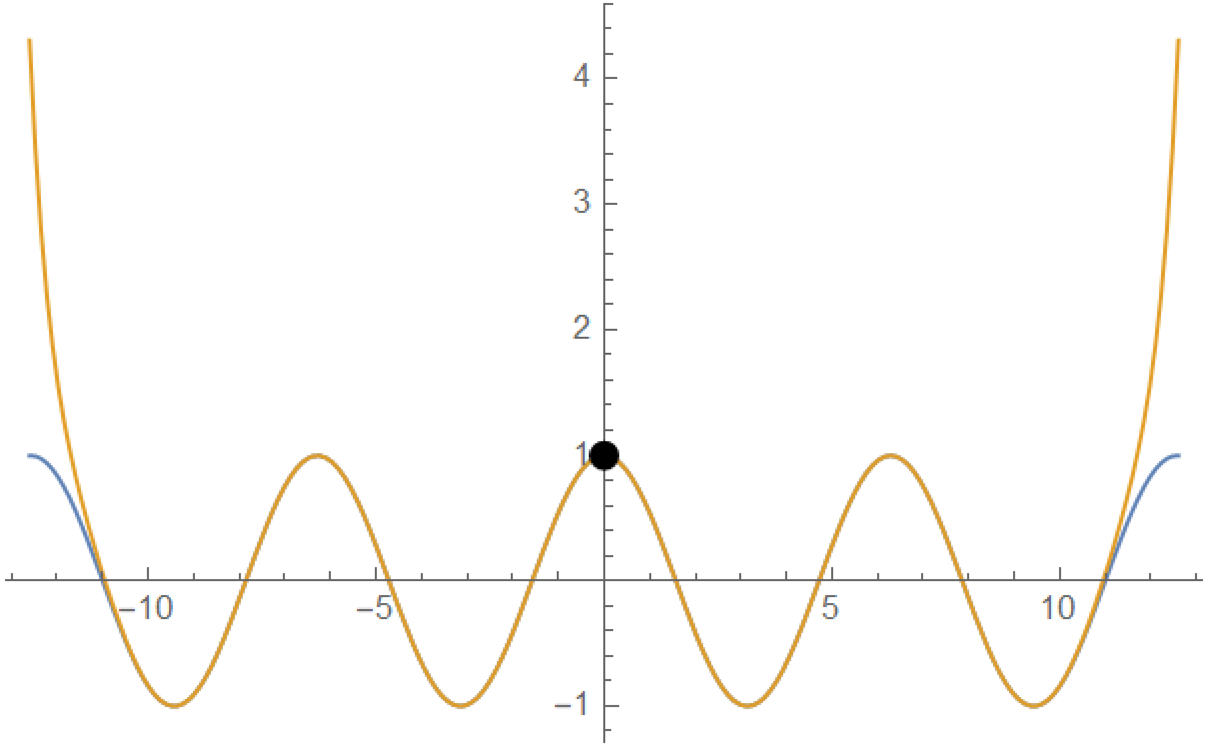
$n = 4$



$n = 6$

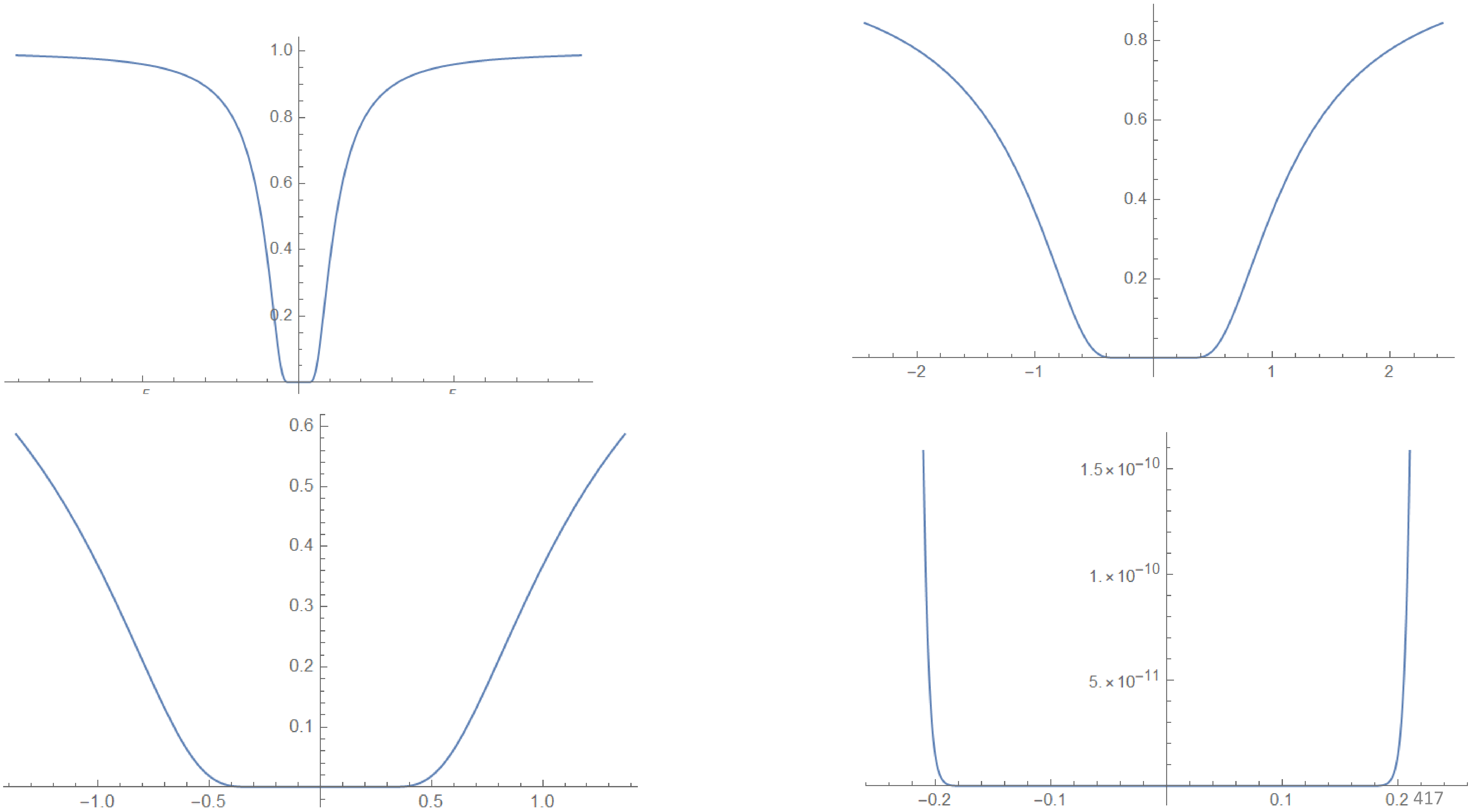


$n = 24$



$n = 28$

```
Manipulate[Plot[Exp[−1/x^2], {x, −c, c}], {c, 10, .25}]
```



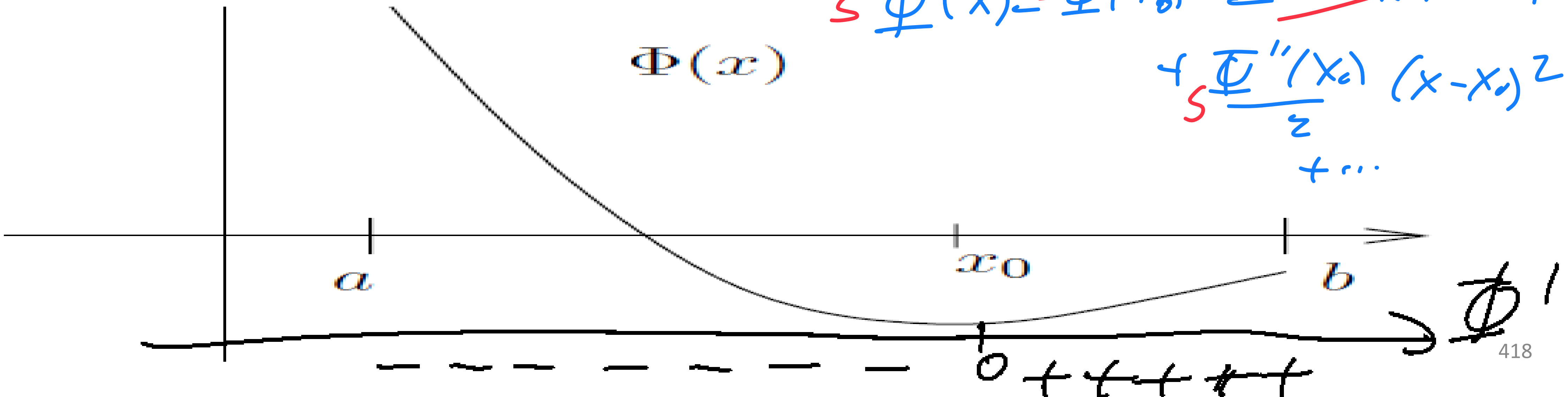
Consider

$S > 0$

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx$$

where the **phase** Φ is real-valued, and both it and the **amplitude** ψ are assumed for simplicity to be indefinitely differentiable. Our hypothesis regarding the minimum of Φ is that there is an $x_0 \in (a, b)$ so that $\Phi'(x_0) = 0$, but $\Phi''(x) > 0$ throughout $[a, b]$ (Figure 2 illustrates the situation.)

$$s\Phi(x) = s\Phi(x_0) + s\cancel{\Phi'(x_0)(x-x_0)}^{0} + s\frac{\Phi''(x_0)}{2}(x-x_0)^2 + \cdots$$



Proposition 2.1 *Under the above assumptions, with $s > 0$ and $s \to \infty$,*

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx = e^{-s\Phi(x_0)}\left[\frac{A}{s^{1/2}} + O\left(\frac{1}{s}\right)\right], \tag{8}$$

where

$$A = \sqrt{2\pi}\frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}.$$

Is this reasonable? $\Phi(x) = \Phi(x_0) + \frac{1}{2}\Phi''(x_0)(x-x_0)^2$

$t = x - x_0$

$$\int_a^b e^{-s\Phi(x_0)}\left[e^{-s\Phi''(x_0)(x-x_0)^2/2}\right]\psi(x)\,dx$$

$\sigma_s^2 = 1/s\Phi''(x_0)$

$\approx \psi(x_0)$ if t near 0

$$e^{-s\Phi(x_0)}\sqrt{\frac{2\pi}{s\Phi''(x_0)}}\int_{a-x_0}^{b-x_0}\frac{1}{\sqrt{2\pi}\sigma}e^{-t^2/2\sigma_s^2}\psi(x_0+t)\,dt$$

Mast when $t=0$, $|t| >\!>\!>$ σ makes small

$$\int_a^b e^{-s\Phi(x)}\,\psi(x)\,dx$$

wlog $\Phi(x_0)=0$

$$\Phi(x) = \tfrac{1}{2}\Phi''(x_0)(x-x_0)^2 + \text{Order}\left((x-x_0)^3\right) \quad \text{near } x_0$$

$$t = x - x_0$$

$$s\,\Phi(x) = \underbrace{s\,\tfrac{1}{2}\Phi''(x_0)\,t^2}_{\text{Size of } t?} + \underbrace{s\,\text{Order}(t^3)}$$

$$s\cdot\frac{s^{2\epsilon}}{s} \sim s^{2\epsilon}$$

$$s\cdot\frac{s^{3\epsilon}}{s^{3/2}} \qquad \frac{s^{3\epsilon}}{s^{1/2}}$$

$$t = \frac{s^{\epsilon}}{s^{1/2}}$$

$$\left(t = \frac{f(s)}{\sqrt{s}}\right)$$

So if $3\epsilon < \tfrac{1}{2}$ Small

So $\epsilon < 1/6$

$$e^{s\Phi(x)} = e^{s\frac{1}{2}\Phi''(x_0)t^2}\cdot e^{s\,\text{Order}(t^3)}$$

This goes to 1 as $s\to\infty$ if $|t| < \dfrac{s^{1/6}}{s^{1/2}}$

$$\int_a^b \sim \;=\; \int_{x_0 - \frac{s^{1/6}}{s^{1/2}}}^{x_0 + s^{1/6}/s^{1/2}} \sim \;+\; \int_{|x-x_0| > \frac{s^{1/6}}{s^{1/2}}} \sim$$

do as before

Show good decay as well beyond $1/\sqrt{s}$ by factor of s^{ε}

Theorem 2.3 *If $|s| \to \infty$ with $s \in S_\delta$, then*

$$\Gamma(s) = e^{s\log s} e^{-s} \frac{\sqrt{2\pi}}{s^{1/2}} \left(1 + O\left(\frac{1}{|s|^{1/2}}\right)\right). \tag{11}$$

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x} = \int_0^\infty e^{-x+s\log x} \frac{dx}{x}$$

Consider

$$\int_a^b e^{-s\Phi(x)} \psi(x)\, dx$$

where the **phase** Φ is real-valued, and both it and the **amplitude** ψ are assumed for simplicity to be indefinitely differentiable. Our hypothesis regarding the minimum of Φ is that there is an $x_0 \in (a, b)$ so that $\Phi'(x_0) = 0$, but $\Phi''(x_0) > 0$ throughout $[a, b]$

Proposition 2.1 *Under the above assumptions, with $s > 0$ and $s \to \infty$,*

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where

$$A = \sqrt{2\pi} \frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}.$$

$x \to sx \qquad dx \to s\,dx$

$x^s = (sx)^s = s^s x^s = s^s e^{s\log x}$

$$\Gamma(s) = s^s \int_0^\infty e^{-sx} e^{s\log x} \frac{dx}{x}$$

$$= s^s \int_0^\infty e^{-s(x-\log x)} \frac{1}{x}\, dx$$

$\Phi(x) = x - \log x$

$\Phi'(x) = 1 - 1/x$

$\Phi'(1) = 0 \qquad x_0 = 1$

$\Phi''(x) = 1/x^2 > 0$

$$s^s \sqrt{2\pi} \frac{1}{s^{1/2} \cdot 1} e^{-s} = \Gamma(s)$$

$\approx (s-1)!$

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Lecture 35: 12-1-23: No class (midterm)

Lecture 36: 12-4-23: The Uncertainty Principle: **https://youtu.be/D8onKzVK9G4**

Plan for the day: Lecture 35: December 10, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Cauchy-Schwarz Inequality
- Fourier transform
- Uncertainty Principle (in mathematics)

General items.

- Generalizations (matrix exponentiation)
- Unreasonable effectiveness of mathematics: Wigner: https://www.maths.ed.ac.uk/~v1ranick/papers/wigner.pdf

Statement of the inequality [edit]

The Cauchy–Schwarz inequality states that for all vectors u and v of an inner product space it is true that

$$|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \cdot \langle \mathbf{v}, \mathbf{v} \rangle, \qquad \text{(Cauchy-Schwarz inequality [written using only the inner product])}$$

where $\langle \cdot, \cdot \rangle$ is the inner product. Examples of inner products include the real and complex dot product; see the examples in inner product. Every inner product gives rise to a norm, called the *canonical* or *induced norm*, where the norm of a vector $\mathbf{u}$ is denoted and defined by:

$$\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

so that this norm and the inner product are related by the defining condition $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$, where $\langle \mathbf{u}, \mathbf{u} \rangle$ is always a non-negative real number (even if the inner product is complex-valued). By taking the square root of both sides of the above inequality, the Cauchy–Schwarz inequality can be written in its more familiar form:[6][7]

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|. \qquad \text{(Cauchy-Schwarz inequality [written using norm and inner product])}$$

Moreover, the two sides are equal if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.[8][9][10]

https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality

For real inner product spaces [edit]

https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality

Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space. Consider an arbitrary pair $u, v \in V$ and the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(t) = \langle tu + v, tu + v\rangle$. Since the inner product is positive-definite, $p(t)$ only takes non-negative values. On the other hand, $p(t)$ can be expanded using the bilinearity of the inner product and using the fact that $\langle u, v\rangle = \langle v, u\rangle$ for real inner products:

$$p(t) = \|u\|^2 t^2 + t\left[\langle u, v\rangle + \langle v, u\rangle\right] + \|v\|^2 = \|u\|^2 t^2 + 2t\langle u, v\rangle + \|v\|^2.$$

Thus, p is a polynomial of degree 2 (unless $u = 0$, which is a case that can be independently verified). Since the sign of p does not change, the discriminant of this polynomial must be non-positive:

$$\Delta = 4\left(\langle u, v\rangle^2 - \|u\|^2\|v\|^2\right) \leqslant 0.$$

The conclusion follows.

Use $x^2 \geq 0$ if $x \in \mathbb{R}$

will do real case: $\|\vec{u} + \lambda\vec{v}\|^2 \geq 0$ and this is $(\vec{u} + \lambda\vec{v}) \cdot (\vec{u} + \lambda\vec{v})$

so $0 \leq \|\vec{u}\|^2 + 2\lambda\, \vec{u}\cdot\vec{v} + \lambda^2\|\vec{v}\|^2$

Goal: $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\|$ Find λ that minimizes

Baker–Campbell–Hausdorff formula:
https://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula

In mathematics, the **Baker–Campbell–Hausdorff formula** is the solution for Z to the equation

$$e^X e^Y = e^Z$$

$$e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

for possibly noncommutative X and Y in the Lie algebra of a Lie group. There are various ways of writing the formula, but all ultimately yield an expression for Z in Lie algebraic terms, that is, as a formal series (not necessarily convergent) in X and Y and iterated commutators thereof. The first few terms of this series are:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots,$$

where "$\cdots$" indicates terms involving higher commutators of X and Y. If X and Y are sufficiently small elements of the Lie algebra $\mathfrak{g}$ of a Lie group G, the series is convergent. Meanwhile, every element g sufficiently close to the identity in G can be expressed as $g = e^X$ for a small X in $\mathfrak{g}$. Thus, we can say that *near the identity* the group multiplication in G—written as $e^X e^Y = e^Z$—can be expressed in purely Lie algebraic terms. The Baker–Campbell–Hausdorff formula can be used to give comparatively simple proofs of deep results in the Lie group–Lie algebra correspondence.

$[X, Y] = XY - YX$, is zero if commute

https://en.wikipedia.org/wiki/Uncertainty_principle

ermöglichen, als es der Gleichung (1) entspricht, so wäre die Quantenmechanik unmöglich. Diese Ungenauigkeit, die durch Gleichung (1) festgelegt ist, schafft also erst Raum für die Gültigkeit der Beziehungen, die in den quantenmechanischen Vertauschungsrelationen

$$pq - qp = \frac{h}{2\pi i}$$

ihren prägnanten Ausdruck finden; sie ermöglicht diese Gleichung, ohne daß der physikalische Sinn der Größen p und q geändert werden müßte.

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

where $\hbar$ is the reduced Planck constant, $h/(2\pi)$.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx, \qquad \forall\, \xi \in \mathbb{R}.$$

15.2. The Fourier transform of the derivative: if $g = df/dx$ then $\hat{g}(\xi) = i\xi \hat{f}(\xi)$. (Integrate by parts).

Proof: All fns are nice: rapid decay at ∞, in L^1, L^2

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)\, e^{-ix\xi}\, dx$$

$$= \int_{-\infty}^{\infty} f'(x)\, e^{-ix\xi}\, dx$$

$u = e^{-ix\xi}$ $\quad dv = f'(x)dx$

$du = -i\xi e^{-ix\xi}dx$ $\quad v = f(x)$

$$= \overset{0}{uv}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\, du$$

$$= i\xi \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx = i\xi\, \hat{f}(\xi)$$

15.3. The Fourier transform under multiplication by x: if $h(x) = xf(x)$ then $d\hat{f}(\xi)/d\xi = -i\hat{h}(\xi)$.

Proof:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$$

$\frac{d}{d\xi}$

$$\frac{d\hat{f}}{d\xi} = \int_{-\infty}^{\infty} f(x) \left[\frac{d}{d\xi} e^{-ix\xi}\right] dx$$

$$= -i \int_{-\infty}^{\infty} [x f(x)]\, e^{-ix\xi}\, dx$$

$$= -i \int_{-\infty}^{\infty} h(x)\, e^{-ix\xi}\, dx$$

$$= -i\, \hat{h}(\xi) \qquad \square$$

15.4. The Fourier transform under translation: find $\hat{g}(\xi)$ if $g(x) = f(x-a)$, a fixed.

Have $\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-ix\xi} dx$

$$= \int_{-\infty}^{\infty} f(x-a) e^{-i(x-a+a)\xi} dx$$

$t = x-a$ $\qquad$ $x: -\infty \to \infty$

$dt = dx$ $\qquad$ $t: -\infty \to \infty$

$$= \int_{-\infty}^{\infty} f(t) e^{-it\xi} \underline{e^{-ia\xi}} dt$$

$$= e^{-ia\xi} \hat{f}(\xi)$$

15.5. The Fourier transform under scaling: find $\hat{h}(\xi)$ if $h(x) = f(\rho x)$, $\rho > 0$.

Consider $\hat{h}(\xi) = \int_{-\infty}^{\infty} h(x)\, e^{-ix\xi}\, dx$

$$= \int_{-\infty}^{\infty} f(\rho x)\, e^{-i\rho x(\xi/\rho)}\, dx$$

$t = \rho x$ so $dt = \rho\, dx$

$x: -\infty \to \infty$ means $t: -\infty \to \infty$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-it(\xi/\rho)}\, dt\, \frac{1}{\rho}$$

$$= \frac{1}{\rho} \hat{f}(\xi/\rho)$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx, \qquad \forall\, \xi \in \mathbb{R}. \quad \textbf{(Eq.1)}$$

15.6. The Fourier transform of a Gaussian function: let $f(x) = e^{-x^2/2}/\sqrt{2\pi}$. Show that $\hat{f}(\xi) = e^{-\xi^2/2}$. (Note that f satisfies the differential equation $df/dx = -xf(x)$. Show that $\hat{f}$ satisfies the same equation (with respect to the ξ variable), by using results of preceding problems. Deduce that $\hat{f}$ is a multiple of $e^{-\xi^2/2}$. Because f has integral $= 1$, it follows that $\hat{f}(0) = 1$.

$$\mathrm{Prob}\{x \in I\} = \int_I |f(x)|^2 \, dx. \qquad E \doteq \int_{\mathbf{R}} x|f(x)|^2 \, dx. \qquad V = \int_{\mathbf{R}} (x - E)^2 |f(x)|^2 \, dx$$

adjust so that $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$

$$\hat{E} = \frac{1}{2\pi} \int_{\mathbf{R}} \xi |\hat{f}(\xi)|^2 \, d\xi \qquad \hat{V} = \frac{1}{2\pi} \int_{\mathbf{R}} (\xi - \hat{E})^2 |\hat{f}(\xi)|^2 \, d\xi$$

If X has density P_X then $E[X^k] = \int_{-\infty}^{\infty} x^k P_X(x) \, dx$

↳ mean is $k=1$, denote by μ

variance is $E[(X-\mu)^2]$, denote by σ^2

Here $P_X = |f(x)|^2$ or $|\hat{f}(\xi)|^2$

Proposition. *If* f *is an element of* $L^2(\mathbf{R})$ *such that* $||f|| = 1$, *then the product of the variances of* f *and of* $\hat{f}$, $V \cdot \hat{V}$, *is at least* $\frac{1}{4}$.

$$Qf(x) = xf(x); \qquad Pf(x) = \frac{1}{i}\frac{df(x)}{dx}$$

$$PQ - QP = -iI; \qquad (Qf, g) = (f, Qg); \qquad (Pf, g) = (f, Pg)$$

Proof: $Q(Pf) = Q\left(\frac{1}{i}f'\right) = \frac{1}{i}x f'(x)$

$P(Qf) = P(xf) = \frac{1}{i}\left[f(x) + x f'(x)\right]$

$(PQ - QP)f = \frac{1}{i}\left[f(x) + xf'(x) - xf'(x)\right] = \frac{1}{i}f(x)$

but $1/i = -i$

so $PQ - QP = -iI$

Rest: $(Qf, g) = \int_{-\infty}^{\infty}\left[x f(x)\right]\bar{g}(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\overline{\left[x g(x)\right]}\,dx = (f, Qg)$

(Pf, g) use integration by parts.

Then the Fourier transform of Pf is $\xi\hat{f}(\xi)$, so

$$V = ||(Q - E)f||^2; \qquad \hat{V} = ||(P - \hat{E})\hat{f}||^2. \tag{16.6}$$

Know $\int_{-\infty}^{\infty} |f|^2 dx = 1$

as $\int_{-\infty}^{\infty} \underbrace{-iI}_{PQ-QP} |f|^2 dx = -i$

$$1 \overset{\text{assumption}}{=} ||f||^2 \overset{\text{def}}{=} (f,f) \overset{①}{=} i(PQf - QPf, f) \overset{②}{=} i[(Qf, Pf) - (Pf, Qf)]$$

$$\overset{③}{=} 2\,\mathrm{Im}(Pf, Qf) \overset{④}{\leq} 2||Qf|| \cdot ||Pf||.$$

V $\hat{V}$

① is true as $PQ - QP = -iI$ so $i(PQ - QP) = I$

② $(PQf - QPf, f) = \langle PQf, f\rangle - \langle QPf, f\rangle$
$= \langle Qf, Pf\rangle - \langle Pf, Qf\rangle$

③ Note $\langle a, b\rangle = \overline{\langle b, a\rangle}$ here $\langle Qf, Pf\rangle - \overline{\langle Qf, Pf\rangle}$
$(a+ib) - (a-ib) = 2ib$ or $2i\,\mathrm{Im}(a+ib)$

④ Cauchy-Schwarz

if $E = \hat{E} = 0$ then $\langle Qf, Qf\rangle = \int_{-\infty}^{\infty} x f(x) \overline{x f(x)}\, dx = V$

want $\int (x-\mu)^2 |f(x)|^2 dx$ if $\mu = 0$ this is the same

Now it is also true that

$$(P - \hat{E}I)(Q - EI) - (Q - EI)(P - \hat{E}I) = -iI$$

so we may repeat the calculation (16.8) with $Q - EI$ in place of Q and $P - \hat{E}I$ in place of P to obtain the desired inequality.

In the usual representation of the wave function, the *position operator* is the operator Q above and the *momentum operator* is hP, where P is the operator above and $h > 0$ is Planck's constant. Thus the inequality proved above gives the quantitative form of the relationship between uncertainty in measurement of position and uncertainty in measurement of velocity known as the *Heisenberg Uncertainty Principle:*

$$\sqrt{V_Q} \cdot \sqrt{V_{hP}} \geq \frac{h}{2}. \tag{16.10}$$

Algebraic: Solve poly of finite degree and integer coeff

Transcendental: all else

Thm: e, π transcendental

Thm: Either $e+\pi$ or $e\pi$ is transcendental

~~Useful fact: $e^{i\pi} = -1$~~

Thm: If α, β are algebraic so is $\alpha+\beta$ and $\alpha\beta$

Assume $e+\pi$, $e\pi$ both algebraic

Then $(e+\pi)^2 - 4e\pi = (e-\pi)^2$ so $(e-\pi)^2$ is alg $\to$ $e-\pi$ alg

$$\left.\begin{matrix} e+\pi & \text{alg} \\ e-\pi & \text{alg} \end{matrix}\right\} \xrightarrow{\text{add}} 2e \text{ alg} \quad \text{contradiction}$$

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Lecture 37: 12-6-23: Eigenvalues: https://youtu.be/47xCLUs12lk

$A\vec{v} = \lambda \vec{v}, \quad \vec{v} \neq \vec{0}$

$\mathrm{Det}(A - \lambda I) = 0$

- See for a longer introduction (or happy to meet and chat):
- Lecture 31: 12/03/21: Eigenvalues and Random Matrix Theory:
  Part I: https://youtu.be/On9hT2ZFpdw (slides)
- Lecture 32: 12/06/21: Random Matrix Theory to L-Functions:
  Part II: https://youtu.be/FoKKIMs9wV8 (slides)

Gershgorin circle theorem

13 languages

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From Wikipedia, the free encyclopedia

In mathematics, the **Gershgorin circle theorem** may be used to bound the spectrum of a square matrix. It was first published by the Soviet mathematician Semyon Aronovich Gershgorin in 1931. Gershgorin's name has been transliterated in several different ways, including Geršgorin, Gerschgorin, Gershgorin, Hershhorn, and Hirschhorn.

Statement and proof [edit]

Let A be a complex $n \times n$ matrix, with entries a_{ij}. For $i \in \{1, \dots, n\}$ let R_i be the sum of the absolute values of the non-diagonal entries in the i-th row:

$$R_i = \sum_{j \neq i} |a_{ij}| .$$

Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc centered at a_{ii} with radius R_i. Such a disc is called a Gershgorin disc.

Theorem. Every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$.

https://en.wikipedia.org/wiki/Gershgorin_circle_theorem

Eigenvalues of Matrices

Defns: A is symmetric if $A = A^T$

A is complex Hermitian if $A = \bar{A}^T$

ex: $\begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}$ real symmetric

$\begin{pmatrix} 3 & 4+i \\ 4-i & 2 \end{pmatrix}$ complex Hermitian

Defn: use H for complex conjugate transpose: $\bar{A}^T = A^H$

$\vec{v} \in \mathbb{C}^n$ then $\|\vec{v}\|^2 = \vec{v}^H \vec{v}$

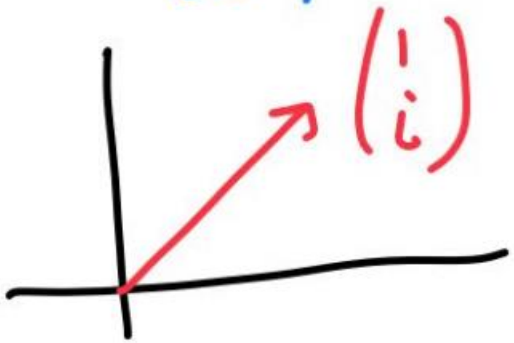
Note H important!

$\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ then $\vec{v}^T \vec{v} = (1 \;\; i)\begin{pmatrix} 1 \\ i \end{pmatrix} = 1^2 + i^2 = 0$

but $\vec{v} \neq \vec{0}$

However $\vec{v}^H \vec{v} = (1 \;\; -i)\begin{pmatrix} 1 \\ i \end{pmatrix} = 1^2 - i^2 = 2$

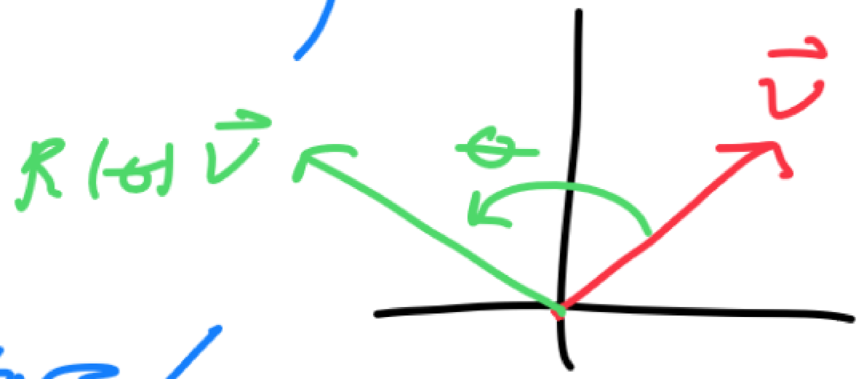
so $\|\vec{v}\| = \sqrt{2}$



Eigenvalues / Eigenvectors of real matrices not necessarily real.

$$A = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

↳ Rotates by θ



Clearly no real eigenvectors / eigenvalues

Thm: If $A = A^H$ then eigenvalues real

Proof: $\vec{v}^H A \vec{v} = \vec{v}^H A^H \vec{v}$ assuming $A\vec{v} = \lambda\vec{v}$

$$\vec{v}^H (A\vec{v}) = (A\vec{v})^H \vec{v}$$

$$\vec{v}^H \lambda \vec{v} = (\lambda \vec{v})^H \vec{v}$$

$$\lambda \vec{v}^H \vec{v} = \bar{\lambda} \vec{v}^H \vec{v}$$

$$\lambda \|\vec{v}\|^2 = \bar{\lambda} \|\vec{v}\|^2$$

as $\|\vec{v}\| \neq 0 \implies \lambda = \bar{\lambda} \implies \lambda \in \mathbb{R}$ $\blacksquare$

<u>Corollary</u>: Eigenvalues of real symm matrices are real.

Importance: can order.

Think energy levels

What of other types of matrices?

Defn: U is unitary if $U^H U = U U^H = I$

Q is orthogonal if $Q^T Q = Q Q^T = I$

(let's take real)

Thm: Eigenvalues of unitary matrices are of the form $e^{i\theta}$ $(\theta \in \mathbb{R})$

Proof $\vec{v}^H U^H U \vec{v} = \vec{v}^H \vec{v} = \vec{v}^H I \vec{v}$, $U\vec{v} = \lambda\vec{v}$

as $U^H U = I$

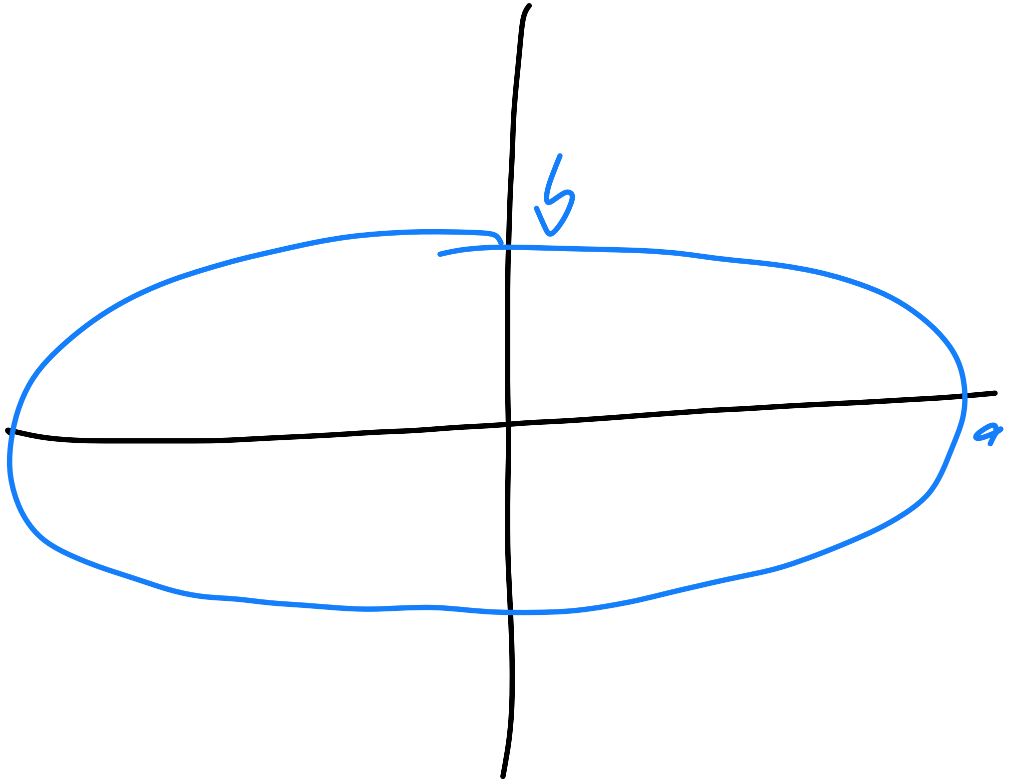
$= (\vec{v}^H U^H)(U\vec{v})$

$= (U\vec{v})^H (U\vec{v})$

$= (\lambda\vec{v})^H(\lambda\vec{v})$

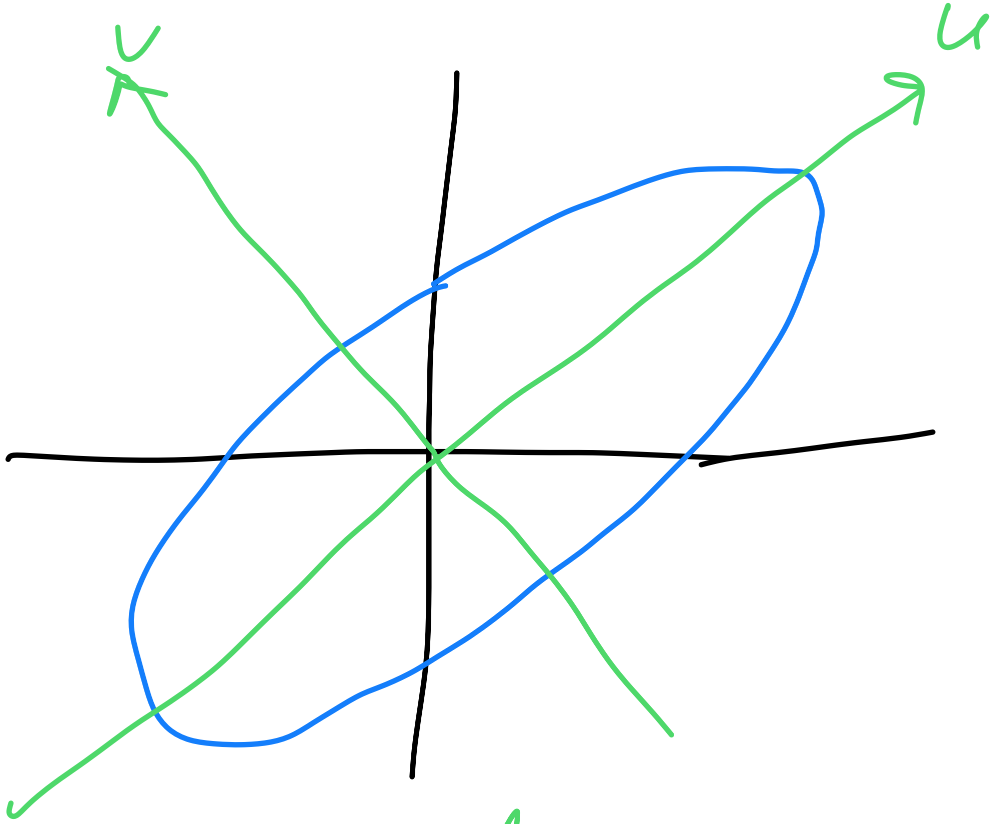
$= \bar{\lambda}\lambda\,\vec{v}^H\vec{v} \Rightarrow |\lambda|^2 \|\vec{v}\|^2 = \|\vec{v}\|^2$

as $\|\vec{v}\|^2 \neq 0 \rightarrow |\lambda| = 1 \Rightarrow \lambda = e^{i\theta}$ $\blacksquare$



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$(x\ \ y)\begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1$$



A

$$(u\ \ v)\begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

and $\begin{pmatrix} u \\ v \end{pmatrix} = Q\begin{pmatrix} x \\ y \end{pmatrix}$

$$(x\ \ y)\, Q^T A\, Q \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

Thm: Eigenvalue Trace Lemma:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

not diagonalizable

$$Tr(A) = \sum_{k=1}^{N} \lambda_k(A)$$

$$Tr(A) = a_{11} + \cdots + a_{NN}$$

Proof: Trivial if A is diagonal as evalues are the diagonal entries.

$$\text{Trace}(S^{-1}AS) = \text{Trace}(ASS^{-1}) = \text{Trace}(A)$$

Every matrix can be put into upper triangular form, and trivially true then.

A and (Q^TAQ)

$A\vec{v} = \lambda\vec{v}$ $\quad (Q^TAQ)Q^T\vec{v} = Q^TA(QQ^Tv)$

$$= Q^T(A\vec{v}) = \lambda(Q^T\vec{v})$$

Method of moments

have $\lambda_1, \dots, \lambda_N$

m^{th}-moment: $\frac{1}{N} \sum_{k=1}^{N} \lambda_k^m$

if know moments, know λ's as only finitely many.

"Pass to the limit..."

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Koloˇglu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability 26 (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

Slides for the rest of the talk here:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/ManhattanToEC_Colloq_MASONIV2020.pdf

We construct the characteristic function[3] of the limiting spectral distribution. Let X_m be a random variable with density f_m. Then (remembering the odd moments vanish)

$$\begin{aligned}
\phi_m(t) &= \mathbb{E}[e^{itX_m}] = \sum_{\ell=0}^{\infty} \frac{(it)^\ell M_{\ell;m}}{\ell!} \\
&= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k;m}}{(2k)!} \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k)!} m^{-(k+1)} (2k-1)!!\, c(k,m)(-t^2)^k. \qquad (3.6)
\end{aligned}$$

In order to obtain a closed form expression, we rewrite the characteristic function as

$$\phi_m(t) = \frac{1}{m} \sum_{k=0}^{\infty} c(k,m) \frac{1}{k!} \left(\frac{-t^2}{2m}\right)^k, \qquad (3.7)$$

using $(2k-1)!! = \frac{(2k)!}{2^k k!}$. The reason for this is that we can interpret the above as a certain coefficient in the convolution of two known generating functions, which can be isolated by a contour integral. Specifically, consider the two functions

$$F(y) := \frac{1}{2y}\left(\left(\frac{1+y}{1-y}\right)^m - 1\right) = \sum_{k=0}^{\infty} c(k,m) y^k \quad \text{and} \quad G(y) := e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}. \qquad (3.8)$$

Note that $\phi_m(t)$ is the function whose power series is the sum of the products of the k^{th} coefficients of $G(-y^2/2m)$ (which is related to the exponential distribution) and $F(y)$ (which is related to the generating function of the $\varepsilon_g(k)$). Thus, we may use a multiplicative convolution to find a formula for the sum. By Cauchy's residue theorem, integrating $F(z^{-1})G(-t^2z/2m)z^{-1}$ over the circle of radius 2 yields

$$\phi_m(t) = \frac{1}{2\pi i m} \oint_{|z|=2} F(z^{-1}) G\left(-\frac{t^2 z}{2m}\right) \frac{dz}{z}, \qquad (3.9)$$

since the constant term in the expansion of $F(z^{-1})G(-t^2z/2m)$ is exactly the sum of the products of coefficients for which the powers of y in $F(y)$ and $G(y)$ are the same.[4] We are integrating along the

circle of radius 2 instead of the unit circle to have the pole inside the circle and not on it. Thus

$$\begin{aligned}
\phi_m(t) &= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}}\right)^m - 1\right) e^{-t^2 z/2m} \frac{dz}{z} \\
&= \frac{1}{4\pi i m} \oint_{|z|=2} \left(\left(\frac{z+1}{z-1}\right)^m - 1\right) e^{-t^2 z/2m} dz \\
&= \frac{e^{-t^2/2m}}{4\pi i m} \oint_{|z|=2} \left(\left(1 + \frac{2}{z-1}\right)^m - 1\right) e^{-t^2(z-1)/2m} dz \\
&= \frac{e^{-t^2/2m}}{4\pi i m} \oint_{|z|=2} \sum_{l=0}^{m} \binom{m}{l} \left(\frac{2}{z-1}\right)^l \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{-t^2}{2m}\right)^s (z-1)^s dz \\
&\quad - \frac{e^{-t^2/2m}}{4\pi i m} \oint_{|z|=2} e^{-t^2(z-1)/2m} dz. \qquad (3.10)
\end{aligned}$$

By Cauchy's Residue Theorem the second integral vanishes and the only surviving terms in the first integral are when $l - s = 1$, whose coefficient is the residue. Thus

$$\begin{aligned}
\phi_m(t) &= \frac{e^{-t^2/2m}}{2m} \sum_{l=1}^{m} \binom{m}{l} 2^l \frac{1}{(l-1)!} \left(\frac{-t^2}{2m}\right)^{l-1} \\
&= \frac{1}{m} e^{-t^2/2m} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} \left(\frac{-t^2}{m}\right)^{l-1} = \frac{1}{m} e^{-t^2/2m} L_{m-1}^{(1)}\left(t^2/m\right), \qquad (3.11)
\end{aligned}$$

which equals the spectral density function of the $m \times m$ GUE (see [Led]).

As the density and the characteristic function are a Fourier transform pair, each can be recovered from the other through either the Fourier or the inverse Fourier transform (see for example [SS1, SS2]). Since the characteristic function is given by

$$\phi_m(t) = \mathbb{E}[e^{itX_m}] = \int_{-\infty}^{\infty} e^{itx} f_m(x)\, dx \qquad (3.12)$$

(where X_m is a random variable with density f_m), the density is regained by the relation

$$f_m(x) = \widehat{\phi_m}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_m(t)\, dt. \qquad (3.13)$$

Taking the Fourier transform of the characteristic function $\phi_m(t)$, and interchanging the sum and the integral, we get

$$\begin{aligned}
f_m(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2/2m}}{m} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} \left(\frac{-t^2}{m}\right)^{l-1} e^{-itx} dt \\
&= -\frac{1}{2\pi} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} (-m)^{-l} \int_{-\infty}^{\infty} t^{2(l-1)} e^{-t^2/2m} e^{-itx} dt \\
&= -\frac{1}{2\pi} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!} (-m)^{-l} I_m. \qquad (3.14)
\end{aligned}$$

Math 383: Complex Analysis: Fall ‘23 (Williams)

Professor Steven J Miller: sjm1@williams.edu

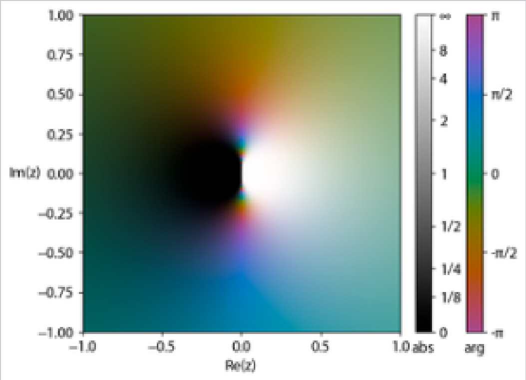
Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/

Lecture 38: 12-8-23: Several Complex Variables: https://youtu.be/tvKfVNy72SY

Little Picard Theorem: If a function $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.

Sketch of Proof: Picard's original proof was based on properties of the modular lambda function, usually denoted by λ, and which performs, using modern terminology, the holomorphic universal covering of the twice punctured plane by the unit disc. This function is explicitly constructed in the theory of elliptic functions. If f omits two values, then the composition of f with the inverse of the modular function maps the plane into the unit disc which implies that f is constant by Liouville's theorem.



Domain coloring plot of the function $\exp(1/z)$, centered on the essential singularity at $z = 0$. The hue of a point z represents the argument of $\exp(1/z)$, the luminance represents its absolute value. This plot shows that arbitrarily close to the singularity, all non-zero values are attained.

This theorem is a significant strengthening of Liouville's theorem which states that the image of an entire non-constant function must be unbounded. Many different proofs of Picard's theorem were later found and Schottky's theorem is a quantitative version of it. In the case where the values of f are missing a single point, this point is called a lacunary value of the function.

Great Picard's Theorem: If an analytic function f has an essential singularity at a point w, then on any punctured neighborhood of w, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.

This is a substantial strengthening of the Casorati–Weierstrass theorem, which only guarantees that the range of f is dense in the complex plane. A result of the Great Picard Theorem is that any entire, non-polynomial function attains all possible complex values infinitely often, with at most one exception.

The "single exception" is needed in both theorems, as demonstrated here:

- e^z is an entire non-constant function that is never 0,
- $e^{\frac{1}{z}}$ has an essential singularity at 0, but still never attains 0 as a value.

https://en.wikipedia.org/wiki/Picard_theorem

Some references:

https://haroldpboas.gitlab.io/courses/650-2013c/notes.pdf

https://arxiv.org/pdf/1507.00562.pdf

https://abel.math.harvard.edu/~knill/////////teaching/severalcomplex_1996/severalcomplex.pdf

https://en.wikipedia.org/wiki/Function_of_several_complex_variables

Review: Definition of the Derivative: One Variable

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Also!

$$\lim_{x \to a} \frac{f(x) - [f(a) + f'(a)(x-a)]}{x-a}$$

tangent line

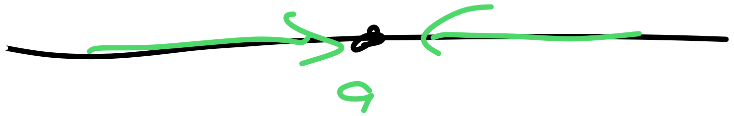
Review: Definition of the Derivative: One Complex Variable

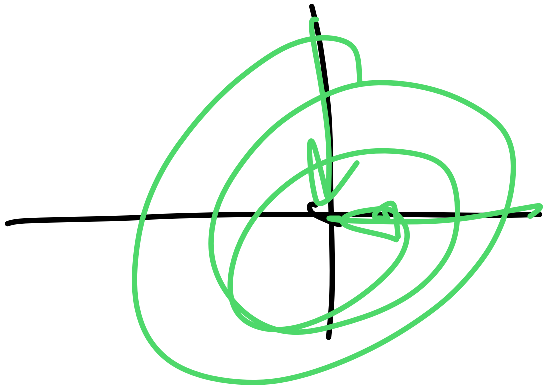
$z = x + iy$

$x \to z$ from previous slide!

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

also have tangent line defn





Question: What should the definition of differentiable (i.e., holomorphic) be for a function of several complex variables?

$f(z_1, z_2, \ldots, z_n)$ so $f: \mathbb{C}^n \to \mathbb{C}$

Try to generalize tangent plane

$$|z-w| = \sqrt{|z_1-w_1|^2 + \cdots + |z_n-w_n|^2}$$

differentiable in each variable

↳ and continuous

POSSIBLE DEFNS OF HOLOMORPHIC

- holomorphic in each variable separately
- continuous as a fn of several variables and holomorphic in each variable separately
- multivariable power series converging in a nbhood of each point.

$\boxed{\text{THM: THE THREE DEFNS ABOVE ARE EQUIVALENT}}$

Polydisc

7 languages

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From Wikipedia, the free encyclopedia

See also: Duocylinder

In the theory of functions of several complex variables, a branch of mathematics, a **polydisc** is a Cartesian product of discs.

More specifically, if we denote by $D(z,r)$ the open disc of center *z* and radius *r* in the complex plane, then an open polydisc is a set of the form

$$D(z_1,r_1)\times\cdots\times D(z_n,r_n).$$

It can be equivalently written as

$$\{w=(w_1,w_2,\dots,w_n)\in\mathbf{C}^n : |z_k-w_k|<r_k,\ \text{for all } k=1,\dots,n\}.$$

One should not confuse the polydisc with the open ball in $\mathbf{C}^n$, which is defined as

$$\{w\in\mathbf{C}^n : \lVert z-w\rVert<r\}.$$

Here, the norm is the Euclidean distance in $\mathbf{C}^n$.

When $n>1$, open balls and open polydiscs are *not* biholomorphically equivalent, that is, there is no biholomorphic mapping between the two. This was proven by Poincaré in 1907 by showing that their automorphism groups have different dimensions as Lie groups.[1]

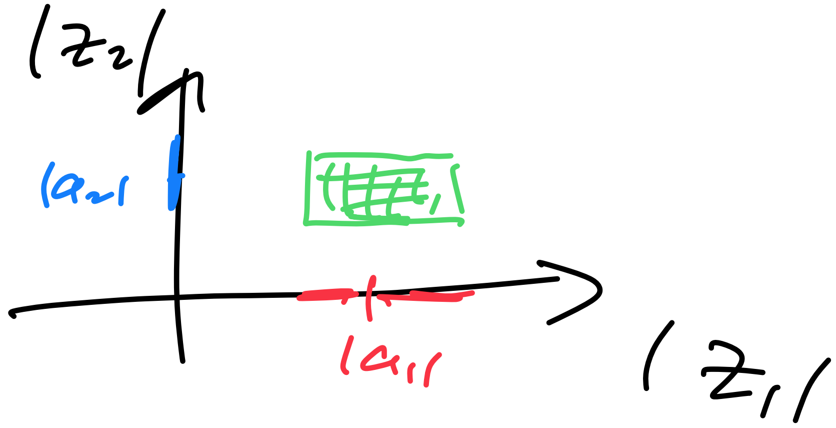
When $n=2$ the term *bidisc* is sometimes used.

https://en.wikipedia.org/wiki/Polydisc

In several variables, a function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if it is holomorphic in each variable separately, and hence if and only if the real part u and the imaginary part v of f satisfiy the Cauchy Riemann equations :

$$\forall i \in \{1, \ldots, n\}, \quad \frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i} \quad \text{and} \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}$$

$|z_1 - a_1| \leq r_1$ and $|z_2 - a_2| \leq r_2$



Cauchy's integral formula I (Polydisc version) [edit]

Prove the sufficiency of two conditions (A) and (B). Let *f* meets the conditions of being continuous and separately homorphic on domain *D*. Each disk has a rectifiable curve γ, γ_ν is piecewise smoothness, class $\mathcal{C}^1$ Jordan closed curve. ($\nu = 1, 2, \ldots, n$) Let D_ν be the domain surrounded by each γ_ν. Cartesian product closure $\overline{D_1 \times D_2 \times \cdots \times D_n}$ is $\overline{D_1 \times D_2 \times \cdots \times D_n} \in D$. Also, take the closed polydisc $\overline{\Delta}$ so that it becomes $\overline{\Delta} \subset D_1 \times D_2 \times \cdots \times D_n$. ($\overline{\Delta}(z,r) = \{\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n; |\zeta_\nu - z_\nu| \leq r_\nu \text{ for all } \nu = 1, \ldots, n\}$ and let $\{z\}_{\nu=1}^n$ be the center of each disk.) Using the Cauchy's integral formula of one variable repeatedly, [note 4]

$$\begin{aligned}
f(z_1, \ldots, z_n) &= \frac{1}{2\pi i} \int_{\partial D_1} \frac{f(\zeta_1, z_2, \ldots, z_n)}{\zeta_1 - z_1}\, d\zeta_1 \\
&= \frac{1}{(2\pi i)^2} \int_{\partial D_2} d\zeta_2 \int_{\partial D_1} \frac{f(\zeta_1, \zeta_2, z_3, \ldots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_1 \\
&= \frac{1}{(2\pi i)^n} \int_{\partial D_n} d\zeta_n \cdots \int_{\partial D_2} d\zeta_2 \int_{\partial D_1} \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)\cdots(\zeta_n - z_n)}\, d\zeta_1
\end{aligned}$$

Because ∂D is a rectifiable Jordanian closed curve[note 5] and *f* is continuous, so the order of products and sums can be exchanged so the iterated integral can be calculated as a multiple integral. Therefore,

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)}\, d\zeta_1 \cdots d\zeta_n \qquad (1)$$

Power series expansion of holomorphic functions on polydisc [edit]

If function *f* is holomorphic, on polydisc $\{z=(z_1,z_2,\dots,z_n)\in\mathbb{C}^n; |z_\nu - a_\nu|<r_\nu, \text{ for all } \nu=1,\dots,n\}$, from the Cauchy's integral formula, we can see that it can be uniquely expanded to the next power series.

$$f(z)=\sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1,\dots,k_n}(z_1-a_1)^{k_1}\cdots(z_n-a_n)^{k_n},$$

$$c_{k_1\cdots k_n}=\frac{1}{(2\pi i)^n}\int_{\partial D_1}\cdots\int_{\partial D_n}\frac{f(\zeta_1,\dots,\zeta_n)}{(\zeta_1-a_1)^{k_1+1}\cdots(\zeta_n-a_n)^{k_n+1}}\,d\zeta_1\cdots d\zeta_n$$

Trick to finding multivariable Taylor Series….

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

$$f(x,y)=f(0,0)+(\nabla f)(0,0)\cdot(x,y)+\frac{1}{2}(x\ y)(Hf)(0,0)\begin{pmatrix}x\\y\end{pmatrix}$$

$$=f(0,0)+\frac{\partial f}{\partial x}(0,0)(x-0)+\frac{\partial f}{\partial y}(0,0)(y-0)+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(0,0)(x-0)^2+2\frac{\partial^2 f}{\partial x\partial y}(0,0)xy+\frac{\partial^2 f}{\partial y^2}(0,0)y^2+\dots\right)$$

Generalizing

$$f(x)=f(0)+\frac{df}{dx}(0)x+\frac{1}{2}\frac{d^2f}{dx^2}(0)x^2+\dots$$

Taylor Expand to 3rd order $f(x,y,z) = \sin(x + y^2 z)$ at $(0,0,0)$

$f(0,0,0) = 0$

$\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ at $(0,0,0)$

then 2nd order, then 3rd order

$g(u) = \sin(u)$ so $g(u) = u - u^3/3! + \ldots$

Take $u = x + y^2 z$

so $f(x,y,z) = (x + y^2 z) - \dfrac{(x+y^2 z)^3}{3!} + \ldots$

$= x + y^2 z - \dfrac{x^3}{6} +$ higher degree

Domain of holomorphy

https://en.wikipedia.org/wiki/Domain_of_holomorphy

6 languages

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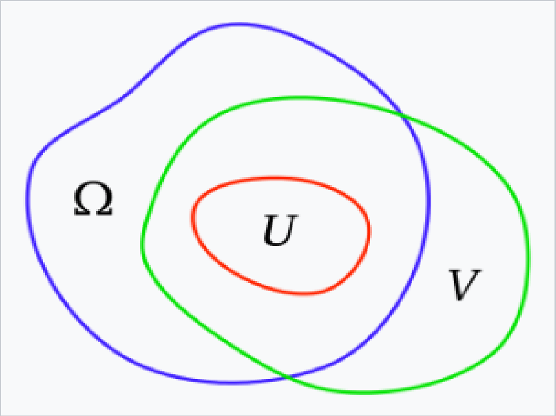
From Wikipedia, the free encyclopedia

In mathematics, in the theory of functions of several complex variables, a **domain of holomorphy** is a domain which is maximal in the sense that there exists a holomorphic function on this domain which cannot be extended to a bigger domain.



The sets in the definition.

Formally, an open set Ω in the n-dimensional complex space $\mathbb{C}^n$ is called a *domain of holomorphy* if there do not exist non-empty open sets $U \subset \Omega$ and $V \subset \mathbb{C}^n$ where V is connected, $V \not\subset \Omega$ and $U \subset \Omega \cap V$ such that for every holomorphic function f on Ω there exists a holomorphic function g on V with $f = g$ on U

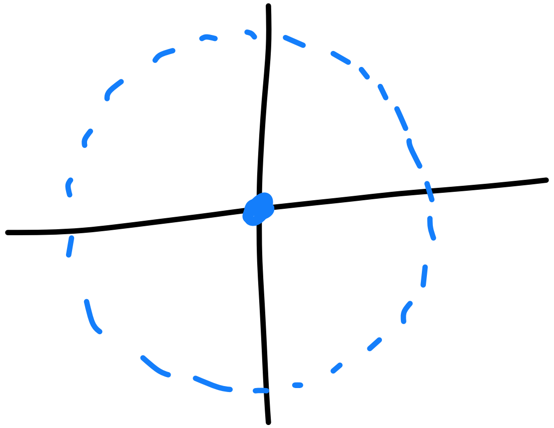
In the $n = 1$ case, every open set is a domain of holomorphy: we can define a holomorphic function with zeros accumulating everywhere on the boundary of the domain, which must then be a natural boundary for a domain of definition of its reciprocal. For $n \geq 2$ this is no longer true, as it follows from Hartogs' lemma.

6 Domains of holomorphy

https://abel.math.harvard.edu/~knill/////////teaching/severalcomplex_1996/severalcomplex.pdf

Definition An open set U in $\mathbf{C}^n$ is a **domain of holomorphy**, if one can not find two nonempty open sets $U_1 \subset U_2$ such that U_2 is connected and not contained in U, $U_1 \subset U_2 \cap U$ and so that for every holomorphic function h on U, there is a holomorphic function h_2 on U_2, which coincides with h on U_1.

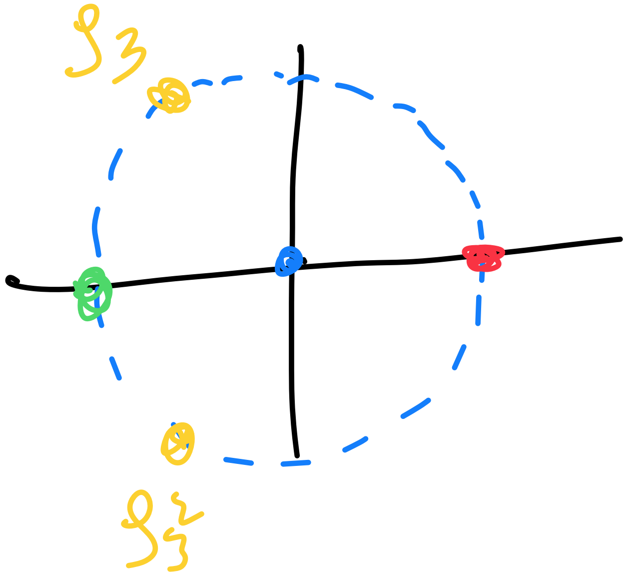
Is the punctured disc / polydisc a domain of holomorphy?



$f(z)$ holo on punctured disc

but not on the disc?

$f(z) = 1/z$



$$\left(\frac{1}{z} + \frac{1}{2}\,\frac{1}{z-1} + \frac{1}{3!}\,\frac{1}{z-(-1)} + \frac{1}{4!}\,\frac{1}{z-\xi_3} + \frac{1}{5!}\,\frac{1}{z-\xi_3^2}\right.$$

THM (NOT MOST GENERAL)

Let $\Delta(0,r)$ be an open polydisc and K any compact set in $\Delta(0,r)$ that does not separate $\Delta(0,r)$ (Think $K=\{0\}$).

Then any function holomorphic on $\Delta(0,r)-K$ extends to be holo on all of $\Delta(0,r)$!

1-Var

$f(z) = \frac{1}{z}$

2-Var

$f(z_1, z_2) = \frac{1}{z_1}$

on polydisc minus $\{0,0\}$

Blows up on more than a point

Try $\frac{1}{z_1^2 + z_2^2}$

blows up at $z_1 = \pm i z_2$

"lines"