Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 02: 9-13-21:
https://youtu.be/8mrJvqbCqB8

Plan for the day: Lecture2: September 13, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Review definition of the complex derivative, examples.
- Cauchy-Riemann Equations.
- Connections with Green's Theorem / Stokes' Theorem.
    - Green's Theorem in a day: https://www.youtube.com/watch?v=Iq-Og1GAtOQ

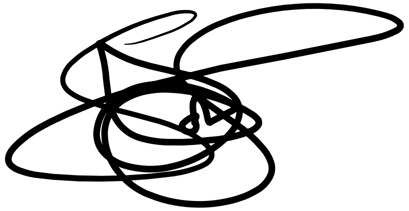
General items.

- Keep track of who is referenced more: Cauchy or Riemann….
- Talk about how to remember results when almost remember….

Complex Deriv

$\rightarrow\leftarrow$ old

def: $f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$



new

Ex: $f(z) = z^n$ then $f'(z) = nz^{n-1}$

$f(z) = \bar{z}$ not diff: $\bar{z} = x - iy$ if $z = x + iy$

holomorphic for complex diff

$f(z) = f(x,y)$ rather than $f(x+iy)$

$f(z) = \underbrace{u(x,y)}_{\text{real}} + i\underbrace{v(x,y)}_{\text{imaginary}}$

$u, v : \mathbb{R}^2 \to \mathbb{R}$

$f : \mathbb{C} \to \mathbb{C}$

Cauchy-Riemann Eqs : f is holomorphic $f(x,y) = u(x,y) + iv(x,y)$

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

Go along real

$$f(z+h) = f(x+h, y)$$

$$= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$z = x + iy$$

$$z+h = x + i(y+h) \quad \text{change is } ih,\ h\in\mathbb{R}$$

$$= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{ih}$$

$$= \frac{1}{i}\frac{\partial f}{\partial y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

EQUAL

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Cauchy-Riemann Eqs

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$u_x = v_y \quad \text{and} \quad v_x = -u_y$$

Thm

If f is holo then C.R. eqs hold, if C.R. holds then f is holo

Cauchy-Riemann Eqs

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$u_x = v_y \text{ and } v_x = -u_y$$

Imagine f is holo, know $u(x,y)$

How free is $v(x,y)$? Only up to a constant

What are the C.R. eqs?

$u_x \Longleftrightarrow v_y$ $\qquad$ $u_y \Longleftrightarrow v_x$

where is the minus sign?

Try $f(z) = z$ $\qquad$ or $\qquad$ $f(z) = z^2 = x^2 - y^2 + 2ixy$

$u(x,y) = x$

$v(x,y) = y$

$f(z) = u(x,y) + iv(x,y)$

$u_x = 1 \qquad u_y = 0$

$v_x = 0 \qquad v_y = 1$

if know there is a minus sign, good

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

$u_x = 2x \qquad u_y = -2y$

$v_x = 2y \qquad v_y = 2x$

Aside: Rational Root Test

$f(x) = a_n x^n + \cdots + a_0 = 0$, $f(p/q) = 0$

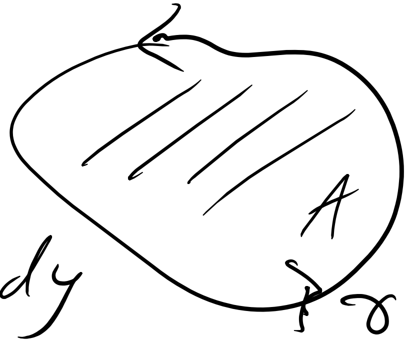
Then $p \mid a_0 \qquad q \mid a_n$

$3x+2$ root is $-2/3$

Green's Thm "P+iQ"

function $P(x,y)$ and $Q(x,y)$

$$\oint_\gamma P\,dx + Q\,dy = \iint_A \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{curl}} dx\,dy$$

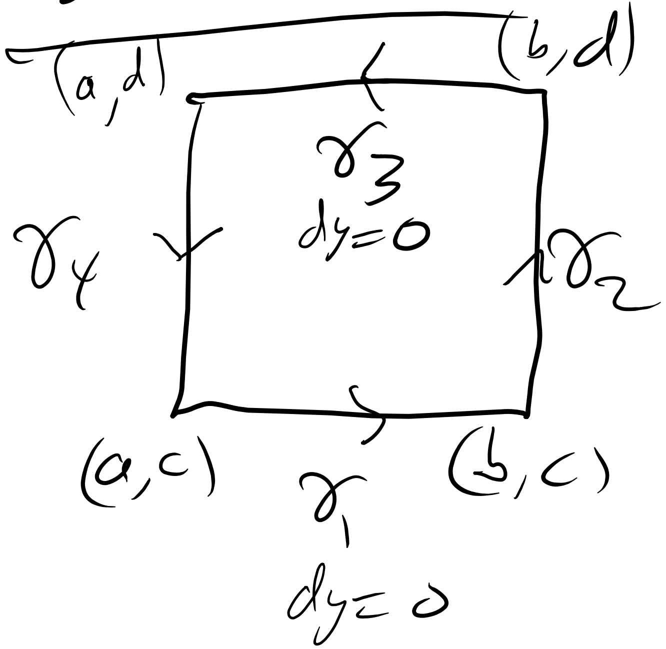


Cauchy-Riemann Eqs

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

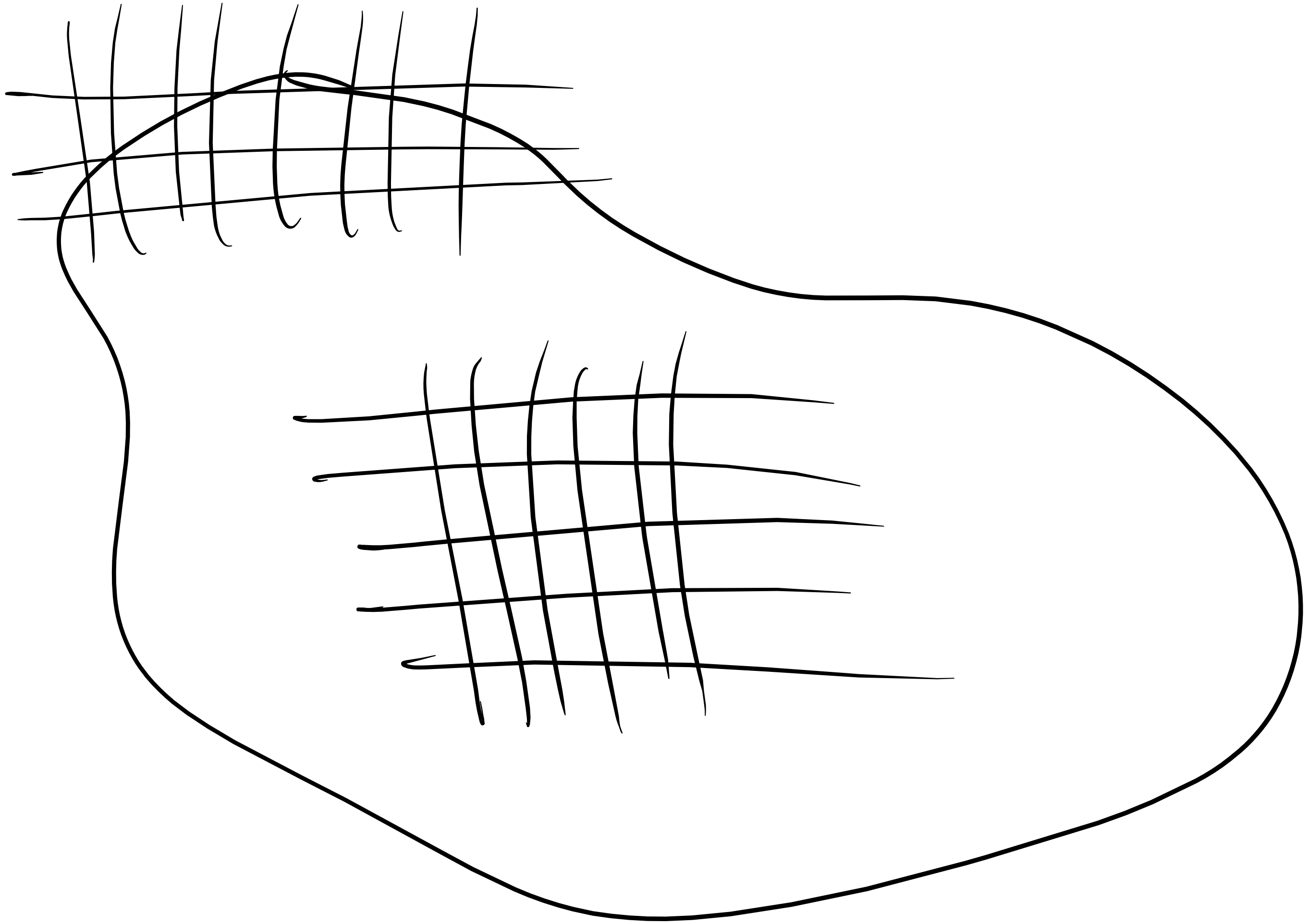
$$u_x = v_y \quad \text{and} \quad v_x = -u_y$$

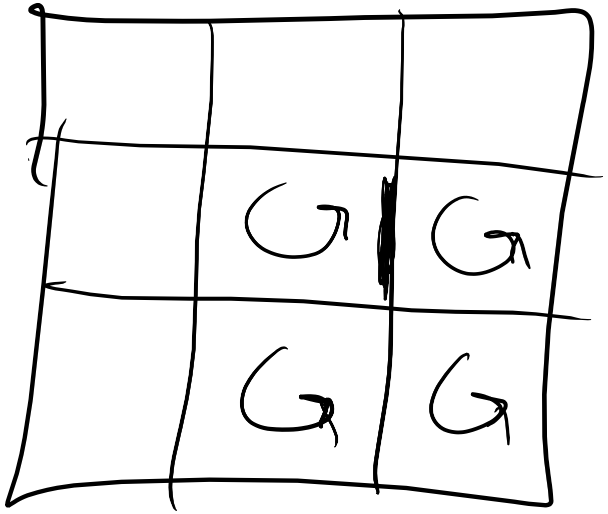
Proof Sketch



$$\int_{y=c}^{d}\int_{x=a}^{b} \frac{\partial Q}{\partial x}\,dx\,dy = \int_{y=c}^{d} Q(x,y)\Big|_{x=a}^{b} dy$$

$$= \int_{y=c}^{d} \Big[\underbrace{Q(b,y)}_{\int_{\gamma_2} Q\,dy} - \underbrace{Q(a,y)}_{\int_{\gamma_4} Q\,dy}\Big] dy = \oint_\gamma Q\,dy$$



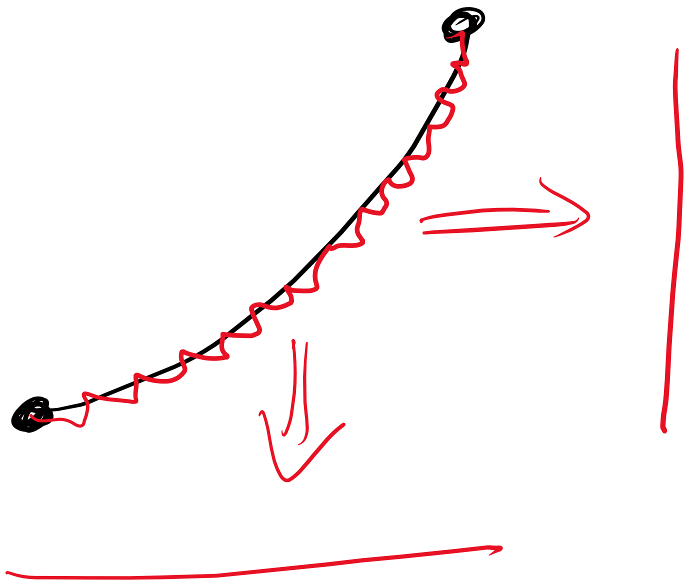


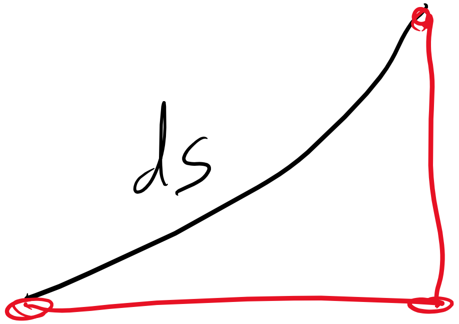
Interior side

once $\uparrow$ once $\downarrow$

Cancels out

Thm: all curves b/w two points have same length





$$ds \neq dx + dy$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds^2 = dx^2 + dy^2$$

Pythagoras!

We need analysis

Green's Thm to Complex

$$\oint_\gamma P\,dx + Q\,dy = \iint_A \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{curl}} dx\,dy$$

Green's Thm!

$f(z) = u + iv \qquad dz = dx + i\,dy$

$$\oint_\gamma f\,dz = \oint_\gamma (u+iv)(dx+i\,dy)$$

$$= \oint_\gamma \underbrace{(u\,dx - v\,dy)}_{P=u \quad Q=-v} + i \oint_\gamma \underbrace{(v\,dx + u\,dy)}_{P=v \quad Q=u} = 0$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \text{ by C.R.}$$

Similarly for other term!

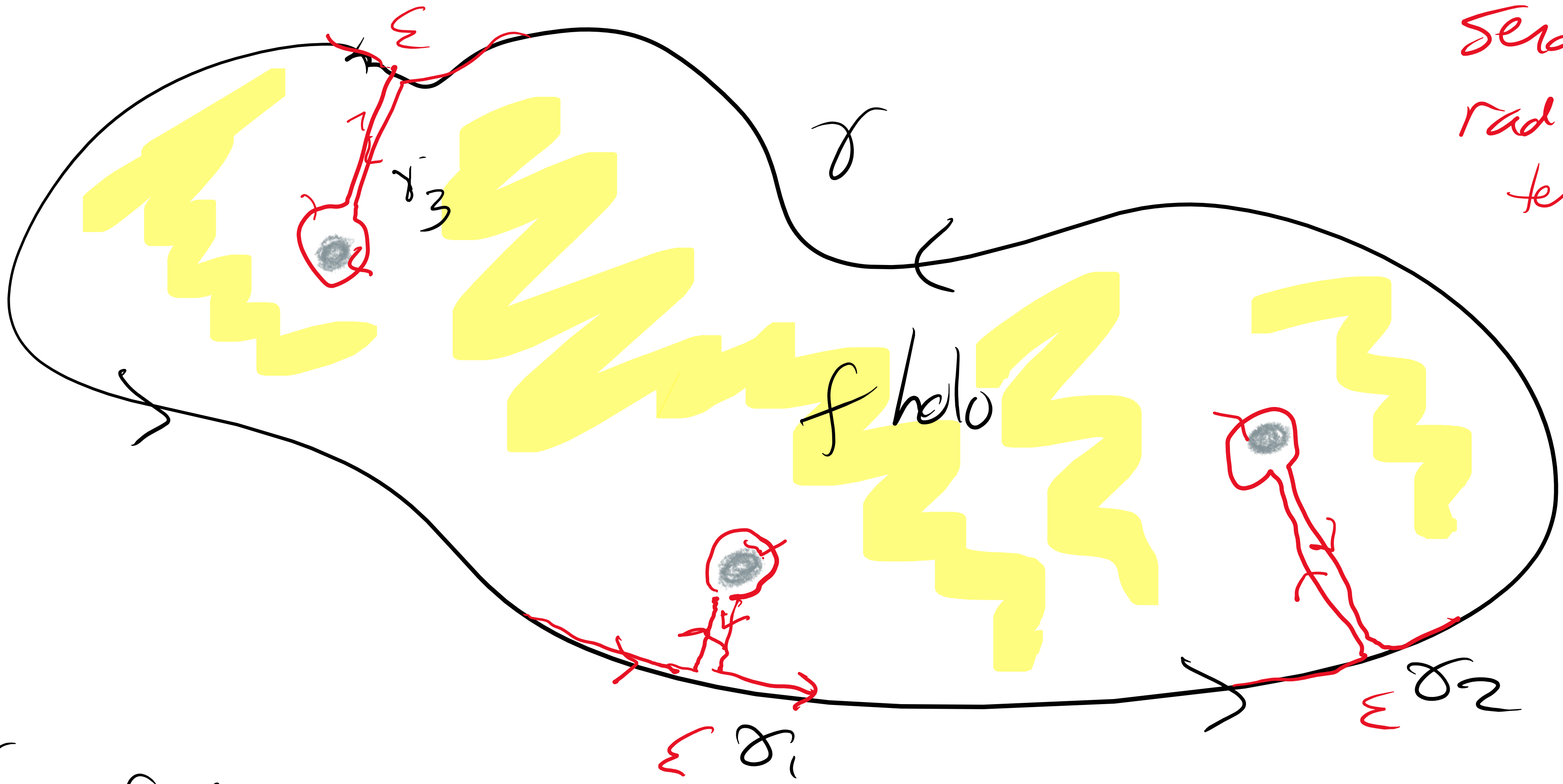
$$\boxed{\int \text{ of holo } f \text{ on closed curve is zero}}$$

Cauchy-Riemann Eqs

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$u_x = v_y \text{ and } v_x = -u_y$$

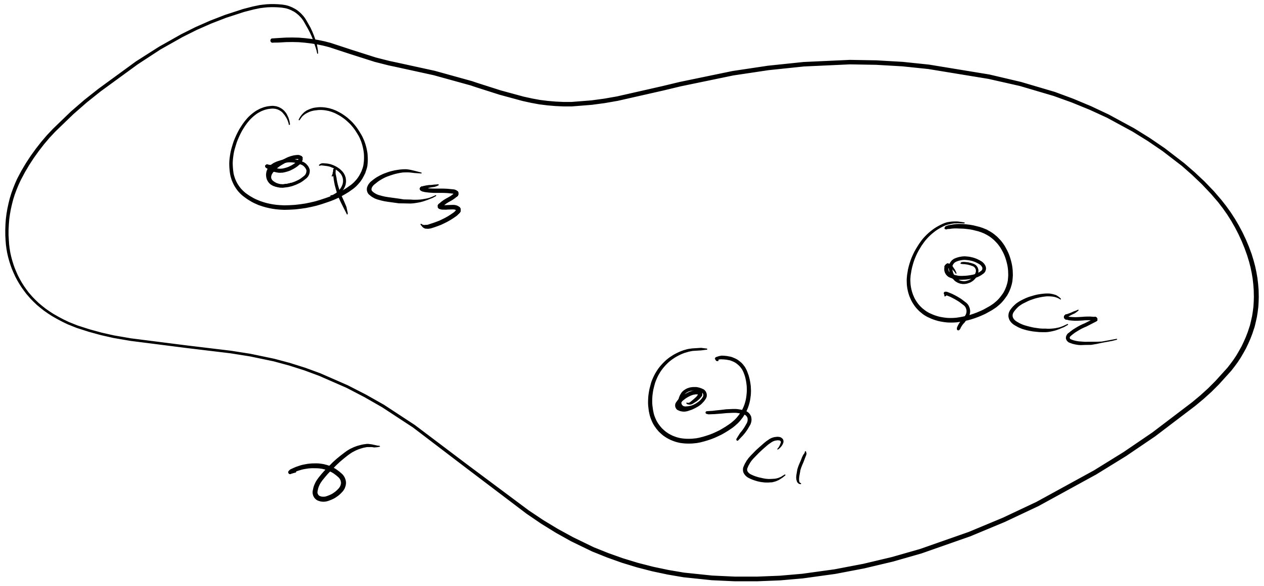
Need f to be non-holo somewhere



send ε → 0
radius of circles
tends to zero

$$0 = \oint_{\gamma+\gamma_1+\gamma_2+\gamma_3} f\,dz \Rightarrow \oint_{\gamma} f = \oint_{-\gamma_1-\gamma_2-\gamma_3} f\,dz$$

neg makes counterclockwise



f holo save bad points

$$\oint_\gamma f = \oint_{C_1} f + \oint_{C_2} f + \oint_{C_3} f$$

"Only" need circles with small radii for our integration curves

Circles : $f(z) = z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$$\oint_{C_r} z^n \, dz$$

$dz = ire^{i\theta} d\theta$

$z = re^{i\theta}$

r fixed

$$= \int_{\theta=0}^{2\pi} r^n e^{in\theta} \cdot ire^{i\theta} d\theta = ir^{n+1} \int_{\theta=0}^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= ir^{n+1} \int_{\theta=0}^{2\pi} \left[\cos((n+1)\theta) + i\sin((n+1)\theta)\right] d\theta$$

$$= \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \text{ any other integer} \end{cases}$$