Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 03: 9-15-21: Review of Real Analysis

Differentiating Term By Term, Analytic Functions, Path Integrals: https://youtu.be/e60Dh8cAIhQ (2017 video)
https://youtu.be/DLyzZhJN58w (2021 lecture)

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$

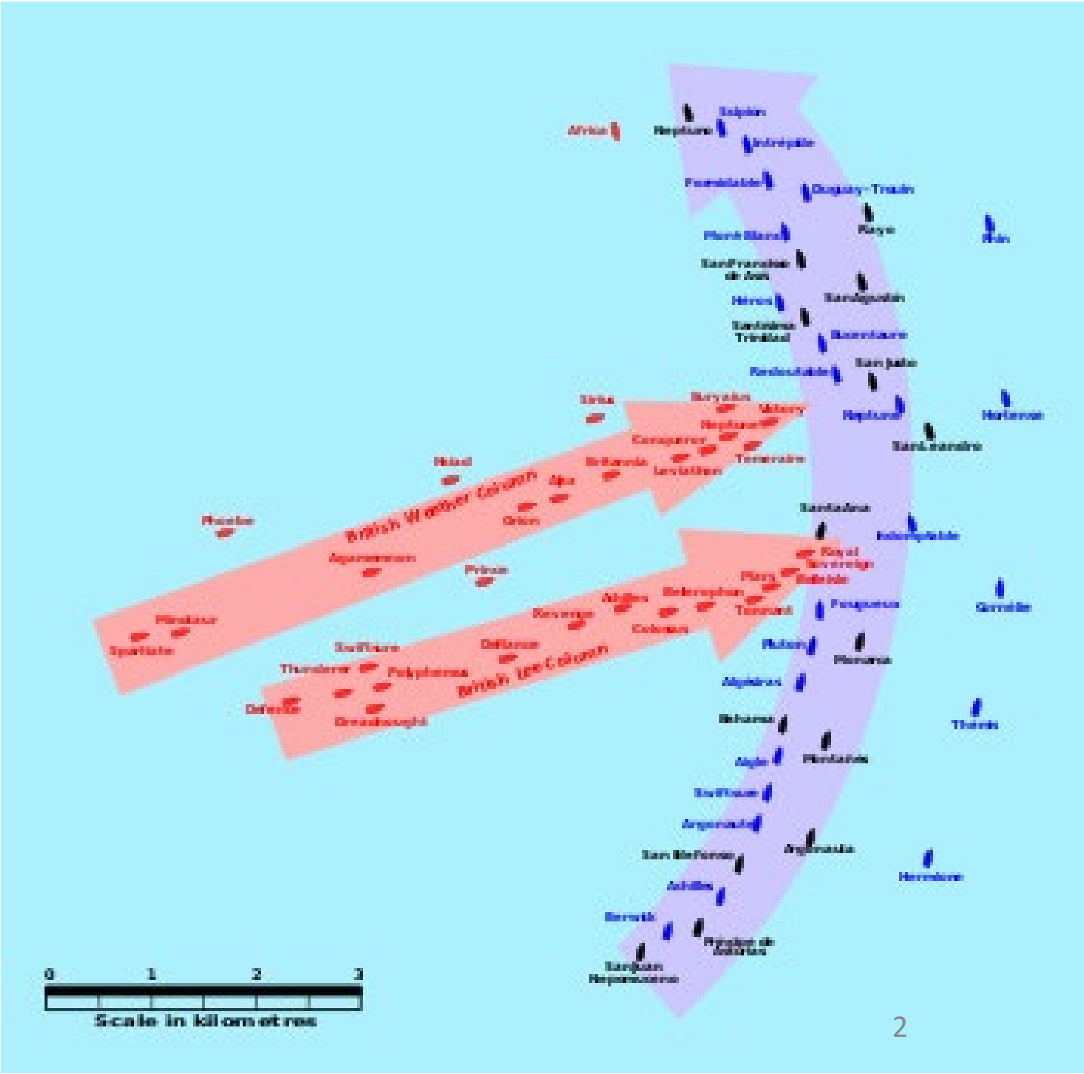
Plan for the day: Lecture 3: September 15, 2021.

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- limsup and liminf
- Convergence (Bolzano-Weierstrass)

General items.

- Sniffing out equations (product rule)



limsup and liminf (from Wikipedia: https://en.wikipedia.org/wiki/Limit_inferior_and_limit_superior)

The **limit inferior** of a sequence (x_n) is defined by

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\Big(\inf_{m\geq n} x_m\Big)$$

or

$$\liminf_{n\to\infty} x_n := \sup_{n\geq 0}\,\inf_{m\geq n} x_m = \sup\{\,\inf\{\,x_m : m\geq n\,\} : n\geq 0\,\}.$$

Similarly, the **limit superior** of (x_n) is defined by

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\Big(\sup_{m\geq n} x_m\Big)$$

or

$$\limsup_{n\to\infty} x_n := \inf_{n\geq 0}\,\sup_{m\geq n} x_m = \inf\{\,\sup\{\,x_m : m\geq n\,\} : n\geq 0\,\}.$$

Alternatively, the notations $\varliminf_{n\to\infty} x_n := \liminf_{n\to\infty} x_n$ and $\varlimsup_{n\to\infty} x_n := \limsup_{n\to\infty} x_n$ are sometimes used.

The limits superior and inferior can equivalently be defined using the concept of subsequential limits of the sequence (x_n).[1] An element ξ of the extended real numbers $\overline{\mathbb{R}}$ is a *subsequential limit* of (x_n) if there exists a strictly increasing sequence of natural numbers (n_k) such that $\xi = \lim_{k\to\infty} x_{n_k}$. If $E \subseteq \overline{\mathbb{R}}$ is the set of all subsequential limits of (x_n), then

$$\limsup_{n\to\infty} x_n = \sup E$$

and

$$\liminf_{n\to\infty} x_n = \inf E.$$

Rearrangement Thm

Assume $\{a_n\}$ converges conditionally, but not absolutely. Show can make limsup/liminf/limit (of the partial sums) whatever you want!

Ex: $(-1)^n/n$: $-1, 1/2, -1/3, 1/4, -1/5, \ldots$

add positives till first exceed 7

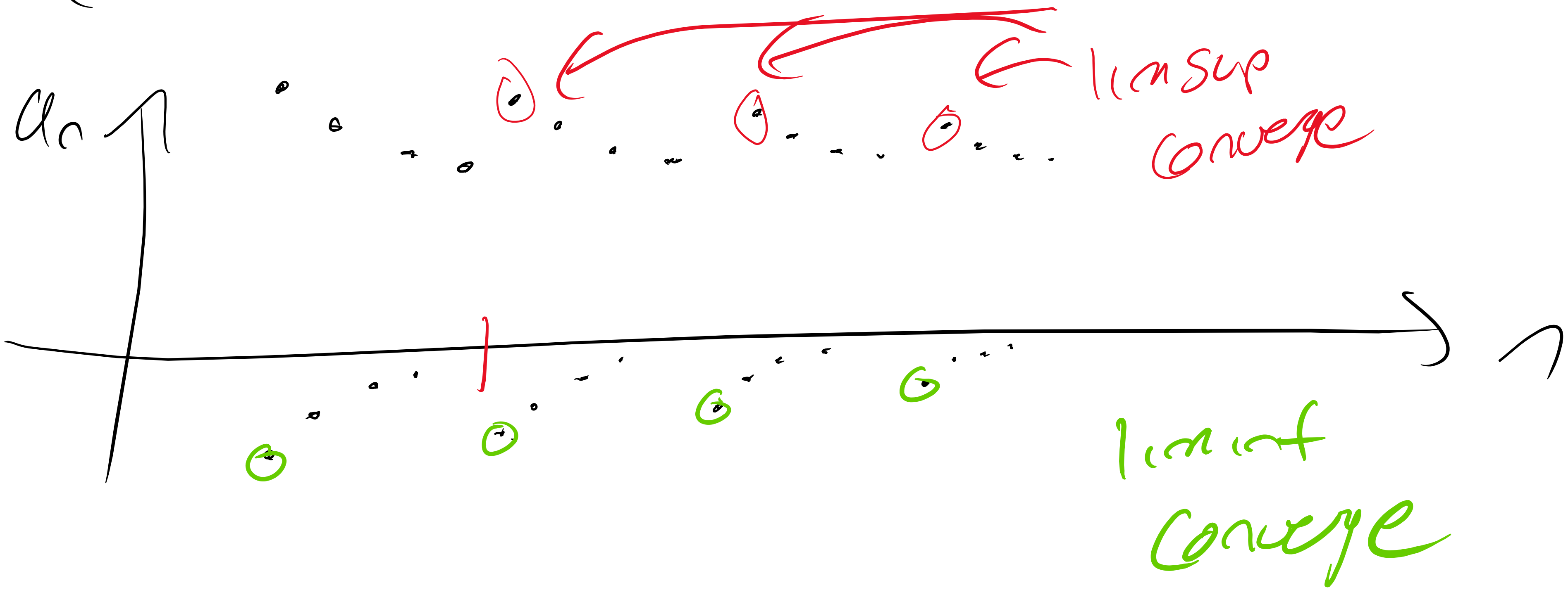
add negatives till just below 3

Shampoo: lather, rinse, repeat

(GNU)

$$a_n = 1 + 2^{-n}(-1)^n \longrightarrow 1$$

$$a_n = (-1)^n + 2^{-n}(-1)^n \text{ : near } 1 \text{ and } -1$$



Converge Conditionally: do ingiven order

Absolutely: converges with absolute values

Ex: $(-1)^n/n = a_n$

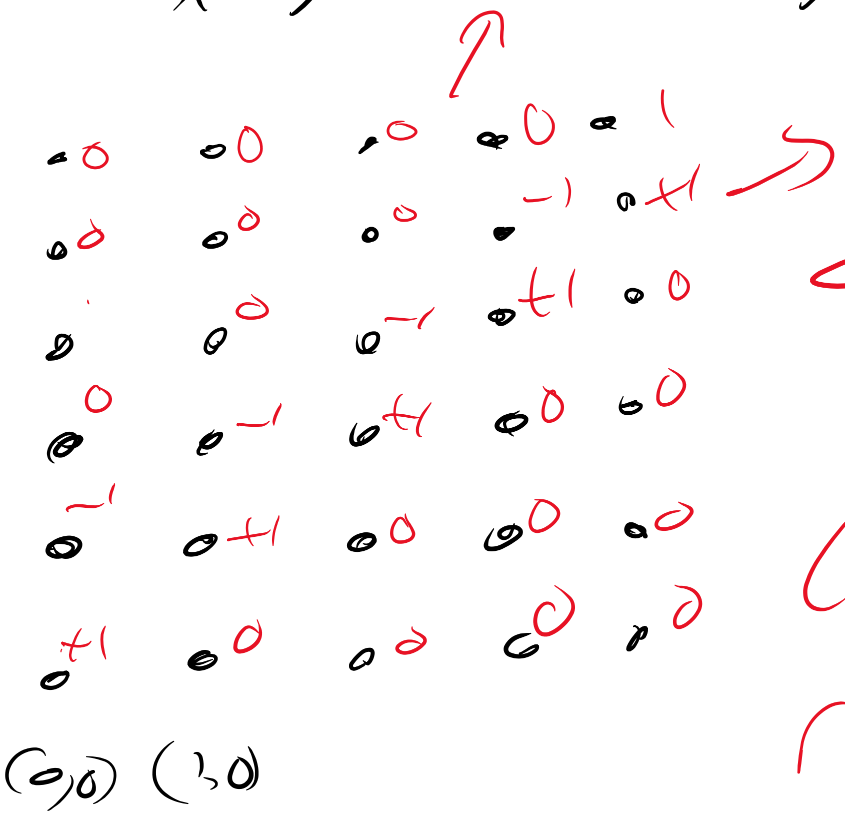
$$\sum_{n=1}^{\infty} a_n = \left(-1+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{4}\right)+\left(-\frac{1}{5}+\frac{1}{6}\right)+\cdots$$

$$-\frac{1}{2n-1}+\frac{1}{2n} = \frac{-1}{(2n-1)(2n)} = \frac{-1}{(2n-1)(2n)} \approx \frac{-1}{4n^2}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{m=1}^{\infty} \frac{-1}{(2m-1)(2m)} \approx -\sum_{m=1}^{\infty} \frac{1}{4m^2} \quad \text{finite}$$

$$\frac{1}{(2m)^2} \le \frac{1}{(2m-1)(2m)} \le \frac{1}{(2m-1)^2}$$

Fubini: $\int_X \int_Y f \, dy \, dx = \int_Y \int_X f \, dx \, dy$ if $\iint |f| < \infty$



$$\sum_m \sum_n a_{m,n} \text{ vs } \sum_n \sum_m a_{m,n}$$

Column then row gives 0

Row then column gives 1

∞ means danger

Does there exist a cont f s.t. $f(x) \geq 0$

$\int_0^\infty f(x)\,dx < \infty$, $\int_0^\infty f(x)^2\,dx < \infty$, ..., $\int_0^\infty f(x)^{2020}\,dx < \infty$

but $\int_0^\infty f(x)^{2021}\,dx = \infty$

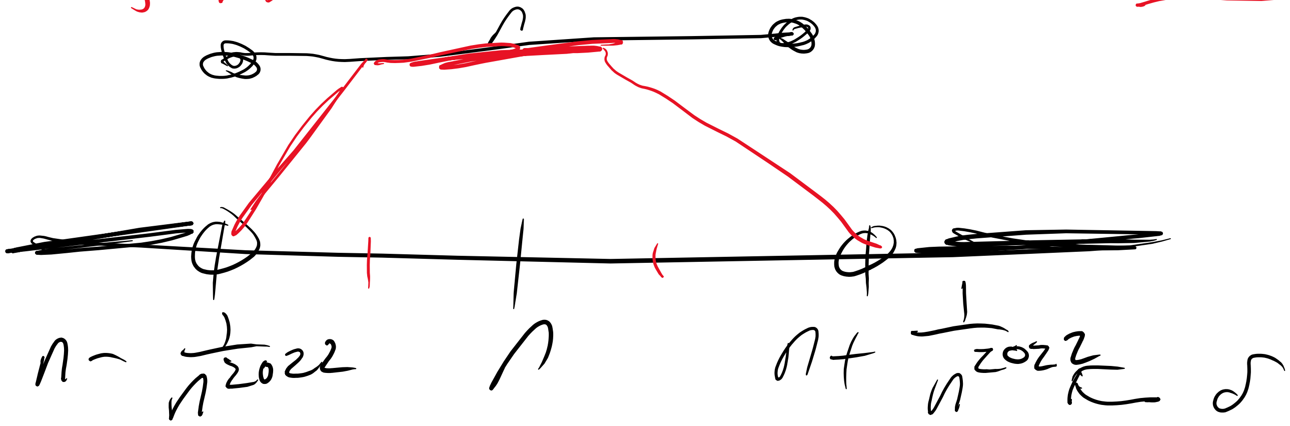
Try $f(x) = x^{-2022}$: NO

Try $f(x) = (x+1)^{-2022}$: NO

Need $f(x)^{2021} > f(x)^{2020}$ SOMEWHERE

Near $x = n$, $f(x) = n$, window of size $2/n^{2022}$

Function is UNBOUNDED!



integral is

$$\int_0^\infty f(x)^m\,dx = \sum_{n=1}^{\infty} \frac{2n^m}{n^{2022}}$$

Bolzano-Weierstrass (from Wikipedia: https://en.wikipedia.org/wiki/Bolzano%E2%80%93Weierstrass_theorem)

Bolzano–Weierstrass theorem

From Wikipedia, the free encyclopedia

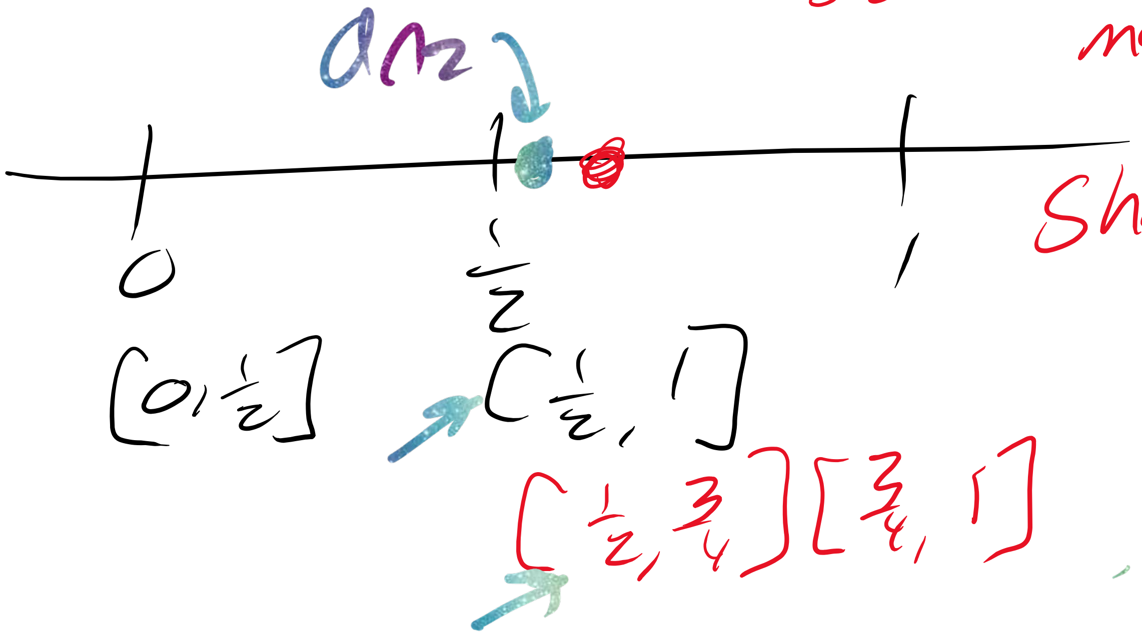
In mathematics, specifically in real analysis, the **Bolzano–Weierstrass theorem**, named after Bernard Bolzano and Karl Weierstrass, is a fundamental result about convergence in a finite-dimensional Euclidean space $\mathbf{R}^n$. The theorem states that each bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.[1] An equivalent formulation is that a subset of $\mathbf{R}^n$ is sequentially compact if and only if it is closed and bounded.[2] The theorem is sometimes called the **sequential compactness theorem**.[3]

Proof in 1-dim

Divide and Conquer

wlog $\{a_n\} \subset [0,1]$

a_{n_1}: smallest index in the subinterval with ∞ many terms.



Shampoo

$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots$

each in $I_1 \supset I_2 \supset I_3 \ldots$ and $|I_n| = 2\,|I_{n+1}|$

$$l_{n_1} \le a_{n_1} \le u_{n_1} \quad \text{where } I_1 = [l_{n_1}, u_{n_1}]$$

$$l_{n_1} \le l_{n_2}, \qquad u_{n_1} \ge u_{n_2}$$

$$l_{n_2} \le a_{n_2} \le u_{n_2} \quad \text{where } I_2 = [l_{n_2}, u_{n_2}]$$

upper bounds

lower bounds

converge to same point

Squeeze or sandwich thm gives $\{a_{n_k}\}$ converge to $\bigcap I_{n_k} = \{x^*\}$

Appendix I: Conditionally convergent and the Rearrangement Theorem.

Consider the alternating harmonic series, where $a(n) = (-1)^n / n$, and choose any two numbers, say 3 and 7. We are going to re-ordering this sequence so that the partial sums form a sequence with limsup of 7 and liminf of 3. We can rearrange the terms to $a(n_1)$, $a(n_2)$, $a(n_3)$ … so that we start with the first positive term, 1/2, and keep adding taking positive terms 1/4, 1/6, 1/8, … until the sum of these just exceeds 7.

We then put in the negative terms, -1, -1/3, -1/5, … until the partial sum dips below 3.

We then put in the positive terms until we are just above 7, then the negatives until we are just below 3. We keep doing this and use up all the terms in our sequence.

Note we always take the largest positive or the smallest negative number available.

If we let s(n) be the sum of the first n terms of this reordered sequence then limsup s(n) is 7 and liminf s(n) is 3.

Appendix II: Strange function.

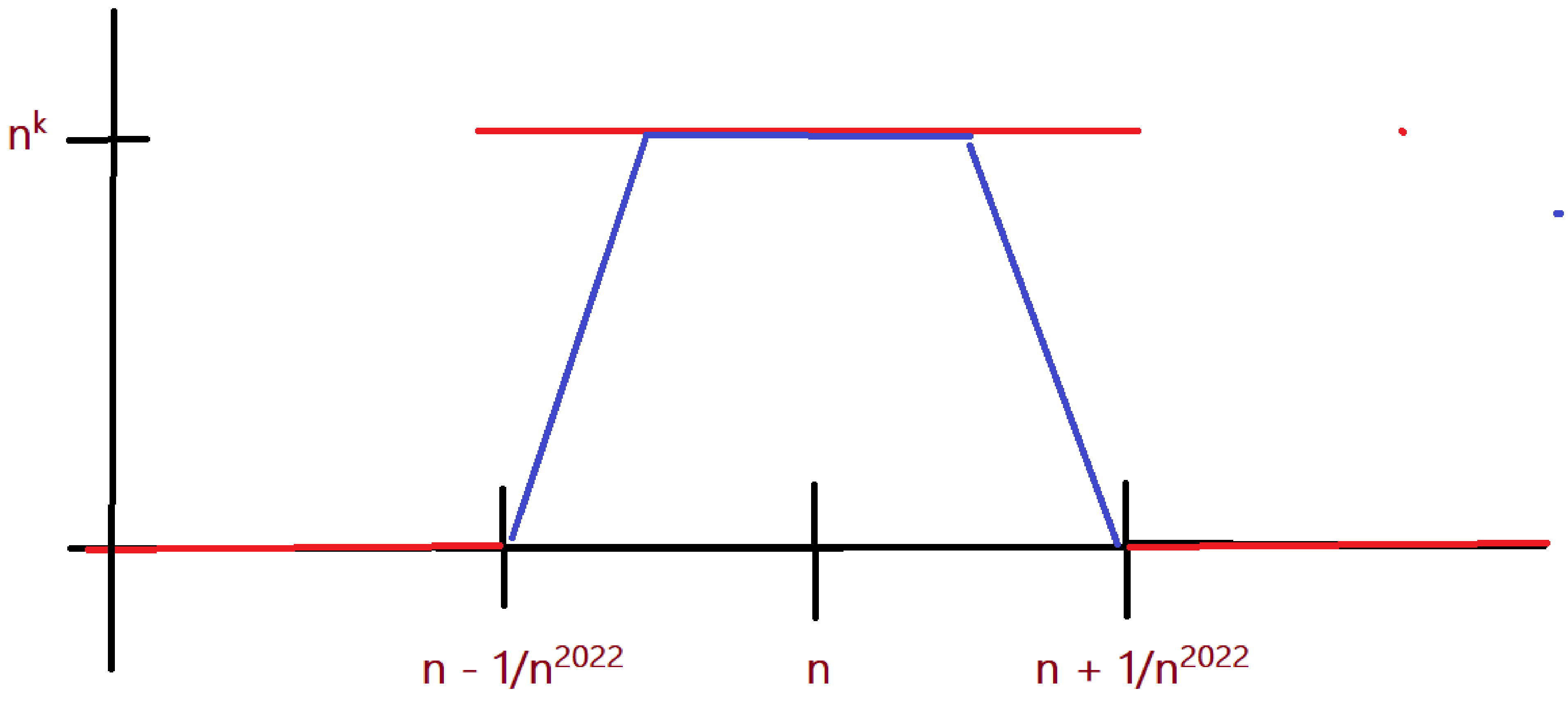
We give a function f such that $||f||_1$, …, $||f||_{2020}$ are finite but $||f||_{2021}$ is not; here $||f||_r$ is the integral of $|f|^r$ from 0 to ∞.

We take a discontinuous non-negative f; it can easily be smoothed.

Let f(x) be zero except in windows near the positive integers n, where f(x) = n for $n - 1/n^{2022} < x < n + 1/n^{2022}$ and 0 otherwise. Thus f is unbounded, it is converging to zero everywhere except at the integers, where it diverges to infinity.

The integral of $f(x)^k$ is just going to be the sum of the areas of the boxes centered at the integers. The box around x=n has height n^k and width $2/n^{2022}$. Thus the integral equals the sum of $2\ n^{k-2022}$. If $k < 2021$ the sum is finite, while if k is 2021 or larger the sum is infinite.

Thus just because an integral converges does not mean the function converges to zero almost everywhere. See next page for a plot….



Not drawn to scale! The function $f(x)^k$ equals zero almost everywhere. It is non-zero in small windows near the integers 1, 2, 3, …. The width of the window decreases with n; at n the window has length $2/n^{2022}$. The function is getting larger and larger as we march down, but on smaller and smaller sets. Note we could make it continuous or differentiable by having f(x) = n from $n - 1/2n^{2022}$ to $n + 1/2n^{2022}$, and then linearly decrease to zero at the endpoints; we illustrate this in blue.