Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 04: 9-17-21: https://youtu.be/6cJjM8PoTzU

Plan for the day: Lecture 4: September 17, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Lecture from 2017: lecture 04: 9/15/17: Primitives, Goursat's Theorem, Goldbach: https://youtu.be/xb9FDV1ldrE

General items.

- Sniffing out equations

Primitives

F is a primitive for f if $F' = f$

↳ unique up to a constant

Love closed curves

area under $y=f(x)$ from $x=a$ to $x=b$

Fundamental Thm of Calc: $\int_a^b f(x)\,dx = F(b) - F(a)$

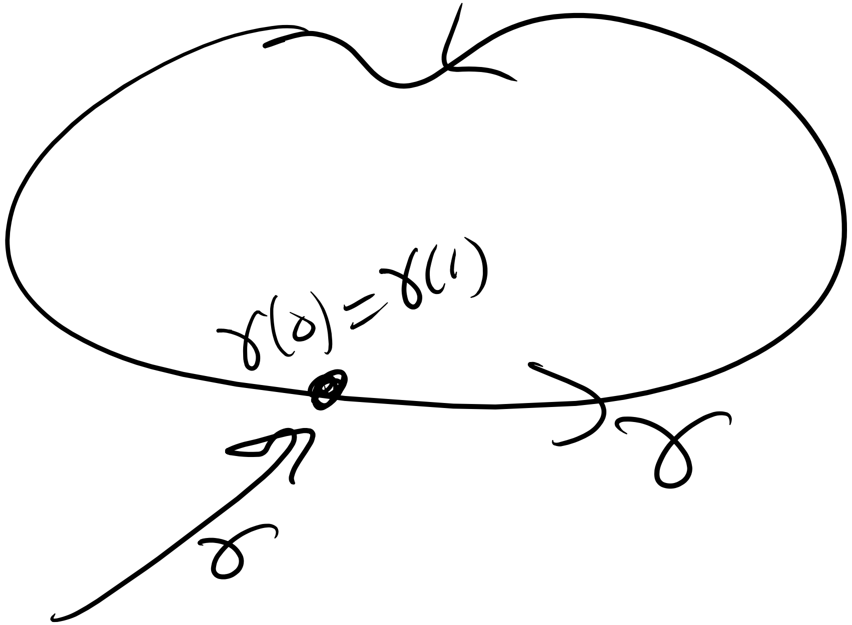
↳ special case of a closed curve: $\int_a^a f(x)\,dx = F(a) - F(a) = 0$

Generalize the FTC

$$\oint_\gamma f(z)\,dz = 0$$

if there is an F st

$$F' = f$$



$[0, 1]$

$$z = \gamma(t)$$

$$dz = \gamma'(t)\,dt$$

$$\int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

$$= F(\gamma(1)) - F(\gamma(0))$$

$$= 0$$

Search for Primitives

$$f(z) = \underbrace{f_{bad}(z)}_{\text{has well understood issues}} + \underbrace{f_{good}(z)}_{\text{has primitive}}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$f(z)$	$F(z)$
e^z	e^z
0	0
1	z
z	$\frac{1}{2} z^2$
z^n	$\frac{1}{n+1} z^{n+1}$
finite poly	other finite poly

Aside: $e^z = \sum_{n=0}^{\infty} z^n/n!$

$$(e^z)' := \lim_{h\to 0} \frac{e^{z+h} - e^z}{h}$$

$$\overset{\text{hw}}{=} e^z \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$e^z e^w = e^{z+w}$$

$$\underbrace{\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!}}$$ vs $$\sum_{k=0}^{\infty} \frac{(z+w)^k}{k!}$$

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{z^l w^{k-l}}{l!\,(k-l)!} \frac{k!}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\sum_{l=0}^{k} \binom{k}{l} z^l w^{k-l}}$$

$$\sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} \qquad \blacksquare$$

Term by term differentiation gives $(e^z)' = e^z$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$ conv abs in Ω

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$$

play games, pull out z, term by term

Smaller in abs value, so converge

(extra $n+1$)

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Use $n^{1/n} \Rightarrow 1$

use $\log(n^{1/n}) = \frac{\log n}{n}$

L'Hopital! $\lim_{n\to\infty} \frac{1/n}{1} = 0$

so exponentiate get 1

Candidate for no primitive

$f(z) = 1/z$

Revisit Real

$f(x) = \frac{1}{x}$ $\quad F(x) = \log x$

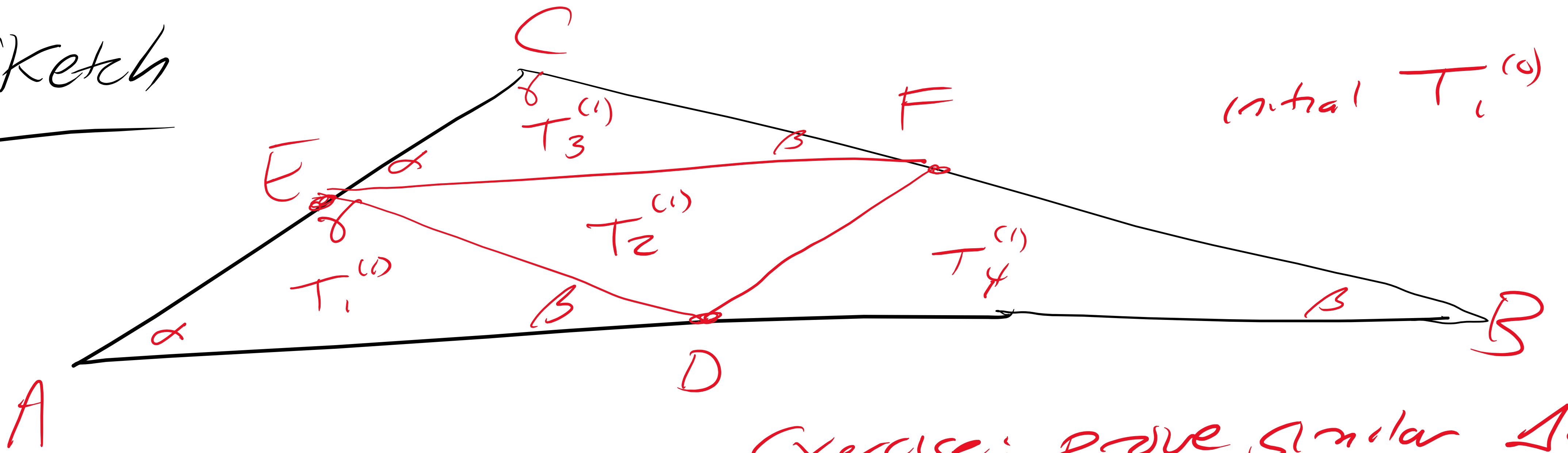
$f(x)$	$f'(x)$	$F(x)$
x^2	$2x$	$x^3/3$
x	1	$x^2/2$
1	0	x
$1/x$	$-1/x^2$	$\log x$
$1/x^2$	$-2/x^3$	$-1/x$

Should be x^0

Goursat's Thm

Open Ω, T triangle in Ω, f holo in Ω, $\oint_T f(z)\,dz = 0$

Sketch



EF // AB

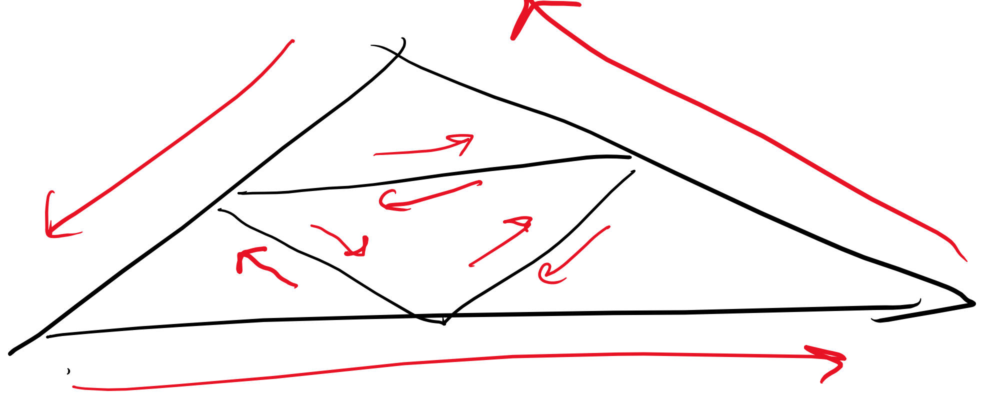
ED // CB

Exercise: prove similar Δs

sides 1/2 the previous level

so $\text{diam}(T_k^{(n+1)}) = \frac{1}{2}\,\text{diam}(T_j^{(n)})$

area would be $\frac{1}{4}$



$$\oint_{T^{(n)}} f = \oint_{4 \text{ triangles at } T^{(n+1)}} f$$

For at least one triangle have

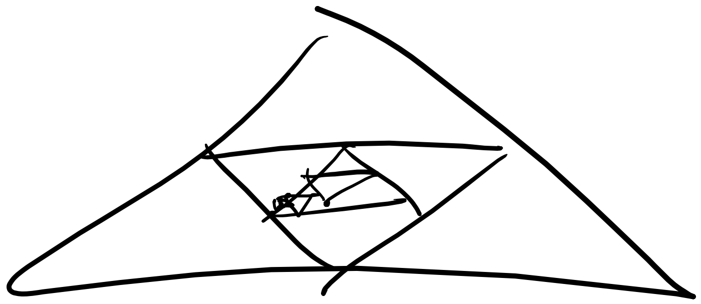
$$\left| \oint_{T^{(n)}} f\,dz \right| \leq 4 \left| \oint_{T_j^{(n+1)}} f\,dz \right|$$

$$|\alpha| = |a+b+c+d| \leq 4 \max(|a|, \ldots, |d|)$$

$$|\alpha| > \max(|a|, \ldots, |d|)$$

Goal is to reduce to small Δs where f is approx const

$$\left|\oint_{T^{(0)}} f\,dz\right| \le 4^n \left|\oint_{T^{(n)}} f\,dz\right|$$



f is holo:

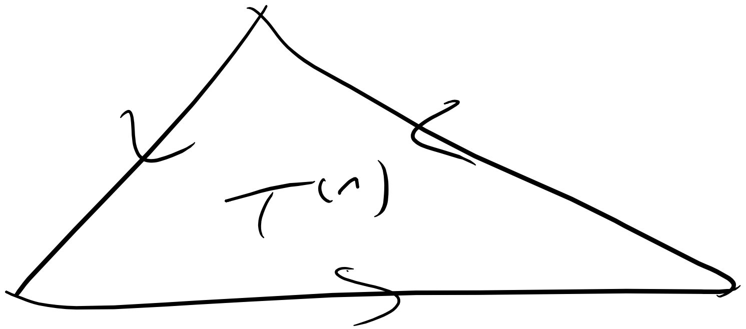
$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f'(z)$$

so $f(z+h) = f(z) + f'(z)\,h + \underbrace{\varepsilon(z,h)}_{\text{Goes to 0 as } h\to 0}\,h$

$$\frac{f(z+h)-f(z)}{h} = f'(z) + \underbrace{\varepsilon(z,h)}_{\substack{\text{lim as } h\to 0\\ \text{must be zero}}}$$

Instead of z and $z+h$ can use z_0 and z

$$f(z) = \underbrace{f(z_0)}_{\substack{\text{primitive} \\ z f(z_0)}} + \underbrace{f'(z_0)(z-z_0)}_{\substack{\text{primitive} \\ f'(z_0)\frac{(z-z_0)^2}{2}}} + \underbrace{\varepsilon(z,z_0)(z-z_0)}_{\substack{\text{not necc} \\ \text{possessing a} \\ \text{primitive}}}$$



$$\oint_{T^{(n)}} f(z)\,dz = 0 + 0 + \underbrace{\oint_{T^{(n)}} \varepsilon(z,z_0)(z-z_0)\,dz}$$

Estimate: $| \quad | \le \underbrace{\max|\varepsilon(z,z_0)|}_{\text{Goes to zero}} * \overset{\text{perimeter}}{\underbrace{\text{Length}(T^{(n)})}_{\le \frac{1}{2^n}\text{Length}(T^{(0)})}} \underbrace{\max|z-z_0|}_{\le \frac{1}{2^n}\text{diam}(T^{(0)})}$

Const $* \; 4^{-n} \; *$ (Something going to zero)

$$\left| \int_{T^{(0)}} f(z)\,dz \right| \leq 4^n 4^{-n} * (\text{something going to zero})$$

$\rightarrow$ it is ZERO!